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A Study on Statistical Convergence of Triple Sequences in Intuitionistic Fuzzy Normed Space

Shailendra Pandit[1*] and Ayaz Ahmad[2]

Abstract. S. Karkaus, K. Demirci, and O. Duman in 2008 studied the statistical convergence of a single sequence over Intuitionistic fuzzy normed space(**IFNS**). M. Mursaleen in 2009, generalized the above work for double sequences over IFNS. The present article is the study of statistical convergence of triple sequence and triple Cauchy sequences on IFNS. In addition, the article includes examples in support of some definitions and theorems. Furthermore, we examined the proof of the completeness of special sequence space.

1. Introduction

In 1951, Fast[10] gave the idea of statistical convergence, which at first, which became very useful tool to deal with the convergence of sequence spaces, where either the idea of ordinary convergence fails or the considered space doesn't stand with purpose, by the idea of natural density. In 2003, B.C. Tripathy[2] studied statistically convergent double sequences, and in 2008, Sahiner and Tripathy [1] published their work on some I-related properties of triple sequences. A lot of work has been done by several authors [3–9] in recent past on statistical convergence of double and triple sequences. In 1965, L. Zadeh[14] gave the notion of fuzzy set theory, which is now very useful in dealing with the problems that couldn't be dealt with in classical set theory. Later on, in 1975, Karmosil and J. michalek[11] generalized the fuzzy set to the fuzzy metric space, which plays a key role in quantum particle physics. Thereafter, many authors have published work on the concept of fuzzy

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metric spaces in their way. Meanwhile, in 1983, Atanassov introduced an Intuitionistic Fuzzy Set as a generalized form of fuzzy set, by assigning the degree of non-membership of an element to the set of study. Subsequently, on this concept, a lot of developments have been made by many authors. The fuzzy set and the intuitionistic fuzzy set are now a wide area of active research.

R. Saadati and Park[17] further studied the Intuitionistic fuzzy topological spaces. Mursaleen and Edely [15] and Tripathy [2] independently introduced the concept of statistical convergence for double sequences. The proposed work is inspired by [15, 18] where statistical convergence of single sequences and double sequences and some topological properties of statistical limit have been studied respectively. We have extended the work to triple sequences in Intuitionistic fuzzy normed spaces (IFNS) and investigate some different topological properties of limit, including completeness of some convergent sequence spaces which are not in earlier work.

2. Basics and Preliminaries

Definition 2.1 ([12])**.** Let $I = [0, 1]$. A binary operation $\circ : I \times I \to I$ is said to be continuous triangular norm(t-norm) if (i) $\circ$ is commutative (ii)$\circ$ is associative (iii)$\circ$ is continuous. (iv) $\gamma \circ 1 = \gamma$ for all $\gamma \in I$.(v) $\gamma_1 \leq \gamma_3$ and $\gamma_2 \leq \gamma_4 \Rightarrow \gamma_1 \circ \gamma_2 \leq \gamma_3 \circ \gamma_4$ for all γ_1, γ_2, γ_3 and $\gamma_4 \in I$

Definition 2.2 ([12])**.** Let $I = [0, 1]$, A binary operation $\diamond : I \times I \to I$ is said to be continuous triangular conorm(t-conorm) if (i) $\diamond$ is commutative (ii) $\diamond$ is associative (iii)$\diamond$ is continuous. (iv) $\gamma \diamond 0 = \gamma$ for all $\gamma \in I$.(v) $\gamma_1 \leq \gamma_3$ and $\gamma_2 \leq \gamma_4 \Rightarrow \gamma_1 \circ \gamma_2 \leq \gamma_3 \circ \gamma_4$ for all γ_1, γ_2, γ_3 and $\gamma_4 \in I$

Definition 2.3 ([18])**.** A 5-tuple ($\mathbb{E}$,$\sigma, \varsigma, \circ, \diamond$) where $\mathbb{E}$ is linear space, $\circ$ is continous t-norm, $\diamond$ is continuous t-conorm and σ, ς are fuzzy sets on $\mathbb{E} \times (0, \infty)$, is said to be Intuitionistic fuzzy normed space (abbreviated as **IFNS**) if for every $\alpha, \beta \in \mathbb{E}$ and $\tau, s \in (0, \infty)$ it fulfils the following subsequent requirements.

(i) $0 \leq \sigma(\alpha, \tau) + \varsigma(\alpha, \tau) \leq 1$.
(ii) $\sigma(\alpha, \tau) > 0$.
(iii) $\sigma(c\alpha, \tau) = \sigma\left(\alpha, \dfrac{\tau}{|c|}\right)$ for all $c \neq 0$.
(iv) $\sigma(\alpha, \tau) = 1$ iff $\alpha = 0$.
(v) $\sigma(\alpha + \beta, \tau + s) \geq \sigma(\alpha, \tau) \circ \sigma(\beta, s)$.
(vi) $\sigma(\alpha, \tau) : (0, \infty) \to [0, 1]$ is a continuous function of τ.
(vii) $\lim_{\tau\to\infty} \sigma(\alpha, \tau) = 1$ and $\lim_{\tau\to 0} \sigma(\alpha, \tau) = 0$.
(viii) $\varsigma(\alpha, \tau) < 1$.

