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Sahand Communications in Mathematical Analysis

Print ISSN: 2322-5807
Online ISSN: 2423-3900
Volume: 19
Number: 3
Pages: 65-76

Sahand Commun. Math. Anal.
DOI: 10.22130/scma.2022.550731.1080

Sahand Communications in Mathematical Analysis (SCMA) Vol. 19 No. 3 (2022), 65-76
http://scma.maragheh.ac.ir
DOI: 10.22130/scma.2022.550731.1080

Conservation Laws and Moving Frame Method of Vaidya-Bonner Space-Time

Davood Farrokhi[1], Rohollah Bakhshandeh-Chamazkoti[2*] and Mehdi Nadjafikhah[3]

Abstract. In the present paper, we compute the conservation laws of the Vaidya-Bonner geodesic space-time metric in a Riemannian space and carry out the moving frame method for this metric. We obtain the connection forms and curvature 2-forms, using the first and second Cartan's structure equations. Finally, the Ricci scalar tensor and the components of Einstein curvature are calculated.

1. Introduction

In [17], Stephen Hawking proved the existence of the thermal radiation of black holes, which makes a striking discovery for further research on black holes. Since the background space-time behaved as fixed and energy conservation, charge conservation and angular momentum conservation were not considered, the derived Hawking radiation spectrums in the further researches are purely thermal [18]. However, it is difficult to research the Hawking effect of the non-static or non-stationary black holes because the calculation about the vacuum expectation value of the re-normalized energy-momentum tensor is very complicated.

The metric of the non-stationary Vaidya-Bonner black hole reads

$$ds^2 = -\left(1 - \frac{M(t)}{r} + \frac{Q(t)}{r^2}\right) dt^2 - 2dtdr + r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right), \tag{1.1}$$

where $M(t)$ is the dynamical mass of the black hole and $Q(t)$ is the electric charge, which both depend on the advanced Eddington time

2020 *Mathematics Subject Classification.* 34B24, 34B27.

Key words and phrases. Conservation laws, Geodesic, Riemannian metric, Moving frame.

Received: 17 March 2022, Accepted: 25 July 2022.

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coordinate t, [15]. If $Q(t)$ equals zero, the non-stationary Vaidya-Bonner black hole is reduced to the Vaidya black hole.

Recently, many papers have been published on the calculation of Lie and Noether symmetries, [14], as well as the conservation laws for space-times that have many applications in physics and astronomy, [1, 3, 4, 11, 16]. In particular, works have been done on the Noether symmetries of the conservation laws for the Vaidiya Metrics, which shows its importance, [5, 8–10, 12, 13].

In [6], we calculated the corresponded Noether symmetries and Lie point symmetries of the Vaidya-Bonner metric. Moreover, a classification of the one-dimensional optimal system of subalgebras for the corresponded Lie symmetries is presented. In this article, we continue the previous work by finding the conservation laws and constructing appropriate manifolds based on moving frame methods. By exploring the relationship between a connection and its curvature, we reveal their geometric interpretation in terms of moving frames. We compute Cartan's first and second structural equations which establish the relationship between a local orthonormal frame, the connection, and its curvature.

2. Noether Symmetry and Conservation Laws

Suppose that (M,g) is a five-dimensional Riemannian manifold equipped with the Vaidya-Bonner metric (1.1). Consider $x=\left(x^1,x^2,x^3,x^4,x^5\right)=(s,t,r,\theta,\varphi)$ as a local coordinate on M and

$$\begin{aligned}\mathbf{X} &= \xi\left(s,\mathbf{x}\right)\frac{\partial}{\partial s}+\sum_{\nu}\eta^{\nu}\left(s,\mathbf{x}\right)\frac{\partial}{\partial x^{\nu}}\\ &= \xi\left(s,t,r,\theta,\varphi\right)\frac{\partial}{\partial s}+\eta^{1}\left(s,t,r,\theta,\varphi\right)\frac{\partial}{\partial t}+\eta^{2}\left(s,t,r,\theta,\varphi\right)\frac{\partial}{\partial r}\\ &\quad+\eta^{3}\left(s,t,r,\theta,\varphi\right)\frac{\partial}{\partial\theta}+\eta^{4}\left(s,t,r,\theta,\varphi\right)\frac{\partial}{\partial\varphi},\end{aligned}$$

