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An Approach to Change the Topology of a Topological Space

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ABSTRACT. In this paper, we are going to introduce an approach to convert the topology of a topological space to another topology (in fact, a coarser topology). For this purpose, we consider a topological space (X, τ) with a closed point p of its points. Using a grill $\mathcal{G}$ on the space (X, τ) and the closure operator associated with τ, we define a Kuratowski closure operator cl_p^* on X to create the desired topology. We then characterize the form of this resulting topology and also determine its relationship to the initial topology of the space.

1. Introduction

Topology as an abstract branch of mathematics can be used in many other sciences, including IT and computer science. If the arrangement of information of a number of data can be defined as a topology, then applying a new condition (or conditions) on the data information can lead to a change in the previous arrangement which can be considered as a changing in topology. Affected by this issue, in this article we intend to change the topology of a topological space.

Denote $\mathcal{P}(X)$ as the power set of X. Then cl as an operator on $\mathcal{P}(X)$ is called a Kuratowski closure operator, if it satisfies the following Kuratowski closure axioms;

(i) $cl(\emptyset) = \emptyset$ (It preserves the empty set),
(ii) $A \subseteq cl(A)$ for any $A \subseteq X$ (It is extensive),
(iii) $cl(cl(A)) = cl(A)$ for any $A \subseteq X$ (It is idempotent),

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(iv) $cl(A \cup B) = cl(A) \cup cl(B)$ for any $A \subseteq X$(It preserves binary unions).

It is well-known that topological spaces are characterized by Kuratowski closure operators. In fact, associated with any topology τ on a set X there is a topological closure operator on the set X, denoted cl_τ or cl_X(in short, cl), which gives for any subset $A \subseteq X$, the smallest closed set clA containing A. Also, on the other hand, corresponding to any topological closure operator cl on a set X, there exists a unique topology, say τ, on the set X in the form of $\tau = \{X - A : cl(A) = A\}$.

In this paper, we are going to introduce a method based on which we can turn the topology of a desired topological space into another topology. Indeed in this method, we try to define a new Kuratowski closure operator cl^* to create a new topology on the set X by using the initial Kuratowski closure operator cl associated with a space (X, τ). It is worth mentioning that in the desired method, we will use a mathematical tool called the grill, which was first introduced by Choquet [3] in 1947 as follows;

A non-empty collection $\mathcal{G}$ of non-empty subsets of X is named a grill, if for every $A, B \subseteq X$ the following properties are true:

(i) if $A \subseteq B \subseteq X$ and $A \in \mathcal{G}$, then $B \in \mathcal{G}$,
(ii) if $A, B \subseteq X$ and $A \cup B \in \mathcal{G}$, then $A \in \mathcal{G}$ or $B \in \mathcal{G}$.

It is worth noting that grills like ideals, nets and filters are very useful tool. It is also seen from the literature that in many situations, grills are more productive and more flexible than these concepts. For instance, we can see its role in proximity spaces ([5]), closure spaces ([2]) and the theory of extension of compactifications ([1], [4]).

For a topological space (X, τ), the following collections are examples of grills on X;

- the collection of all uncountable subsets of X,
- the collection of all subsets of X whose closure has nonempty interior,
- for $A \subseteq X$ the collection $\{B \subseteq X : B \cap A \neq \emptyset\}$ and specially for every point p of X the collection $\{A \subseteq X; p \in A\}$,
- and also, $\mathcal{G} = \mathcal{P}(X) \setminus \{\emptyset\}$.

We denote the family of all grills on X with $\mathcal{G}(X)$ and note that the maximal element of $\mathcal{G}(X)$ is $\mathcal{G} = \mathcal{P}(X) \setminus \{\emptyset\}$.

Remark 1.1. Here we remind the reader that the condition (i) requires $X \in \mathcal{G}$ for any $\mathcal{G} \in \mathcal{G}(X)$ and from condition (ii) it follows that $A^c \in \mathcal{G}$ when $A \notin \mathcal{G}$ for any $A \subseteq X$ and $\mathcal{G} \in \mathcal{G}(X)$.