(ix) $\varsigma(c\alpha,\tau)=\varsigma\left(\alpha,\dfrac{\tau}{|c|}\right)$ for all $c\neq 0$.
(x) $\varsigma(\alpha,\tau)=0$ iff $\alpha=0$.
(xi) $\varsigma(\alpha+\beta,\tau+s)\leq\varsigma(\alpha,\tau)\diamond\sigma(\beta,s)$.
(xii) $\varsigma(\alpha,\tau):(0,\infty)\to[0,1]$ is a continuous function of τ.
(xiii) $\lim_{\tau\to\infty}\varsigma(\alpha,\tau)=0$ and $\lim_{\tau\to 0}\varsigma(\alpha,\tau)=1$.

The doublet (σ,ς) gives Intuitionistic fuzzy norm (abbreviated as **IFN**) and defined by.

$$(\sigma,\varsigma)=\{(\alpha,\tau):\sigma(\alpha,\tau),\varsigma(\alpha,\tau):\alpha\in\mathbb{E}\}$$

Remark 2.4 ([15])**.** If $\mathbb{E}$ is a norm linear space then every norm on $\mathbb{E}$ defines an Intuitionistic fuzzy norm.

Definition 2.5. A triple sequence $a=(a_{lkj})$ of elements of IFNS $(\mathbb{E},\sigma,\varsigma,\circ,\diamond)$ is said to be convergent to a number ξ, if there exist $l_0,k_0,j_0\in\mathbb{N}$ such that for all $l\geq l_0$, $k\geq k_0$, $j\geq j_0$, and $\tau\in(0,\infty)$ we have $1-\sigma(a_{lkj}-\xi,\tau)<\varepsilon$ and $\varsigma(a_{lkj}-\xi,\tau)<\varepsilon$ for every $\varepsilon\in(0,1)$.

Lemma 2.6 ([12])**.** *A sequence (a_n) over IFNS $(\mathbb{E},\sigma,\varsigma,\circ,\diamond)$ is convergent to l with respect to IFN (σ,ς) iff $\sigma(a_n-l,\tau)\to 1$ and $\varsigma(a_n-l,\tau)\to 0$ as $n\to\infty$.*

Definition 2.7. Let $A\subseteq\mathbb{N}\times\mathbb{N}\times\mathbb{N}$ and $\Lambda(l,k,j)=\{(r,q,p)\in A:r\leq l;q\leq k;p\leq j\}$ then triple natural density of A is defined as,

$$\delta_3(A)=\lim_{l,k,j\to\infty}\frac{|\Lambda(l,k,j)|}{lkj};$$

where vertical bars stand for cardinality of the set, provided that defined limit exist in Pringsheim's sense.

Example 2.8. Let us suppose $A=\{(r,q,3p):r,q,p\in\mathbb{N}\}$ then

$$\begin{aligned}\delta_3(A)&=\lim_{l,k,j\to\infty}\frac{|\Lambda(l,k,j)|}{lkj}\\&=\lim_{l,k,j\to\infty}\frac{lkj}{3lkj}\\&=\frac{1}{3},\end{aligned}$$

i.e. the triple natural density of A is $\frac{1}{3}$.

Consider another set $B=\{(r^2,q^2,p^2):r,q,p\in\mathbb{N}\}$ then the triple natural density of B is given by

$$\delta_3(B)=\lim_{l,k,j\to\infty}\frac{|\Lambda(l,k,j)|}{lkj}$$

$$= \lim_{l,k,j\to\infty} \frac{\sqrt{lkj}}{lkj}$$
$$= 0.$$

Remark 2.9. Every finite subset of $\mathbb{N}\times\mathbb{N}\times\mathbb{N}$ has triple natural density equals 0.

3. Statistical Convergence

Definition 3.1. A real triple sequence $a = (a_{lkj})$ is said to be statistically convergent to a number ξ if for all $\varepsilon > 0$ the triple natural density of the set $\{(l,k,j) \in \mathbb{N}^3 : |a_{lkj} - \xi| \geq \varepsilon\} = 0$.

Symbolically, we write it as $st_3 - \lim\ a = \xi$

Remark 3.2. Every real triple sequence $a = (a_{lkj})$ which converges to a number l in Pringsheim's sense is also statistically convergent to the same limit (see Remark 2.9.)

Definition 3.3. A triple sequence $a = (a_{lkj})$ of the elements of IFNS $(\mathbb{E},\sigma,\varsigma,\circ,\diamond)$ is said to be statistically convergent to ξ with respect to Intuitionistic fuzzy norm (σ,ς) if for all $\varepsilon \in (0,1)$ and for some $\tau > 0$ the triple natural density of the set $\{(l,k,j) \in \mathbb{N}^3 : 1-\sigma(a_{lkj}-\xi,\tau) \geq \varepsilon$ and $\varsigma(a_{lkj}-\xi,\tau) \geq \varepsilon\} = 0$.