as a vector field is defined on tangent space M. One can find the first prolongation, [7], of the vector field $\mathbf{X}$ with the following form

$$\mathbf{X}^{[1]}=\mathbf{X}+\sum_{\nu}\left(\eta^{\nu}_{,s}+\eta^{\nu}_{,\mu}\dot{x}^{\nu}-\xi_{,s}\dot{x}^{\nu}-\xi_{,s}\dot{x}^{\mu}\dot{x}^{\nu}\right)\frac{\partial}{\partial\dot{x}^{\nu}}. \tag{2.1}$$

The infinitesimal generator $\mathbf{X}$ is a Noether point symmetry of the Lagrangian

$$\begin{aligned}\mathcal{L} &= \sum_{\mu,\nu}g_{\mu}\nu\left(x^{\sigma}\right)\dot{x}^{\mu}\dot{x}^{\nu}\\ &= -\left(1-\frac{M(t)}{r}+\frac{Q(t)}{r^2}\right)\dot{t}^2-2\dot{t}\dot{r}+r^2\left(\dot{\theta}^2+\sin^2\theta\,\dot{\varphi}^2\right),\end{aligned} \tag{2.2}$$

if there exists a gauge function, $A\left(s, x^{\mu}\right)$, such that

$$\mathbf{X}^{[1]}\mathcal{L}+\left(D_s\xi\right)\mathcal{L}=D_sA,$$

where

$$D_s=\frac{\partial}{\partial s}+\sum_{\mu}\dot{x}^{\mu}\frac{\partial}{\partial x^{\mu}}.$$

Usually, in the calculation of Noether symmetries (for simplicity), the gauge function A is considered equal to zero. By substituting in the following equation

$$\mathbf{X}^{[1]}\mathcal{L}+\left(D_s\xi\right)\mathcal{L}=0,$$

we will obtain

(2.3)

$$\begin{aligned}
&\ddot{t}+\frac{2Q(t)-rM(t)}{4r^3}\,\dot{t}^2+\frac{r\sin^2\theta}{2}\,\dot{\varphi}^2+\frac{r}{2}\,\ddot{\theta}^2=0,\\
&\ddot{r}-\frac{2}{8r^5}\left(\left(rM'(t)+4Q'(t)\right)\left(rM(t)-2Q(t)\right)\left(rM(t)-r^2-Q(t)\right)\right)\dot{t}^2\\
&\qquad+\frac{rM(t)-2Q(t)}{4r^3}\,\dot{t}\dot{r}+\frac{M(t)-r^2-Q(t)}{4r}\,\dot{\theta}^2\\
&\qquad+\frac{\left(M(t)r-r^2-Q(t)\right)\sin^2\theta}{4r}\,\ddot{\varphi}=0,\\
&\ddot{\theta}+\frac{2}{r}\dot{r}\dot{\theta}-\sin\theta\,\cos\theta\,\ddot{\varphi}^2=0,\\
&\ddot{\varphi}+\frac{1}{r}\dot{r}\dot{\varphi}+2\cot\theta\,\dot{\theta}\dot{\varphi}=0.
\end{aligned}$$

We conclude the corresponding Noether symmetries of metric (1.1) as follows

$$\begin{aligned}
&\mathbf{v}_1=\frac{\partial}{\partial s},\qquad \mathbf{v}_2=\frac{\partial}{\partial t},\qquad \mathbf{v}_3=\frac{\partial}{\partial\varphi},\\
&\mathbf{v}_4=-\cos\varphi\,\frac{\partial}{\partial\theta}+\sin\varphi\,\cot\theta\,\frac{\partial}{\partial\varphi},\\
&\mathbf{v}_5=\sin\varphi\,\frac{\partial}{\partial\theta}+\cos\varphi\,\csc\theta\,\frac{\partial}{\partial\varphi}.
\end{aligned}\tag{2.4}$$