Here, let us to provide a brief description of the method.
Let (X, τ) be an arbitrary topological space, $p \in X$ and $\mathcal{G}$ be a grill

on (X,τ), we define an operator cl_p^* on X based on the grill $\mathcal{G}$ and the closure operator cl associated with τ, as follows;

$$cl_p^*(A) = \begin{cases} clA, & \text{if } clA \notin calG, \\ clA \cup \{p\}, & \text{if } clA \in calG. \end{cases}$$

Considering a suitable condition about the point p in the next section, it will be shown that the operator cl_p^* satisfies Kuratowski closure axioms, that is, the operator can constitute a topology, say, τ_p^* on X. We then completely determine this topology and show that this is a coarser topology than τ. Also, using several different grills, we provide some examples of τ^*.

Finally we add, for ease of work in the absence of any ambiguity, we will show the symbols of cl_p^* and τ_p^* with cl^* and τ^* respectively.

2. Main Results

As proposed in the introduction, this section intends to provide a method to construct another topology on a topological space. To this end, we first give the general construction of a new Kuratowski closure operator from the old in any topological space.

We start with an agreement; a point $p \in X$ is said to be closed if $\{p\}$ is a closed subset of X.

Theorem 2.1. *Let (X,τ) be a topological space and p be a closed point of it. Also, let $\mathcal{G}$ be a grill on (X,τ). Then the operator $cl^* : \mathcal{P}(X) \to \mathcal{P}(X)$ defined by*

$$cl^*(A) = \begin{cases} clA & \text{if } clA \notin calG, \text{ for } A \subseteq X \\ clA \cup \{p\} & \text{if } clA \in calG, \text{ for } A \subseteq X \end{cases}$$

is a Kuratowski closure operator, inducing a topology τ^* on X.

Proof. From $cl(\emptyset) = \emptyset \notin \mathcal{G}$ we have $cl^*(\emptyset) = \emptyset$, and it is clear that $A \subseteq cl^*(A)$ for any $A \subseteq X$.

We now verify that for any $A, B \subseteq X$, $cl^*(A \cup B) = cl^*(A) \cup cl^*(B)$. Let A and B are two subsets of X and consider two cases;

- (i) $cl(A \cup B) \notin \mathcal{G}$,
- (ii) $cl(A \cup B) \in \mathcal{G}$.

In case (i) we have

$$\begin{aligned} cl^*(A \cup B) &= cl(A \cup B) \\ &= clA \cup clB, \end{aligned}$$

and also, $clA = cl^*A$ and $clB = cl^*B$, because $cl(A \cup B) \notin \mathcal{G}$ implies that $clA, clB \notin \mathcal{G}$. So, in this case

$$cl^*(A \cup B) = clA \cup clB$$

$$= cl^*(A) \cup cl^*(B).$$

In case (ii) we have

$$\begin{aligned} cl^*(A \cup B) &= cl(A \cup B) \cup \{p\} \\ &= clA \cup clB \cup \{p\}, \end{aligned}$$

also $clA \cup clB \cup \{p\} = cl^*A \cup cl^*B$, because $cl(A \cup B) \in \mathcal{G}$ implies that $clA \in \mathcal{G}$ or $clB \in \mathcal{G}$. So, in this case

$$\begin{aligned} cl^*(A \cup B) &= clA \cup clB \cup \{p\} \\ &= cl^*(A) \cup cl^*(B). \end{aligned}$$

We next show that $cl^*(cl^*(A)) = cl^*(A)$, for any $A \subseteq X$.

Here, let A be a subset of X and consider two cases;

- ∘ (i) $clA \notin \mathcal{G}$,
- ∘ (ii) $clA \in \mathcal{G}$.