Equivalently,

$$\lim_{r,q,p\to\infty} \frac{1}{rqp} \left|\{(l,k,j) \in \mathbb{N}^3 : l \leq r; k \leq q; j \leq p; 1-\sigma(a_{lkj}-\xi,\tau) \geq \varepsilon \right.$$
$$\left. \text{and}\quad \varsigma(a_{lkj}-\xi,\tau) \geq \varepsilon\}\right|.$$

Where vertical bars denotes the cardinality of the respective set.

Example 3.4. Let $\mathbb{E}$=$\mathbb{R}^3$ where $(\mathbb{R}^3, \|.\|_2)$ is a normed linear space and $\mathbb{R}$ denotes the set of real numbers. Let $a \circ b = ab$ and $a \diamond b = a+b-ab$ consider the IFNS $(\mathbb{E},\sigma,\varsigma,\circ,\diamond)$ where $\sigma(v,\tau) = \dfrac{\tau}{\tau+\|v\|_2}$ and $\varsigma(v,\tau) = \dfrac{\|v\|_2}{\tau+\|v\|_2}$, where $\|v\|_2$ gives the second norm of vector v and τ is some positive real number.

Now define the sequence $a = (a_{lkj})$ as

$$a = a_{lkj}$$
$$= \begin{cases} \left(l, \frac{1}{k}, \frac{1}{j}\right)^t, & l = p^2, \text{ where } p, kj \in \mathbb{N}, \\ \left(\frac{1}{l^2}, \frac{1}{k+j}, \frac{1}{j^2}\right)^t, & \text{otherwise,} \end{cases}$$

We have, $M = \{(l,k,j) \in \mathbb{N}^3 : 1-\sigma(a_{lkj}-0,\tau) \geq \varepsilon$ and $\varsigma(a_{lkj}-0,\tau) \geq \varepsilon\}$ then $M = \{(l^2,k,j) \in \mathbb{N}^3 : l^2 \leq r, k \leq q, j \leq p \in \mathbb{N}\}$.

Therefore,

$$\begin{aligned}\delta_3(M) &= \lim_{r,q,p\to\infty} \frac{\sqrt{r}qp}{rqp} \\ &= 0.\end{aligned}$$

Hence $st_3^{(\sigma,\varsigma)} - \lim a = 0$.

Theorem 3.5. *Let* $(\mathbb{E},\sigma,\varsigma,\circ,\diamond)$ *be an IFNS, if the sequence* $a = (a_{lkj})$ *converges to a number* ξ *in Pringsheim's sense then the sequence* (a_{lkj}) *statistically converges to the same limit* ξ*. But converse not true in general.*

Proof. Let $(\sigma,\varsigma) - \lim a = \xi$. There exist l_0, k_0, and $j_0 \in \mathbb{N}$. Such that for all $(l,k,j) \in \mathbb{N}^3$ and $l \geq l_0, k \geq k_0, j \geq j_0$, $1-\sigma(a_{lkj}-\xi,\tau) < \varepsilon$ and $\varsigma(a_{lkj}-\xi,\tau) < \varepsilon$ which implies the set $M = \{(l,k,j) \in \mathbb{N}^3 : 1-\sigma(a_{lkj}-\xi,\tau) \geq \varepsilon$ and $\varsigma(a_{lkj}-\xi,\tau) \geq \varepsilon\}$ is a finite set. Now (by Remark **??**) we get $\delta_3(M) = 0$ which forces. $st_3^{(\sigma,\varsigma)} - \lim a = \xi$. For converse part, we considering an example.

Let $(\mathbb{R},\sigma,\varsigma,\circ,\diamond)$ be an IFNS. if we define a triple sequence $a = (a_{lkj})$ of real numbers over considered IFNS, defined as

$$\begin{aligned} a &= a_{lkj} \\ &= \begin{cases} l, & l,k \text{ and } j \text{ are perfect cube} \\ 2^{-(l+k+j)}, & \text{otherwise,} \end{cases}\end{aligned}$$

Define, $\sigma(v,\tau) = \dfrac{\tau}{\tau+|v|}$ and $\varsigma(v,\tau) = \dfrac{|v|}{\tau+|v|}$.

We have

$$\begin{aligned} &M = \{(l,k,j) \in \mathbb{N}^3 : 1-\sigma(a_{lkj}-l,\tau) \geq \varepsilon \\ &\varsigma(a_{lkj}-l,\tau) \geq \varepsilon\} = \{(l,k,j) \in \mathbb{N}^3 : l,\ m, \text{and } n \text{ are perfect cube}\}\end{aligned}$$

Now, $\Lambda(r,q,p) = \{(l,k,j) : l \leq r, k \leq q; j \leq p\}$, $|\Lambda(r,q,p)| = r^{\frac{1}{3}}q^{\frac{1}{3}}p^{\frac{1}{3}}$, which yields

$$\begin{aligned}\delta_3(M) &= \lim_{r,q,p\to\infty} \frac{|\Lambda(r,q,p)|}{rqp} \\ &= (rqp)^{-\frac{2}{3}} \to 0.\end{aligned}$$