Now, we want to determine the conservation laws for systems of Euler-Lagrange equations using Noether's theorem. This theorem relies on the availability of a Lagrangian and the corresponding Noether symmetries, which leaves the action integral invariant. Now the conservation laws of the system (2.3) are given by

$$D_sT_i=0.$$

Theorem 2.1. *If* $\mathbf{X}$ *be a Noether point symmetry of a Lagrangian* $\mathcal{L} = (s, x^\mu, \dot{x}^\mu)$*, then*

$$T = \bar{\xi}\mathcal{L} + \left(\bar{\eta}^\mu - \dot{x}^\mu\bar{\xi}\right)\frac{\partial\mathcal{L}}{\partial\dot{x}^\mu} - A,$$

is the first integral of $\mathbf{X}$*, where* $A = A(s, x^\mu)$ *is the gauge function.*

The conserved flow of Noether symmetries (2.4), is given by

$$T = \bar{\xi}\mathcal{L}+\left(\bar{\eta}^1 - \dot{t}\bar{\xi}\right)\frac{\partial\mathcal{L}}{\partial\dot{t}}+\left(\bar{\eta}^2 - \dot{t}\bar{\xi}\right)\frac{\partial\mathcal{L}}{\partial\dot{r}}+\left(\bar{\eta}^3 - \dot{\theta}\bar{\xi}\right)\frac{\partial\mathcal{L}}{\partial\dot{\theta}}+\left(\bar{\eta}^4 - \dot{\varphi}\bar{\xi}\right)\frac{\partial\mathcal{L}}{\partial\dot{\varphi}}-A.$$

By assumption $A = 0$, the conserved flows corresponding to the Noether symmetries (2.4) are listed in Table 1.

TABLE 1. Noether symmetries and conservation laws for metric (1.1).

Noether symmetry	Conservation law
$\mathbf{v}_1 = \dfrac{\partial}{\partial s}$	$T_1 = \frac{1}{r^2}(-r^4\left(\sin^2\theta\dot{\varphi}^2 + \dot{\theta}^2\right) + \big(Q(t)$ $-M(t)r + r^2)\dot{t}^2 + 2r\left(r\dot{r} + M(t)\right)\dot{t}\big)$
$\mathbf{v}_2 = \dfrac{\partial}{\partial t}$	$T_2 = -2\dot{t}$
$\mathbf{v}_3 = \dfrac{\partial}{\partial\varphi}$	$T_3 = 2r^2\sin^2\theta\,\dot{\varphi}$
$\mathbf{v}_4 = -\cos\varphi\dfrac{\partial}{\partial\theta} + \sin\varphi\,\cot\theta\dfrac{\partial}{\partial\varphi}$	$T_4 = 2r^2\left(\sin\theta\,\sin\varphi\,\cos\theta\,\dot{\varphi} - \cos\varphi\,\dot{\theta}\right)$
$\mathbf{v}_5 = \sin\varphi\dfrac{\partial}{\partial\theta} + \cos\varphi\,\csc\theta\dfrac{\partial}{\partial\varphi}$	$T_5 = 2r^2\left(\sin\theta\,\cos\varphi\,\dot{\varphi} + \sin\varphi\,\dot{\theta}\right)$

3. MOVING FRAME METHOD

3.1. **Moving Frames in Euclidean Spaces.** Consider $\mathbb{R}^N$ equipped with the standard inner product $\langle , \rangle$ and U as an open domain in $\mathbb{R}^N$. A moving frame in $U \subseteq \mathbb{R}^N$ means choosing an orthonormal base $\{\mathbf{e}_1(x), \ldots, \mathbf{e}_N(x)\}$ for all T_xU, $x \in U$. Taking exterior derivatives we find

$$dx = \sum_A \omega_A\mathbf{e}_A,$$

$$d\mathbf{e}_A = \sum_B \omega_{BA}\mathbf{e}_B,$$

where ω_A's and ω_{AB}'s are 1-forms. The orthonormality condition implies

$$\begin{aligned} 0 &= d\langle \mathbf{e}_A, \mathbf{e}_B \rangle \\ &= \langle d\mathbf{e}_A, \mathbf{e}_B \rangle + \langle \mathbf{e}_A, d\mathbf{e}_B \rangle, \end{aligned}$$

and consequently

$$\omega_{AB} + \omega_{BA} = 0.$$

That is, the matrix-valued 1-form $\omega = (\omega_{AB})$ takes values in the Lie algebra SO(N). Using $d^2 x = 0$ and $d^2 \mathbf{e}_A = 0$, we obtain

$$\begin{aligned} d\omega_A + \sum_B \omega_{AB} \wedge \omega_B = 0, \\ d\omega_{AB} + \sum_C \omega_{AC} \wedge \omega_{CB} = 0. \end{aligned}$$

