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Non-Instantaneous Impulsive Fractional Integro-Differential Equations with State-Dependent Delay

Nadia Benkhettou[1], Abdelkrim Salim[2], Khalida Aissani[3], Mouffak Benchohra[4] and Erdal Karapinar[5*]

Abstract. This paper deals with the existence and uniqueness of the mild solution of the fractional integro-differential equations with non-instantaneous impulses and state-dependent delay. Our arguments are based on the fixed point theory. Finally, an example to confirm of the results is provided.

1. Introduction

The main objective of this work is to study the existence of mild solutions for a class of fractional integro-differential equations with state-dependent delay and non-instantaneous impulses described by the form

$$\begin{cases} {}^{C}D_{\vartheta}^{\zeta}\mathfrak{p}(\vartheta)+\mathcal{Z}\mathfrak{p}(\vartheta)=\int_0^{\vartheta}a(\vartheta,\varrho)\,\Psi\left(\varrho,\mathfrak{p}_{\eta(\varrho,\mathfrak{p}_{\varrho})},\mathfrak{p}(\varrho)\right)d\varrho, \\ \qquad\qquad\qquad \vartheta\in(\varrho_j,\vartheta_{j+1}]\subset\Theta,\ j=0,\ldots,\nu, \\ \mathfrak{p}(\vartheta)=\Phi_j\left(\vartheta,\mathfrak{p}_{\eta(\vartheta,\mathfrak{p}_{\vartheta})},\mathfrak{p}(\vartheta)\right), \qquad \vartheta\in(\vartheta_j,\varrho_j],\ j=1,\ldots,\nu, \\ \mathfrak{p}_0=\beta\in\chi, \end{cases} \tag{1.1}$$

where ${}^{C}D_{\vartheta}^{\zeta}$ is the Caputo fractional derivative of order $0<\zeta<1$, $\Psi:\Theta\times\chi\times\Xi\longrightarrow\Xi$, $\Theta=[0,\mathfrak{w}]$, $\mathfrak{w}>0$ and $\eta:\Theta\times\chi\to(-\infty,\mathfrak{w}]$ are appropriated functions, $a:\Omega\to\mathbb{R}\,(\Omega=\{(\vartheta,\varrho)\in\Theta\times\Theta:\vartheta\geq\varrho\})$. $(\Xi,\|\cdot\|)$ is a real Banach space. Here $0=\vartheta_0=\varrho_0<\vartheta_1\leq\varrho_1\leq\vartheta_2<\cdots<\vartheta_{\nu-1}\leq$

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* Corresponding author.

$\varrho_\nu \leq \vartheta_\nu \leq \vartheta_{\nu+1} = \mathfrak{w}$ are pre-fixed numbers, $\mathcal{Z} : D(\mathcal{Z}) \subset \Xi \to \Xi$ is the infinitesimal generator of an analytic semigroup $\{\mathcal{S}(\vartheta)\}_{\vartheta\geq 0}$ of uniformly bounded linear operators on Ξ and $\Phi_j \in \mathfrak{C}((\vartheta_j, \varrho_j] \times \chi \times \Xi, \Xi)$; $j = 1, 2, \ldots, \nu$. For any continuous function $\mathfrak{p}$ defined on $(-\infty, \mathfrak{w}]$ and any $\vartheta \in \Theta$, we denote by $\mathfrak{p}_\vartheta$ the element of χ given by $\mathfrak{p}_\vartheta(\varepsilon) = \mathfrak{p}(\vartheta + \varepsilon)$, for $\varepsilon \in (-\infty, 0]$, where $\beta \in \chi$ to be specified later.

There have been many applications of fractional differential and integral equations including economics, engineering, neutron transport, science, bioengineering, applied mathematics, radiative transfer, and the kinetic theory of gases, among others [13, 16, 17]. Recent years have seen significant progress in ordinary and partial fractional differential equations see the papers [1–9, 14, 20, 24, 28, 38–40], and the sources within. Moreover, fractional functional differential equations with state-dependent delay are commonly used as models of equations in applications, and various authors have investigated these types of equations. [10–12, 18, 26].

The study of impulsive differential equations has gained more attention in recent years due to its applications. Most of the research papers dealt with the existence of solutions for equations with instantaneous impulses, (for more details one can see [15, 27, 31, 35, 37]). In [22, 29, 32, 34, 36], the authors investigated several types of impulsive differential equations with non-instantaneous impulses.

This paper is organized into four sections. Section 2 provides some basic notations, definitions and theorems. The existence of the mild solutions to the problem (1.1) is presented in Section 4. Section 4 is concerned with an example.