$st_3^{(\sigma,\varsigma)} - \lim a = 0$. But there does not exist l_0, k_0, and $j_0 \in \mathbb{N}$ such that $1-\sigma(a_{lkj},\tau) < \varepsilon$ and $\varsigma(a_{lkj},\tau) < \varepsilon$ for all $l \geq l_0, k \geq k_0, j \geq j_0$. Hence $(\sigma,\varsigma) - \lim a$ doesn't exist. □

Lemma 3.6. *Let* $(\mathbb{E},\sigma,\varsigma,\circ,\diamond)$ *be an IFNS , then for all* $\varepsilon>0$ *and* $\tau>0$ *the following are equivalents.*

(i) $st_3^{(\sigma,\varsigma)}-\lim(a_{lkj})=\xi$.
(ii) $\delta_3\{(l,k,j)\in\mathbb{N}^3 : 1-\sigma(a_{lkj}-\xi,\tau)\geq\varepsilon\}=\delta_3\{(l,k,j)\in\mathbb{N}^3 : \varsigma(a_{lkj}-\xi,\tau)\geq\varepsilon\}=0$.
(iii) $\delta_3\{(l,k,j)\in\mathbb{N}^3 : 1-\sigma(a_{lkj}-\xi,\tau)<\varepsilon\}=\delta_3\{(l,k,j)\in\mathbb{N}^3 : \varsigma(a_{lkj}-\xi,\tau)<\varepsilon\}=1$.
(iv) $st_3-\lim\sigma(a_{lkj}-\xi,\tau)=1$ *and* $st_3-\lim\varsigma(a_{lkj}-\xi,\tau)=0$.

4. Algebra of Statistical Limit

Theorem 4.1. *Statistical limit of triple sequence* $a=(a_{lkj})$ *over* $(\mathbb{E},\sigma,\varsigma,\circ,\diamond)$ *with respect to IFN* (σ,ς) *is unique.*

Proof. If possible, suppose that $st_3^{(\sigma,\varsigma)}-\lim a\to l_1$ and $st_3^{(\sigma,\varsigma)}-lima\to l_2$ then for every $\varepsilon>0$ choose, $\lambda>0$ such that $(1-\lambda)\circ(1-\lambda)>1-\varepsilon$ and $\lambda\diamond\lambda<\varepsilon$ then for some $\tau>0$ we have

$$\begin{aligned}
P_1(\lambda,\tau)&=\{(l,k,j)\in\mathbb{N}^3 : 1-\sigma(a_{lkj}-l_1,\tau)\geq\lambda\},\\
P_2(\lambda,\tau)&=\{(l,k,j)\in\mathbb{N}^3 : 1-\sigma(a_{lkj}-l_2,\tau)\geq\lambda\},\\
Q_1(\lambda,\tau)&=\{(l,k,j)\in\mathbb{N}^3 : \varsigma(a_{lkj}-l_1,\tau)\geq\lambda\},\\
Q_2(\lambda,\tau)&=\{(l,k,j)\in\mathbb{N}^3 : \varsigma(a_{lkj}-l_2,\tau)\geq\lambda\}.
\end{aligned}$$

By previous lemma, for some $\tau>0$, we have

$$\delta_3(P_1(\lambda,\tau))=\delta_3(Q_1(\lambda,\tau))=0,\qquad \delta_3(P_2(\lambda,\tau))=\delta_3(Q_2(\lambda,\tau))=0,$$

now we defining,
$R_{\sigma,\varsigma}(\lambda,\tau)=\{P_1(\lambda,\tau)\cup P_2(\lambda,\tau)\}\cap\{Q_1(\lambda,\tau)\cup Q_2(\lambda,\tau)\}$ which implies $\delta_3(R_{\sigma,\varsigma}(\lambda,\tau))=0$. Then $\delta_3(\mathbb{N}^3-R_{\sigma,\varsigma}(\lambda,\tau))=1$. Now if $(r,q,p)\in\mathbb{N}^3-R_{\sigma,\varsigma}(\lambda,\tau)$ then $(r,q,p)\in\mathbb{N}^3-\{P_1(\lambda,\tau))\cup P_2(\lambda,\tau)\}$ or $(r,q,p)\in\mathbb{N}^3-\{Q_1(\lambda,\tau))\cup Q_2(\lambda,\tau)\}$.
Case-I. If $(r,q,p)\in\mathbb{N}^3-\{P_1(\lambda,\tau))\cup P_2(\lambda,\tau)\}$ then

$$\begin{aligned}
1-\sigma(l_1-l_2,\tau)&\leq 1-\sigma(l_1-a_{rqp},\tau)\circ\sigma(a_{rqp}-l_2,\tau)\\
&<1-(1-\lambda)\circ(1-\lambda)\\
&<1-(1-\varepsilon)\\
&=\varepsilon.
\end{aligned}$$

Case-II. If $(r,q,p)\in\mathbb{N}^3-\{Q_1(\lambda,\tau))\cup Q_2(\lambda,\tau)\}$ then

$$\begin{aligned}
\varsigma(l_1-l_2,\tau)&\leq\varsigma(l_1-a_{rqp},\tau)\diamond\varsigma(a_{rqp}-l_2,\tau)\\
&<\lambda\diamond\lambda\\
&<\varepsilon.
\end{aligned}$$

since $\tau > 0$ was arbitrary. Hence, by both of the above cases, we conclude that $1 - \sigma(l_1 - l_2, \tau) < \varepsilon$ and $\varsigma(l_1 - l_2, \tau) < \varepsilon$; $\forall\ \varepsilon > 0$. which yields the result. $\square$