These equations are often called the *structure equations* for Euclidean space or more precisely for the group of rigid motions of Euclidean space. The second set of equations is also known as the structure equations for the orthogonal group.

3.2. **Moving Frames in semi-Riemannian Manifold.** Let M be an n-dimensional semi-Riemannian manifold and $(x_1, \cdots, x_n)$ be a local coordinate at the point $p \in M$, equipped with metric

$$g = \sum_{i,j} g_{ij}\, dx^i \otimes dx^j.$$

Then $\left\{ \frac{\partial}{\partial x_i}\Big|_p \right\}_{i=1}^n$ is a basis for tangent space T_pM and by assumption $X_i = \frac{\partial}{\partial x_i}$, we can write

$$\begin{aligned} \nabla_{X_k} X_j &= \sum_i \Gamma^i_{kj} X_i \\ &= \sum_i \omega^i_j \left(X_k \right) X_i, \end{aligned}$$

where ∇ is an affine connection and Γ^i_{kj} is the Christopher coefficients

$$\omega^i_j = \sum_k \Gamma^i_{kj}\, dx^k.$$

Therefore, we have Cartan's structure equations as follows

$$\begin{aligned} \Theta^i &= d\theta^i + \sum_j \omega^i_j \wedge \theta^j, \\ \Omega^i_j &= d\omega^i_j + \sum_k \omega^i_k \wedge \omega^k_j, \end{aligned}$$

thus, we can obtain the torsion 2-forms Θ^i and curvature 2-forms Ω^i_j from the above equations. Now, let us use the curvature forms Ω^i_j to define tensor components $\bar{R}^i_{jkl}$ by

$$\Omega^i_j = \frac{1}{2}\sum_{k,l} \bar{R}^i_{jkl}\, \theta^k \wedge \theta^l,$$

where $\bar{R}^i_{jkl} = \bar{R}^i_{jlk}$ and we have

$$\begin{aligned} \langle \theta^i, \bar{R}\left(X_k, X_l\right) X_j \rangle &= \langle \theta^i, \Omega^j_s \left(X_k, X_l\right) X_s \rangle \\ &= \Omega^i_j \left(X_k, X_l\right) \\ &= \bar{R}^i_{jkl}. \end{aligned} \tag{3.1}$$

One can compute the components of the curvature tensor by (3.1), thus the components of the Ricci tensor are given by

$$\begin{aligned} R_{\mu\nu} &= \bar{R}^\lambda_{\mu\lambda\nu} \\ &= \sum_\lambda \Omega^\lambda_\mu \left(X_\lambda, X_\mu\right), \end{aligned} \tag{3.2}$$

and the Einstein tensor G is a 2-tensor by the following formula

$$G_{ij} = R_{ij} - \frac{1}{2} g_{ij} R,$$

where R_{ij} is the Ricci tensor, g_{ij} is the metric tensor and R is the Ricci scalar curvature. Now, we want to compute the Ricci tensor, Einstein tensor and Ricci curvature of metric (1.1), using the moving frame method, [2].