2. Preliminaries

Consider the space $\mathfrak{C}(\Theta, \Xi)$ of the Ξ-valued continuous functions on Θ and the Banach space $L(\Xi)$ of bounded and linear operators on Ξ. Consider the space $L^1(\Theta, \Xi)$ of $\Xi-$valued Bochner integrable functions on Θ with the norm

$$\|\mathfrak{p}\|_{L^1} = \int_0^{\mathfrak{w}} \|\mathfrak{p}(\vartheta)\| \, d\vartheta.$$

We denote by $L^\infty(\Theta, \mathbb{R})$ the Banach space of the essentially bounded measurable functions with the norm

$$\|\mathfrak{p}\|_{L^\infty} = \inf\{\epsilon > 0 : |\mathfrak{p}(\vartheta)| \leq \epsilon, \text{ a.e. } \vartheta \in \Theta\}.$$

Definition 2.1 ([19])**.** Let $\zeta > 0$ and $\Psi : \Theta \to \Xi$ be an integrable function. The Riemann-Liouville integral is defined by:

$$I_\vartheta^\zeta \Psi(\vartheta) = \frac{1}{\Gamma(\zeta)} \int_0^\vartheta \frac{\Psi(\varrho)}{(\vartheta - \varrho)^{1-\zeta}} d\varrho.$$

Definition 2.2 ([33])**.** The Caputo derivative of order ζ for a function $\Psi : \Theta \to \Xi$ can be written as

$$\begin{aligned} D_{\vartheta}^{\zeta}\Psi(\vartheta) &= \frac{1}{\Gamma(n-\zeta)}\int_0^{\vartheta} \frac{\Psi^{(n)(\varrho)}}{(\vartheta-\varrho)^{\zeta+1-n}}d\varrho \\ &= I^{n-\zeta}\Psi^{(n)}(\vartheta), \end{aligned}$$

where $\vartheta > 0, n-1 \leq \zeta < n$. If $0 \leq \zeta < 1$, then

$$D_{\vartheta}^{\zeta}\Psi(\vartheta) = \frac{1}{\Gamma(1-\zeta)}\int_0^{\vartheta} \frac{\Psi'(\varrho)}{(\vartheta-\varrho)^{\zeta}}d\varrho.$$

Definition 2.3. A function $\Psi : \Theta \times \chi \times \Xi \longrightarrow \Xi$ is Carathéodory if it verifies:

(i) for each $\vartheta \in \Theta$, the function $\Psi(\vartheta, \cdot, \cdot) : \chi \times \Xi \longrightarrow \Xi$ is continuous;
(ii) for each $(\mathfrak{q}, w) \in \chi \times \Xi$, the function $\Psi(\cdot, \mathfrak{q}, w) : \Theta \to \Xi$ is measurable.

In this study, we will use the proposed intuitive definition of the phase space by Hale and Kato [21]. In particular, χ will be a linear space of functions translating $(-\infty, 0]$ into Ξ with a seminorm $\|.\|_{\chi}$, and verifies the following:

($\mathcal{C}d_{A_1}$): If $\mathfrak{p} : (-\infty, \mathfrak{w}] \longrightarrow \Xi$ is continuous on Θ and $\mathfrak{p}_0 \in \chi$, then $\mathfrak{p}_{\vartheta} \in \chi$ and $\mathfrak{p}_{\vartheta}$ is continuous in $\vartheta \in \Theta$ and

$$\|\mathfrak{p}(\vartheta)\| \leq \alpha_1 \|\mathfrak{p}_{\vartheta}\|_{\chi}, \tag{2.1}$$

where $\alpha_1 \geq 0$.

($\mathcal{C}d_{A_2}$): There exists a continuous function $C_1(\vartheta) > 0$ and a locally bounded function $C_2(\vartheta) \geq 0$ in $\vartheta \geq 0$ such that

$$\|\mathfrak{p}_{\vartheta}\|_{\chi} \leq C_1(\vartheta) \sup_{\varrho \in [0,\vartheta]} \|\mathfrak{p}(\varrho)\| + C_2(\vartheta)\|\mathfrak{p}_0\|_{\chi}, \tag{2.2}$$

for $\vartheta \in [0, \mathfrak{w}]$ and $\mathfrak{p}$ as in ($\mathcal{C}d_{A_1}$).

($\mathcal{C}d_{A_3}$): The space χ is complete.

Remark 2.4. Condition (2.1) in ($\mathcal{C}d_{A_1}$) is equivalent to $\|\beta(0)\| \leq \alpha_1 \|\beta\|_{\chi}$, for all $\beta \in \chi$.