Theorem 4.2. *If the sequences $a = (a_{lkj})$ and $b = (b_{lkj})$ over IFNS $(\mathbb{E},\sigma,\varsigma,\circ,\diamond)$ statistically convergent with respect to IFN (σ,ς) to l_1 and l_2 respectively. Then with respect to same IFN (σ,ς),the sequence $(a_{lkj} + b_{lkj})$ converges to $l_1 + l_2$.*

Proof. We are given, $st_3^{(\sigma,\varsigma)} - \lim a = l_1$ and $st_3^{(\sigma,\varsigma)} - \lim b = l_2$; then $P = \{(l,k,j) \in \mathbb{N}^3 : 1 - \sigma(a_{lkj} - l_1, \tau) \geq \varepsilon$ or $\varsigma(a_{lkj} - l_1, \tau) \geq \varepsilon\}$ and $Q = \{(l,k,j) \in \mathbb{N}^3 : 1 - \sigma(b_{lkj} - l_2, \tau) \geq \varepsilon$ or $\varsigma(b_{lkj} - l_2, \tau) \geq \varepsilon\}$ with $\delta_3(Q) = \delta_3(P) = 0$, which implies $\delta_3(P^c) = \delta_3(\mathbb{N}^3 - P) = 1$ and $\delta_3(Q^c) = \delta_3(\mathbb{N}^3 - Q) = 1$, where,

$$P^c = \left\{(l,k,j) \in \mathbb{N}^3 : 1 - \sigma(a_{lkj} - l_1, \tau) < \varepsilon \text{ and } \varsigma(a_{lkj} - l_1, \tau) < \varepsilon\right\},$$
$$Q^c = \left\{(l,k,j) \in \mathbb{N}^3 : 1 - \sigma(b_{lkj} - l_2, \tau) < \varepsilon \text{ and } \varsigma(b_{lkj} - l_2, \tau) \geq \varepsilon\right\}.$$

Now, for $\varepsilon > 0$ choose, $\lambda \in (0,1)$ such that $(1-\varepsilon) \circ (1-\varepsilon) > 1 - \lambda$ and $\varepsilon \diamond \varepsilon < \lambda$ for $(l,k,j) \in (\mathbb{N}^3 - P) \cap (\mathbb{N}^3 - Q) = \mathbb{N}^3 - (P \cup Q)$ we have

$$\begin{aligned} 1 - \sigma\left(a_{lkj} + b_{lkj} - l_1 - l_2, \tau\right) &\leq 1 - \sigma\left(a_{lkj} - l_1, \frac{\tau}{2}\right) \circ \sigma\left(b_{lkj} - l_2, \frac{\tau}{2}\right) \\ &< 1 - (1-\varepsilon) \circ (1-\varepsilon) \\ &< 1 - (1-\lambda) \\ &= \lambda. \end{aligned}$$

Also,

$$\begin{aligned} \varsigma\left(a_{lkj} + b_{lkj} - l_1 - l_2, \tau\right) &\leq \varsigma\left(a_{lkj} - l_1, \frac{\tau}{2}\right) \diamond \varsigma\left(b_{lkj} - l_2, \frac{\tau}{2}\right) \\ &< \varepsilon \diamond \varepsilon \\ &< \lambda, \end{aligned}$$

and we also have, $\delta_3(P \cup Q) \leq \delta_3(P) + \delta_3(Q)$ which implies $\delta_3(P \cup Q) = 0$ equivalently, $\delta_3 \mathbb{N}^3 - (P \cup Q)) = 1$, which yields, $\delta_3\{(l,k,j) \in \mathbb{N}^3 : 1 - \sigma(a_{lkj} + b_{lkj} - l_1 - l_2, \tau) < \lambda$ or $\varsigma(a_{lkj} + b_{lkj} - l_1 - l_2, \tau) < \lambda\} = 1$ then $\delta_3\{(l,k,j) \in \mathbb{N}^3 : 1 - \sigma(a_{lkj} + b_{lkj} - l_1 - l_2, \tau) \geq \lambda$ or $\varsigma(a_{lkj} + b_{lkj} - l_1 - l_2, \tau) \geq \lambda\} = 0$, for all ε (since ε was arbitrary). Hence $st_3^{(\sigma,\varsigma)} - \lim(a+b) = l_1 + l_2$. $\square$

Theorem 4.3. *If the sequences $a = (a_{lkj})$ over IFNS $(\mathbb{E},\sigma,\varsigma,\circ,\diamond)$ statistically convergent with respect to IFN (σ,ς) to ξ. Then with respect to same IFN (σ,ς), the sequence (ca_{lkj}),where c is a scalar, converges to $c\xi$.*

Proof. We have, $st_3^{(\sigma,\varsigma)} - \lim(a) = \xi$, $\delta_3(P) = \delta_3\{(l,k,j) \in \mathbb{N}^3 : 1 - \sigma(a_{lkj} - \xi, \tau) \geq \epsilon$ or $\varsigma(a_{lkj} - \xi, \tau) \geq \varepsilon\} = 0$ another hand, $\delta_3(\mathbb{N}^3 - P) = 1$.