Proposition 3.1. *The nonzero components of the connection matrix* $\left(\omega^j_i\right)$ *of the Vaidya-Bonner space-time metric* (1.1) *are*

$$\begin{aligned} \omega^1_2 &= \frac{rM(t) - 2Q(t)}{2r^2\sqrt{r^2 - rM(t) + Q(t)}}\, \theta^1, \\ \omega^2_1 &= \frac{r\left(rM'(t) - Q'(t)\right)}{2\sqrt{\left(r^2 - rM(t) + Q(t)\right)^3}}\, \theta^2, \\ \omega^3_2 &= -\omega^2_3 = \frac{\sqrt{r^2 - rM(t) + Q(t)}}{r^2}\, \theta^3, \\ \omega^4_2 &= -\omega^2_4 = \frac{\sqrt{r^2 - rM(t) + Q(t)}}{r^2}\, \theta^4, \\ \omega^4_3 &= -\omega^3_4 = \frac{\cot\theta}{r}\, \theta^4. \end{aligned} \tag{3.3}$$

Proof. We can choose the coframe

$$\theta^1 = \sqrt{1 - \frac{M(t)}{r} + \frac{Q(t)}{r^2}}\, dt, \qquad \theta^2 = \frac{dr}{\sqrt{1 - \frac{M(t)}{r} + \frac{Q(t)}{r^2}}}, \tag{3.4}$$

$$\theta^3 = r\, d\theta, \qquad \theta^4 = r \sin\theta\, d\varphi,$$

for Vaidya-Bonner space-time metric (1.1) such that

$$g = -\theta^1 \otimes \theta^1 - \theta^1 \otimes \theta^2 - \theta^2 \otimes \theta^1 + \theta^3 \otimes \theta^3 + \theta^4 \otimes \theta^4.$$

Using the coframe (3.4), we have

$$dt = \frac{1}{\sqrt{1 - \frac{M(t)}{r} + \frac{Q(t)}{r^2}}}\, \theta^1, \quad dr = \sqrt{1 - \frac{M(t)}{r} + \frac{Q(t)}{r^2}}\, \theta^2, \tag{3.5}$$

$$d\theta = \frac{1}{r}\, \theta^3, \qquad d\varphi = \frac{1}{r \sin\theta}\, \theta^4.$$

Now, by differentiating (3.4), we find the following 2-forms

$$\begin{aligned} d\theta^1 &= \frac{rM(t) - Q(t)}{2r^2 \sqrt{r^2 - rM(t) + Q(t)}}\, \theta^1 \wedge \theta^2, \\ d\theta^2 &= \frac{r\,(rM'(t) - Q'(t))}{2\sqrt{(r^2 - rM(t) + Q(t))^3}}\, \theta^2 \wedge \theta^1, \\ d\theta^3 &= \frac{\sqrt{r^2 - rM(t) + Q(t)}}{r^2}\, \theta^3 \wedge \theta^2, \\ d\theta^4 &= \frac{\sqrt{r^2 - rM(t) + Q(t)}}{r^2}\, \theta^4 \wedge \theta^2 + \frac{\cot\theta}{r}\, \theta^4 \wedge \theta^3, \end{aligned} \tag{3.6}$$

that are the first structure equations. The equation (3.6) leads to (3.3). □

Proposition 3.2. *The curvature 2-forms of the Vaidya-Bonner space-time metric* (1.1) *are given by*

$$\Omega^1_1 = -\Omega^2_2 = \frac{(rM(t) - 2Q(t))\,(rM'(t) - Q'(t))}{4r\,(r^2 - rM(t) + Q(t))^2}\, \theta^1 \wedge \theta^2,$$

$$\Omega^1_2 = -\Omega^2_1 = \frac{3Q(t) - rM(t)}{r^4}\, \theta^2 \wedge \theta^1,$$

$$\Omega^1_3 = -\Omega^3_1 = \frac{2Q(t) - rM(t)}{2r^4}\, \theta^1 \wedge \theta^3,$$

$$\Omega^1_4 = -\Omega^4_1 = \frac{2Q(t) - rM(t)}{2r^4}\, \theta^1 \wedge \theta^4,$$

$$\Omega_3^2 = -\Omega_2^3 = \frac{2Q(t) - rM(t)}{2r^4}\ \theta^2 \wedge \theta^3 + \frac{rM'(t) - Q'(t)}{2r\left(r^2 - rM(t) + Q(t)\right)}\ \theta^1 \wedge \theta^3,$$

$$\Omega_4^2 = -\Omega_2^4 = \frac{2Q(t) - rM(t)}{2r^4}\ \theta^2 \wedge \theta^4 + \frac{rM'(t) - Q'(t)}{2r\left(r^2 - rM(t) + Q(t)\right)}\ \theta^1 \wedge \theta^4,$$