In case (i) we have

$$\begin{aligned} cl^*(cl^*(A)) &= cl^*(clA)\,(since\, cl(clA) \\ &= clA \notin \mathcal{G}) \\ &= clA \\ &= cl^*A, \end{aligned}$$

while in case (ii); we have

$cl^*(cl^*(A)) = cl^*((clA) \cup \{p\})$(due to the additive property of cl^* shown above)$= cl^*(clA) \cup cl^*(\{p\}) = ((clA) \cup \{p\}) \cup cl(\{p\})$(due to closedness p in hypothesis)$= clA \cup \{p\} = cl^*A$.

It follows that cl^* is a Kuratowski closure operator on X which gives rise to a topology, say τ^*, on X. □

Dual to the topological closure operator is the topological interior operator, in the following we determine the form of operator int^* as dual of cl^*.

Theorem 2.2. *Let (X, τ) be a topological space and cl^* be the operator constructed in Theorem 2.1. Then for any subset A of X the interior operator int^* as dual of the operator cl^* has the form of*

$$int^*(A) = \begin{cases} intA, & \text{if } p \notin A \text{ or } p \in A \text{ with } cl(X \setminus A) \notin \mathcal{G}, \\ (intA) \setminus \{p\}, & \text{if } p \in A \text{ with } cl(X \setminus A) \in \mathcal{G}. \end{cases}$$

Proof. Let A be a subset of X and consider two cases (i): $p \in A$ and (ii) $p \notin A$.

In case (i), considering $cl(X \setminus A) \notin \mathcal{G}$ we have $cl^*(X \setminus A) = cl(X \setminus A)$, so

$$\begin{aligned} int^*(A) &= X \setminus cl^*(X \setminus A) \\ &= X \setminus cl(X \setminus A) \end{aligned}$$

$$= intA,$$

while considering $cl(X \setminus A) \in \mathcal{G}$ leads to $cl^*(X \setminus A) = cl(X \setminus A) \cup \{p\}$, and so

$$\begin{aligned} int^*A &= X \setminus cl^*(X \setminus A) \\ &= X \setminus [cl(X \setminus A) \cup \{p\}] \\ &= (X \setminus cl(X \setminus A)) \cap (X \setminus \{p\}) \\ &= (intA) \setminus \{p\}. \end{aligned}$$

In case (ii) we have $p \in X \setminus A$. Here, considering $cl(X \setminus A) \notin \mathcal{G}$ leads to $cl^*(X \setminus A) = cl(X \setminus A)$, so

$$\begin{aligned} int^*(A) &= X \setminus cl^*(X \setminus A) \\ &= X \setminus cl(X \setminus A) \\ &= intA, \end{aligned}$$

while considering $cl(X \setminus A) \in \mathcal{G}$ requires that $cl^*(X \setminus A) = cl(X \setminus A) \cup \{p\}(\text{as } p \in X \setminus A) = cl(X \setminus A)$ and so

$$\begin{aligned} int^*(A) &= X \setminus cl^*(X \setminus A) \\ &= X \setminus cl(X \setminus A) \\ &= intA. \end{aligned}$$

Therefore, according to the above we have the following formula

$$int^*A = \begin{cases} intA, & \text{if } p \in A,\ cl(X \setminus A) \notin \mathcal{G}, \\ (intA) \setminus \{p\}, & \text{if } p \in A,\ cl(X \setminus A) \in \mathcal{G}, \\ intA, & \text{if } p \notin A. \end{cases}$$

and we are done. □

By determining the set $\{A \subseteq X : int^*A = A\}$ as the fixed points of the operator int^*, τ^*-open sets are identified and therefore the topology τ^* is determined.

Corollary 2.3. *From results of Therorem 2.2, we have* $\tau^* = \{A \in \tau : p \notin A\} \cup \{A \in \tau : p \in A,\ X \setminus A \notin \mathcal{G}\}$.