Case I. If $c \neq 0$; for $(l,k,j) \in P$ for all $k \neq 0$. setting $\tau = \dfrac{t}{|c|} > 0$ then,

$$\begin{aligned} 1 - \sigma(ca_{lkj} - rl, t) &= 1 - \sigma\left(a_{lkj} - \xi, \frac{t}{|c|}\right) \\ &= 1 - \sigma(a_{lkj} - \xi, \tau) \\ &\geq \varepsilon, \end{aligned}$$

and

$$\begin{aligned} \varsigma(ca_{lkj} - c\xi, t) &= \varsigma\left(a_{lkj} - \xi, \frac{t}{|c|}\right) \\ &= \varsigma(a_{lkj} - \xi, \tau) \\ &\geq \varepsilon. \end{aligned}$$

Which gives, $\delta_3\{(l,k,j) \in \mathbb{N}^3 : 1 - \sigma(ca_{lkj} - c\xi, t) \geq \varepsilon$ or $\varsigma(ca_{lkj} - c\xi, t) \geq \varepsilon\} = 0$

Case II. If $c = 0$. In this case theorem is obvious.

Thus theorem is established. $\square$

5. Statistical Cauchy Sequences

Definition 5.1 ([16]). Let $(\mathbb{E},\sigma,\varsigma,\circ,\diamond)$ be an IFNS. Then a triple sequence (a_{lkj}) is said to be statistically Cauchy with respect to Intuitionistic fuzzy norm (σ,ς) if for every $\varepsilon > 0$ and for $\tau > 0$ there exist l, k, and $j \in \mathbb{N}$ such that for all $r \geq l$, $q \geq k$, $p \geq j$. we have $\delta_3(\{(r,q,p) \in \mathbb{N}^3 : 1 - \sigma(a_{rqp} - a_{lkj}, \tau) \geq \varepsilon$ or $\varsigma(a_{rqp} - a_{lkj}, \tau) \geq \varepsilon\}) = 0$.

Theorem 5.2. *A triple sequence $a = (a_{lkj})$ over IFNS $(\mathbb{E},\sigma,\varsigma,\circ,\diamond)$ is statistically convergent with respect to IFN (σ,ς) if and only if it is statistically Cauchy with respect to same IFN (σ,ς).*

Proof. Let (a_{lkj}) is statistically convergent to α with respect to (σ,ς) which implies that for all $0 < \varepsilon < 1$ and $\tau > 0$ we have. $\delta_3(\mathbb{M}) = 0$; where, $\mathbb{M} = \{(l,k,j) \in \mathbb{N}^3 : 1 - \sigma(a_{lkj} - \alpha, \tau) \geq \varepsilon$ or $\varsigma(a_{lkj} - \alpha, \tau) \geq \varepsilon\}$ then $\delta_3(\mathbb{M}^c) = 1$. Where $\mathbb{M}^c = \{(r,q,p) \in \mathbb{N}^3 : 1 - \sigma(a_{rqp} - \alpha, \tau) < \varepsilon$ and $\varsigma(a_{lkj} - \alpha, \tau) < \varepsilon\}$.

Choose (r,s,t) and $(u,v,w) \in \mathbb{M}^c$ and choose some $0 < \lambda < 1$ such that $(1-\varepsilon) \circ (1-\varepsilon) > 1 - \lambda$ and $\varepsilon \diamond \varepsilon < \lambda$ then

$$\begin{aligned} 1 - \sigma(a_{rst} - a_{uvw}, \tau) &= 1 - \sigma(a_{rst} - \alpha + \alpha - a_{uvw}, \tau) \\ &\leq 1 - \sigma(a_{rst} - \alpha, \tau) \circ \sigma(a_{uvw} - \alpha, \tau) \\ &< 1 - (1-\varepsilon) \circ (1-\varepsilon) \\ &< 1 - \lambda, \end{aligned}$$

and

$$\begin{aligned}\varsigma(a_{rst}-a_{uvw},\tau) &= \varsigma(a_{rst}-\alpha+\alpha-a_{uvw},\tau)\\ &\leq \varsigma(a_{rst}-\alpha,\tau)\diamond\varsigma(a_{uvw}-\alpha,\tau)\\ &\leq \varepsilon\diamond\varepsilon\\ &< \lambda.\end{aligned}$$

Since (r,s,t) and (u,v,w) were arbitrary, $\mathbb{M}^c\subseteq K$. Where $K=\{(r,s,t)\in\mathbb{N}^3: 1-\sigma(a_{rst}-a_{uvw},\tau)<\lambda \text{ and } \varsigma(a_{rst}-a_{uvw},\tau)<\lambda\}$ then $\delta_3(\mathbb{M}^c)\leq\delta_3(K)\leq 1$, moreover, $1\leq\delta_3(K)\leq 1$ implies $\delta_3(K)=1$, or $\delta_3(K^c)=0$. Where $K^c=\{(r,s,t)\in\mathbb{N}^3: 1-\sigma(a_{rst}-a_{uvw},\tau)\geq\lambda$ or $\varsigma(a_{rst}-a_{uvw},\tau)\geq\lambda\}$.