$$\Omega_4^3 = -\Omega_3^4 = \frac{rM(t) - Q(t)}{r^4}\ \theta^3 \wedge \theta^4,$$

$$\Omega_3^3 = -\Omega_4^4 = 0.$$

Proof. Consider the connection matrix $\left(\omega_i^j\right)$ with the nonzero components (3.3). Now, we compute the differential of 1-forms (3.3) as follows

$$\begin{aligned}
(3.7)\quad d\omega_2^1 &= \frac{3Q(t) - rM(t)}{r^4}\ \theta^2 \wedge \theta^1,\\
d\omega_1^2 &= \frac{r^2\left(rM'(t) - Q'(t)\right)\left(rM'(t) - Q'(t)\right)}{\left(rM(t) - Q(t) - r^2\right)^3}\ \theta^1 \wedge \theta^2,\\
d\omega_2^3 &= \frac{rM(t) - 2Q(t)}{2r^4}\ \theta^2 \wedge \theta^3 + \frac{Q'(t) - rM'(t)}{2r\left(r^2 - rM(t) + Q(t)\right)}\ \theta^1 \wedge \theta^3,\\
d\omega_2^4 &= \frac{rM(t) - 2Q(t)}{2r^4}\ \theta^2 \wedge \theta^4 + \frac{Q'(t) - rM'(t)}{2r\left(r^2 - rM(t) + Q(t)\right)}\ \theta^1 \wedge \theta^4\\
&\quad + \frac{\cot\theta\sqrt{r^2 - rM(t) + Q(t)}}{r^3}\ \theta^3 \wedge \theta^4,\\
d\omega_3^4 &= \frac{-1}{r^2}\ \theta^3 \wedge \theta^4.
\end{aligned}$$

Now, we may obtain the Cartan second structure equations

$$\Omega_1^1 = d\omega_1^1 + \sum_{k=1}^{4} \omega_k^1 \wedge \omega_1^k = \frac{\left(rM(t) - 2Q(t)\right)\left(rM'(t) - Q'(t)\right)}{4r\left(r^2 - rM(t) + Q(t)\right)^2}\ \theta^1 \wedge \theta^2,$$

$$\Omega_2^1 = d\omega_2^1 + \sum_{k=1}^{4} \omega_k^1 \wedge \omega_2^k = \frac{3Q(t) - rM(t)}{r^4}\ \theta^2 \wedge \theta^1,$$

$$\Omega_3^1 = d\omega_3^1 + \sum_{k=1}^{4} \omega_k^1 \wedge \omega_3^k = \frac{2Q(t) - rM(t)}{2r^4}\ \theta^1 \wedge \theta^3,$$

$$\Omega_4^1 = d\omega_4^1 + \sum_{k=1}^{4} \omega_k^1 \wedge \omega_4^k = \frac{2Q(t) - rM(t)}{2r^4}\ \theta^1 \wedge \theta^4,$$

$$\Omega_3^2 = d\omega_3^2 + \sum_{k=1}^{4} \omega_k^2 \wedge \omega_3^k = \frac{2Q(t) - rM(t)}{2r^4}\ \theta^2 \wedge \theta^3$$
$$+ \frac{rM'(t) - Q'(t)}{2r(r^2 - rM(t) + Q(t))}\ \theta^1 \wedge \theta^3,$$

$$\Omega_4^2 = d\omega_4^2 + \sum_{k=1}^{4} \omega_k^2 \wedge \omega_4^k = \frac{2Q(t) - rM(t)}{2r^4}\ \theta^2 \wedge \theta^4$$
$$+ \frac{rM'(t) - Q'(t)}{2r(r^2 - rM(t) + Q(t))}\ \theta^1 \wedge \theta^4,$$

$$\Omega_4^3 = d\omega_4^3 + \sum_{k=1}^{4} \omega_k^3 \wedge \omega_4^k = \frac{rM(t) - Q(t)}{r^4}\ \theta^3 \wedge \theta^4.$$