Proof. To determine τ^*, we note that $A \in \tau^*$ if and only if $int^*A = A$. Considering $cl(X \setminus A) \notin \mathcal{G}$ in case (i) in the proof of the Therorem 2.2, leads to $int^*A = intA$, so $A \in \tau^*$ if and only if $A \in \tau$, while considering $cl(X \setminus A) \in \mathcal{G}$ in case (i), leads to $int^*A = (intA) \setminus \{p\}$, so $A \in \tau^*$ if and only if $A = (intA) \setminus \{p\}$ and this is impossile. Therefore, no subset A of X, which includes p and is valid under the condition $cl(X \setminus A) \in \mathcal{G}$, can belong to τ^*. In case (ii) of the proof of the Therorem 2.2, because

$int^*A = intA$, hence $A \in \tau^*$ if and only if $A \in \tau$. So according to the above we have

$$\tau^* = \{A \in \tau : p \notin A\} \cup \{A \in \tau : p \in A,\ X \setminus A = cl(X \setminus A) \notin \mathcal{G}\}. \quad \square$$

In the following remark, the relationship between τ and τ^* has been determined.

Remark 2.4. In general, τ^* is coarser than τ, because $\tau = \{A \in \tau : p \notin A\} \cup \{A \in \tau : p \in A\}$ and clearly, $\{A \in \tau : p \in A,\ X \setminus A \notin \mathcal{G}\} \subseteq \{A \in \tau : p \in A\}$.

In the following example, we show that by choosing the appropriate grills, τ^* and τ can be matched in some topological spaces.

Example 2.5. Let τ denotes the cofinite topology on a (an infinite) set X and $\mathcal{G}$ be the grill of all infinte subsets of X, then $\tau^* = \{A \in \tau : p \notin A\} \cup \{A \in \tau : p \in A,\ X \setminus A \notin \mathcal{G}\} = \{A \in \tau : p \notin A\} \cup \{A \in \tau : p \in A,\ X \setminus A$ is finite$\}$(since here, $A \in \tau$ if and only if $X \setminus A$ is finite)$=$ $\{A \in \tau : p \notin A\} \cup \{A \in \tau : p \in A,\ X \setminus A$ is closed in $(X, \tau)\} = \{A \in \tau : p \notin A\} \cup \{A \in \tau : p \in A\} = \tau$.
Here, by selecting X as an uncountable set equipped with cocountable topology τ and grill $\mathcal{G} = \{A \subseteq X : A$ is a uncountable subset of $X\}$, we will have the same result again.

In the next example, we determine τ^* for two different grills.

Example 2.6. In this example, we have calculated τ^* for two grills $\mathcal{P}(X) \setminus \{\emptyset\}$ and $\{A \subseteq X : intclA \neq \emptyset\}$.

$a)$ If we put $\mathcal{G}_1 = \mathcal{P}(X) \setminus \{\emptyset\}$, then

$$\begin{aligned}\tau_1^* &= \{A \in \tau : p \notin A\} \cup \{A \in \tau : p \in A,\ X \setminus A \notin \mathcal{P}(X) \setminus \{\emptyset\}\} \\ &= \{A \in \tau : p \notin A\} \cup \{A \in \tau : p \in A,\ X \setminus A = \emptyset\} \\ &= \{A \in \tau : p \notin A\} \cup \{X\}.\end{aligned}$$

$b)$ If $\mathcal{G}_2 = \{A \subseteq X : intclA \neq \emptyset\}$, then

$$\begin{aligned}\tau_2^* &= \{A \in \tau : p \notin A\} \cup \{A \in \tau : p \in A,\ intcl(X \setminus A) = \emptyset\} \\ &= \{A \in \tau : p \notin A\} \cup \{A \in \tau : p \in A,\ clA = X\}.\end{aligned}$$

In the previous example, we saw that $\tau_1^* \subseteq \tau_2^*$, in this regard, we have the following proposition.