Since $\lambda>0$, was arbitrary, $\delta_3\left(\{(r,s,t)\in\mathbb{N}^3: 1-\sigma(a_{rst}-a_{uvw},\tau)\geq\varepsilon\right.$ or $\left.\varsigma(a_{rst}-a_{uvw},\tau)\geq\varepsilon\}\right)=0$
which yields (a_{lkj}) is statistically Cauchy with respect to IFN (σ,ς).

Conversely, for some l_0,k_0 and $j_0\in\mathbb{N}$ we have, $\delta_3(\{(r,s,t)\in\mathbb{N}^3: 1-\sigma(a_{rst}-a_{l_0k_0j_0},\tau)\geq\varepsilon$ or $\varsigma(a_{rst}-a_{l_0k_0j_0},\tau)\geq\varepsilon\})=0$, which implies $\delta_3(K)=1$ Where, $K=\{(l,k,j)\in\mathbb{N}^3: 1-\sigma(a_{lkj}-a_{l_0k_0j_0},\tau)<\varepsilon$ and $\varsigma(a_{lkj}-a_{l_0k_0j_0},\tau)<\varepsilon\})=0$. Let $\alpha\in(\mathbb{E},\sigma,\varsigma,\circ,\diamond)$ such that $1-\sigma(a_{l_0k_0j_0}-\alpha,\tau)<\varepsilon$ and $\varsigma(a_{l_0k_0j_0}-\alpha,\tau)<\varepsilon$ for every $\varepsilon>0$.

Consider, $G=\{(u,v,w)\in\mathbb{N}^3: 1-\sigma(a_{uvw}-\alpha,\tau)<\varepsilon$ and $\varsigma(a_{uvw}-\alpha,\tau)<\varepsilon\}$ And letting $(r,q,p)\in K$

$$\begin{aligned}1-\sigma(a_{rqp}-\alpha,\tau) &= 1-\sigma(a_{rqp}-a_{l_0k_0j_0}+a_{l_0k_0j_0}-\alpha,\tau)\\ &< 1-(1-\varepsilon)\circ(1-\varepsilon)\\ &< 1-(1-\lambda)\\ &< \lambda.\end{aligned}$$

Again

$$\begin{aligned}\varsigma(a_{rqp}-\alpha,\tau) &= \varsigma(a_{rqp}-a_{l_0k_0j_0}+a_{l_0k_0j_0}-\alpha,\tau)\\ &\leq \varsigma\left(a_{rqp}-a_{l_0k_0j_0},\frac{\tau}{2}\right)\diamond\varsigma\left(a_{l_0k_0j_0}-\alpha,\frac{\tau}{2}\right)\\ &< \varepsilon\diamond\varepsilon\\ &< \lambda,\end{aligned}$$

then $(r,q,p)\in G$ another hand, $K\subseteq G\Rightarrow\delta_3(K)\leq\delta_3(G)$, $1\leq\delta_3(G)\leq 1\Rightarrow\delta_3(G)=1$ which gives $\delta_3(G^c)=0$ hence, $\delta_3\{(u,v,w): 1-\sigma(a_{uvw}-\alpha,\tau)\geq\varepsilon$ or $\varsigma(a_{uvw}-\alpha,\tau)\geq\varepsilon\}=0$. Then $st_3^{(\sigma,\varsigma)}-\lim a=\alpha$. Thus proof is completed. □

We now introduce some special spaces and then proved its completeness which is immediate effect of above definitions and results.

${}_3l_\infty=\{x=(x_{lkj}): x$ is a bounded triple sequence of real numbers.$\}$

${}_3l_{\infty S} = \{(a_{lkj}) \in {}_3l_\infty$: for some naturals l_0, k_0, j_0 we have $\delta_3\{(l,k,j) \in \mathbb{N}^3 : 1-\sigma(a_{lkj} - a_{l_0k_0j_0}, \tau) \geq \epsilon$ or $\varsigma(a_{lkj} - a_{l_0k_0j_0}, \tau) \geq \epsilon\} = 0\}$.

Theorem 5.3. *The space ${}_3l_{\infty S}$ of statistically Cauchy triple bounded sequences is complete.*

Proof. Let $\left(a^n_{lkj}\right)_{n\geq 1}$ is a Cauchy sequence in ${}_3l_{\infty S}$ which implies there exist $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ and $\varepsilon > 0$ we have

$$1-\sigma\left(a^n_{lkj} - a^{n_0}_{lkj}, \tau\right) < \varepsilon, \qquad \varsigma\left(a^n_{lkj} - a^{n_0}_{lkj}, \tau\right) < \varepsilon.$$

Also, $\left(a^n_{lkj}\right)$ is a statistically Cauchy sequence for each $n \in \mathbb{N}$.