□

Proposition 3.3. *The Ricci scalar curvature of space-time metric* (1.1) *equals to*

$$R = \frac{(2Q(t) - rM(t))\,(rM'(t) - Q'(t))}{2r\,(r^2 - rM(t) + Q(t))^2} + \frac{10Q(t) - 4rM(t)}{r^4}.$$

Proof. According to formula (3.2), the components of the Ricci tensor in the Vaidya-Bonner space-time metric are calculated in the following equations

(3.8)

$$\begin{aligned}
R_{11} &= \Omega_1^1\,(X_1, X_1) - \Omega_2^1\,(X_1, X_2) - \Omega_3^1\,(X_1, X_3) - \Omega_4^1\,(X_1, X_4) \\
&= \frac{(rM(t) - 2Q(t))\,(rM'(t) - Q'(t))}{4r\,(r^2 - rM(t) + Q(t))^2} + \frac{4rM(t) - 10Q(t)}{2r^4}, \\
R_{22} &= \Omega_2^1\,(X_1, X_2) + \Omega_2^2\,(X_2, X_2) - \Omega_3^2\,(X_2, X_3) - \Omega_4^2\,(X_2, X_4) \\
&= \frac{(rM(t) - 2Q(t))\,(rM'(t) - Q'(t))}{4r\,(r^2 - rM(t) + Q(t))^2} + \frac{Q'(t) - rM'(t)}{2r\,(r^2 - rM(t) + Q(t))} \\
&\quad + \frac{4Q(t) - rM(t)}{2r^4}, \\
R_{33} &= \Omega_3^1\,(X_1, X_3) + \Omega_3^2\,(X_2, X_3) + \Omega_3^3\,(X_3, X_3) - \Omega_4^3\,(X_3, X_4) \\
&= \frac{(rM'(t) - Q'(t))}{2r\,(r^2 - rM(t) + Q(t))} + \frac{6Q(t) - 4rM(t)}{2r^4}, \\
R_{44} &= \Omega_4^1\,(X_1, X_4) + \Omega_4^2\,(X_2, X_4) + \Omega_4^3\,(X_3, X_4) + \Omega_4^4\,(X_4, X_4) \\
&= \frac{(rM'(t) - Q'(t))}{2r\,(r^2 - rM(t) + Q(t))} + \frac{Q(t)}{r^4},
\end{aligned}$$

Since (1.1) has the following form

$$R=\sum_{i,j} g_{ij}\,\theta^i\otimes\theta^j,$$

thus, the Ricci scalar becomes

$$\begin{aligned}R&=\sum_{i,j} g_{ij}R_{ij}\\&=\frac{(2Q(t)-rM(t))\,(rM'(t)-Q'(t))}{2r\left(r^2-rM(t)+Q(t)\right)^2}+\frac{10Q(t)-4rM(t)}{r^4}.\end{aligned}$$

□

Corollary 3.4. *The component of the Einstein tensor* $G=(G_{ij})$ *of metric* (1.1) *is*

$$\begin{aligned}G_{11}&=R_{11}+\frac{1}{2}R=0,\\G_{22}&=R_{22}-\frac{1}{2}R=\frac{Q'(t)-rM'(t)}{r\left(r^2-rM(t)+Q(t)\right)}+\frac{2rM(t)-4Q(t)}{r^4},\\G_{33}&=R_{33}-\frac{1}{2}R=\frac{(rM(t)-2Q(t))\,(rM'(t)-Q'(t))}{4r\left(r^2-rM(t)+Q(t)\right)^2}\\&\quad+\frac{rM'(t)-Q'(t)}{2r\left(r^2-rM(t)+Q(t)\right)}+\frac{-2Q(t)}{2r^4},\\G_{44}&=R_{44}-\frac{1}{2}rR=\frac{(rM(t)-2Q(t))\,(rM'(t)-Q'(t))}{4r\left(r^2-rM(t)+Q(t)\right)^2}\\&\quad+\frac{rM'(t)-Q'(t)}{2r\left(r^2-rM(t)+Q(t)\right)}+\frac{2rM(t)-4Q(t)}{r^4},\end{aligned}$$

and all other components of G *vanish identically.*

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