Proposition 2.7. *Let $\mathcal{G}_1$ and $\mathcal{G}_2$ be satisfied the conditions of grills on a space (X, τ) such that $\mathcal{G}_1 \subseteq \mathcal{G}_2$, then $\tau_2^* \subseteq \tau_1^*$.*

Proof. If $\mathcal{G}_1 \subseteq \mathcal{G}_2$, then for any $B \subseteq X$, $B \notin \mathcal{G}_2$ implies that $B \notin \mathcal{G}_1$. Putting $B = X \setminus A$, we have $\{A \in \tau : p \in A,\ X \setminus A \notin \mathcal{G}_2\} \subseteq \{A \in \tau : p \in A,\ X \setminus A \notin \mathcal{G}_1\}$ and therefore $\tau_2^* \subseteq \tau_1^*$. $\square$

Based on the above proposition, we have the following corollary.

Corollary 2.8. *For any grill $\mathcal{G}$ defined on a topological space (X,τ), we have* $\{A \in \tau : p \notin A\} \cup \{X\} \subseteq \tau^* \subseteq \tau$.

Proof. Clearly, $\mathcal{P}(X) \setminus \{\emptyset\} \supseteq \mathcal{G}$ for any grill $\mathcal{G} \in \mathcal{G}(X)$. Now, from Example 2.6, Proposition 2.7 and Remark 2.4 we get the result. □

The next example shows that using two different closed points in our method may lead to equivalent topologies.

Example 2.9. Let (X,τ) be a topological space with $X = \{a,b,c,d\}$ and $\tau = \{\emptyset, \{a\}, \{a,b\}, \{a,b,c\}, \{a,b,d\}, X\}$. Also, let $\mathcal{G} = \{\{a\}, \{a,b\}, \{a,c\}, \{a,d\}, \{a,b,c\}, \{a,b,d\}, \{a,c,d\}, \{a,b,c,d\}\}$ be a grill on (X,τ). Then for closed points c and d of (X,τ), we have

$$\{A \in \tau : c \notin A\} = \{\emptyset, \{a\}, \{a,b\}, \{a,b,d\}, X\},$$

and

$$\{A \in \tau : d \notin A\} = \{\emptyset, \{a\}, \{a,b\}, \{a,b,c\}, X\},$$

also,

$$\{A \in \tau : c \in A, X \setminus A \notin \mathcal{G}\} = \{\{a,b,c\}, \{a,b,c,d\}\},$$

and

$$\{A \in \tau : d \in A, X \setminus A \notin \mathcal{G}\} = \{\{a,b,d\}, \{a,b,c,d\}\}.$$

So we will have,

$$\tau_c^* = \{\emptyset, \{a\}, \{a,b\}, \{a,b,c\}, \{a,b,d\}, \{a,b,c,d\}\},$$

and

$$\tau_d^* = \{\emptyset, \{a\}, \{a,b\}, \{a,b,c\}, \{a,b,d\}, \{a,b,c,d\}\}.$$

Therefore, it is possible to obtain the same topology for different closed points.

Proposition 2.10. *The set $\{A \in \tau : p \notin A\}$ as a part of τ^*, is a topology on $X \setminus \{p\}$, in fact it is the subspace topology from τ on $X \setminus \{p\}$.*

Proof. Clearly $\emptyset \in \{A \in \tau : p \notin A\}$ and $X \setminus \{p\}$ (since we choose p as a closed point of τ) is in $\{A \in \tau : p \notin A\}$. Now, let A_1 and A_2 are in $\{A \in \tau : p \notin A\}$, then $A_1 \cap A_2 \in \tau$ and also, $p \notin A_1 \cap A_2$. So $A_1 \cap A_2 \in \{A \in \tau : p \notin A\}$.