Now, fixing l, k, j and cosidering the sequence of real numbers. $\left(a^n_{lkj}\right) = a^1_{lkj}, a^2_{lkj}, a^3_{lkj}, \ldots,$ Which supposed converges to (z_{lkj}). (due to completeness of $\mathbb{R}$). Since l, k, j were arbitrary. Then $\left(a^n_{lkj}\right)$ converges to (z_{lkj}) for all l, k and j. Hence, there exist $n_\varepsilon \in \mathbb{N}$ such that for all $n \geq n_\varepsilon$. We have, $1-\sigma(a^n_{lkj} - z_{lkj}, \tau) < \varepsilon$ and $\varsigma(a^n_{lkj} - z_{lkj}, \tau) < \varepsilon$.

Again for, $n \geq max\{n_0, n_\varepsilon\}$ the sequence $\left(a^n_{lkj}\right)$ is statistically Cauchy. Which implies that there exist $l_0, k_0, j_0 \in \mathbb{N}$ such that

$\delta_3\left\{(l,k,j) \in \mathbb{N}^3 : 1-\sigma\left(a^n_{lkj} - a^n_{l_0k_0j_0}, \tau\right) \geq \varepsilon \text{ or } \varsigma(a^n_{lkj} - a^n_{l_0k_0j_0}, \tau) \geq \varepsilon\right\} = 0.$

Then $\delta_3(\mathbb{M}) =$. Where, $\mathbb{M} = \Big\{(l,k,j) \in \mathbb{N}^3 : 1-\sigma\left(a^n_{lkj} - a^n_{l_0k_0j_0}, \tau\right) < \varepsilon$ or $\varsigma\left(a^n_{lkj} - a^n_{l_0k_0j_0}, \tau\right) < \varepsilon\Big\}$.

Now choose $\lambda \in (0,1)$ such that $(1-\varepsilon)\circ(1-\varepsilon) > 1-\lambda$ and $\varepsilon \diamond \varepsilon < \lambda$. Letting

$K^c = \{(p,q,r) \in \mathbb{N}^3 : 1-\sigma(z_{pqr} - z_{p_0q_0r_0}, \tau) \geq \varepsilon$ or $\varsigma(z_{pqr} - z_{p_0q_0r_0}, \tau) \geq \varepsilon\}$.

Now we wish to prove $\delta_3(K^c) = 0$ for some naturals l_0, k_0 and j_0, for the requirement, we have $K = \{(p,q,r) \in \mathbb{N}^3 : 1-\sigma(z_{pqr} - z_{p_0q_0r_0}, \tau) < \varepsilon$ and $\varsigma(z_{pqr} - z_{p_0q_0r_0}, \tau) < \varepsilon\}$.

Put $l_0 \geq p_0$; $k_0 \geq q_0$; $j_0 \geq r_0$; and let $(r,s,t) \in \mathbb{M}$.
Then

$$\begin{aligned}
&1-\sigma(z_{rst} - z_{p_0q_0r_0}, \tau)\\
&\quad = 1-\sigma\left(z_{rst} - a^n_{rst} + a^n_{rst} - a^n_{p_0q_0r_0} + a^n_{p_0q_0r_0} - z_{p_0q_0r_0}, \tau\right)\\
&\quad \leq 1-\sigma\left(z_{rst} - a^n_{rst}, \frac{\tau}{3}\right)\circ\sigma\left(a^n_{rst} - a^n_{p_0q_0r_0}, \frac{\tau}{3}\right)\circ\sigma\left(a^n_{p_0q_0r_0} - z_{p_0q_0r_0}, \frac{\tau}{3}\right)\\
&\quad < 1-(1-\varepsilon)\circ(1-\varepsilon)\circ(1-\varepsilon)\\
&\quad < 1-(1-\lambda)\\
&\quad < \lambda.
\end{aligned}$$

Also, we have

$$\begin{aligned}
&\varsigma(z_{rst} - z_{p_0q_0r_0}, \tau \\
&\quad = \varsigma\left(z_{rst} - a^n_{rst} + a^n_{rst} - a^n_{p_0q_0r_0} + a^n_{p_0q_0r_0} - z_{p_0q_0r_0}, \tau\right) \\
&\quad \leq \varsigma\left(z_{rst} - a^n_{rst}, \frac{\tau}{3}\right) \diamond \varsigma\left(a^n_{rst} - a^n_{p_0q_0r_0}, \frac{\tau}{3}\right) \diamond \varsigma\left(a^n_{p_0q_0r_0} - z_{p_0q_0r_0}, \frac{\tau}{3}\right) \\
&\quad < \varepsilon \diamond \varepsilon \diamond \varepsilon \\
&\quad < \lambda,
\end{aligned}$$

which implies that $\mathbb{M} \subseteq K \Rightarrow K^c \subseteq \mathbb{M}^c$ therefore, $\delta_3(K^c) \leq \delta_3(\mathbb{M}^c) = 0$. Then $\delta_3(K^c) = 0$. Hence, $(z_{lkj}) \in {}_3l_{\infty S}$. Which completes the proof. □

6. Conclusion

The work intends to generalize the statistical convergence of sequences to triple sequences in the setting of intuitionistic fuzzy normed spaces. Algebraic properties of the limits of this convergence have been analyzed. The relation between usual convergence and statistical convergence has been studied for triple sequences along with topological properties of the spaces.

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