Also if for an arbitrary indexing set K, $\{A_k : k \in K\}$ be a subcollection of $\{A \in \tau : p \notin A\}$, then for all $k \in K$ we have $A_k \in \tau$ and $p \notin A_k$. So $\cup_{k\in K} A_k \in \tau$ and $p \notin \cup_{k\in K} A_k$ for each $k \in K$, hence $\cup_{k\in K} A_k \in \{A \in \tau : p \notin A\}$. Thus, we show that the set $\{A \in \tau : p \notin A\}$ forms a topology on $X \setminus \{p\}$, but as $X \setminus \{p\}$ is an open set in (X,τ), then we infer that the topology $\{A \in \tau : p \notin A\}$ is the subspace topology from τ on $X \setminus \{p\}$. □

Proposition 2.11. *If we put* $T = \{A \in \tau : p \in A, X \setminus A \notin \mathcal{G}\}$ *then* $T \cup \{\emptyset\}$ *is a topology on* X.

Proof. Clearly $\emptyset \in T \cup \{\emptyset\}$ and also, as $p \in X \in \tau$ and $X \setminus X = \emptyset \notin \mathcal{G}$, we get $X \in T$. Now, if we have $A_1, A_2 \in T$, then $p \in A_1 \cap A_2 \in \tau$ and also, $X \setminus (A_1 \cap A_2) = (X \setminus A_1) \cup (X \setminus A_2) \notin \mathcal{G}$. Therefore $A_1 \cap A_2 \in T$.

On the other hand, if for an arbitrary indexing set K we put $\{A_k : k \in K\} \subseteq T$, then clearly $p \in \cup_{k\in K} A_k \in \tau$ and $X \setminus \cup_{k\in K} A_k = \cap_{k\in K}(X \setminus A_k) \subseteq X \setminus A_1 \notin \mathcal{G}$, so $\cup_{k\in K} A_k \in T$. □

From Proposition 2.10 we have the following corollary.

Corollary 2.12. *Let* (X, τ) *is a topological space with a grill* $\mathcal{G}$ *on it. Then the collection* $\tau \cup \{A \cup \{p\} : A \in \tau,\ cl(X \setminus A) \notin \mathcal{G}\}$ *is a topology on* X^**, where* $X^* = X \cup \{p\}$ *for* $p \notin X$.

Definition 2.13 ([1])**.** A topological space (Y, μ) is called an extension of a space (X, τ) if (Y, μ) contains (X, τ) as a dense subspace. The extension (Y, μ) is called a one-point extension of (X, τ) if $Y \setminus X$ is a singleton.

According to the previous definition, we get the next corollary from Proposition 2.10.

Corollary 2.14. *Let* (X, τ) *be a connected topological space. Then, each space* (X, τ^*) *associted to each grill* $\mathcal{G} \in \mathcal{G}(X)$ *can be counted as a one-point extension of the (sub)space* $X \setminus \{p\}$*. Moreover between them,* (X, τ^*) *for* $\tau^* = \{A \in \tau : p \notin A\} \cup \{X\}$ *is the smallest.*

As the final point, we investigate the relationship between the set $D^*(X) = \{A \subseteq X : A \text{ is dense in } (X, \tau^*)\}$ and $D(X) = \{A \subseteq X : A \text{ is dense in } (X, \tau)\}$.

Let $\emptyset \neq A \subsetneq X$, we consider two cases (i) $p \in A$ and (ii) $p \notin A$.

In case (i), whether clA belongs to $\mathcal{G}$ or not, we have $cl^*A = clA$. Also, in case (ii), for any subset $\emptyset \neq A \subsetneq X$ that $clA \notin \mathcal{G}$ the equality of $cl^*A = clA$ remains. So, in these cases, collections $D^*(X)$ and $D(X)$ share their elements. But, in case (ii) for sets A that $clA \in \mathcal{G}$, if p is a limit point of the set A, then $cl^*A = clA \cup \{p\} = clA$ and thus in this case again $D^*(X)$ and $D(X)$ share their elements, while if $p \notin clA$, then $cl^*A = clA \cup \{p\}$, so we have $A \in D^*(X)$ if and only if $clA \supseteq X \setminus \{p\}$.

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