We consider the phase space $\mathfrak{C}_{\mu} \times L^p(\Upsilon, \Xi)$ of Lebesgue-measurable and continuous functions $\xi : (-\infty, 0] \to \Xi$. The seminorm in $\|.\|_{\chi}$ is given by

$$\|\xi\|_{\chi} = \sup_{\varepsilon \in [-\mu, 0]} \|\xi(\varepsilon)\| + \left(\int_{-\infty}^{-\mu} \Upsilon(\varepsilon)\|\xi(\varepsilon)\|^p d\varepsilon\right)^{\frac{1}{p}}.$$

The space $\chi = \mathfrak{C}_\mu \times L^p(\Upsilon, \Xi)$ verifies $(\mathcal{C}d_{A_1})$, $(\mathcal{C}d_{A_2})$ and $(\mathcal{C}d_{A_3})$. Further, for $\mu = 0$ and $p = 2$, this space coincides with $\mathfrak{C}_0 \times L^2(\Upsilon, \Xi)$. (see [25], for more details).

Consider the space

$$\mathcal{P}_{\mathfrak{C}}(\Theta, \Xi) = \Big\{ \mathfrak{p} : \Theta \to \Xi, \mathfrak{p} \in \mathfrak{C}((\vartheta_j, \vartheta_{j+1}], \Xi), j = 0, 1, \dots, \nu \\ \text{and } \mathfrak{p}\left(\vartheta_j^+\right), \mathfrak{p}\left(\vartheta_j^-\right) \text{ exist with, } \mathfrak{p}\left(\vartheta_j^-\right) = \mathfrak{p}(\vartheta_j), j = 1, \dots, \nu \Big\}.$$

Obviously, $\mathcal{P}_{\mathfrak{C}}(\Theta, \Xi)$ is a Banach space with the norm

$$\|\mathfrak{p}\|_{PC} = \sup_{\vartheta \in \Theta} \|\mathfrak{p}(\vartheta)\|_{\Xi}.$$

3. Existence and Uniqueness Results

Definition 3.1. The function $\mathfrak{p} : (-\infty, \mathfrak{w}] \to \Xi$ is a mild solution of (1.1) if $\mathfrak{p}_0 = \beta$ on $(-\infty, \mathfrak{w}]$, $\mathfrak{p}|_{[0,\mathfrak{w}]} \in \mathcal{P}_{\mathfrak{C}}([0, \mathfrak{w}], \Xi)$ and $\mathfrak{p}$ verifies
(3.1)

$$\mathfrak{p}(\vartheta) = \begin{cases} \displaystyle\int_0^\vartheta \int_0^\varrho \mathfrak{T}_2(\vartheta - \varrho) a(\varrho, \kappa) \Psi\left(\kappa, \mathfrak{p}_{\eta(\kappa, \mathfrak{p}_\kappa)}, \mathfrak{p}(\kappa)\right) d\kappa d\varrho, \\ \quad + \mathfrak{T}_1(\vartheta) \beta(0), \qquad \vartheta \in [0, \vartheta_1], \\ \Phi_j\left(\vartheta, \mathfrak{p}_{\eta(\kappa, \mathfrak{p}_\kappa)}, \mathfrak{p}(\vartheta)\right), \qquad \vartheta \in (\vartheta_j, \varrho_j], j = 1, 2, \dots, \nu, \\ \displaystyle\int_0^\vartheta \int_0^\varrho \mathfrak{T}_2(\vartheta - \varrho) a(\varrho, \kappa) \Psi\left(\kappa, \mathfrak{p}_{\eta(\kappa, \mathfrak{p}_\kappa)}, \mathfrak{p}(\kappa)\right) d\kappa d\varrho \\ \quad + \mathfrak{T}_1(\vartheta - \varrho_j) \Phi_j\left(\vartheta, \mathfrak{p}_{\eta(\kappa, \mathfrak{p}_\kappa)}, \mathfrak{p}(\vartheta)\right), \\ \qquad \vartheta \in (\varrho_j, \vartheta_{j+1}], \end{cases}$$

where

$$\mathfrak{T}_1(\vartheta) = \int_0^\infty \rho_\zeta(\varphi) \mathcal{S}\left(\vartheta^\zeta \varphi\right) d\varphi,$$
$$\mathfrak{T}_2(\vartheta) = \zeta \int_0^\infty \varphi \vartheta^{\zeta - 1} \rho_\zeta(\varphi) \mathcal{S}\left(\vartheta^\zeta \varphi\right) d\varphi,$$

and ρ_ζ is a probability density function defined on $(0, \infty)$, where

$$\rho_\zeta(\varphi) = \frac{1}{\zeta} \varphi^{-1 - \left(\frac{1}{\zeta}\right)} \mathfrak{y}_\zeta\left(\varphi^{-\frac{1}{\zeta}}\right) \\ \geq 0,$$

so that

$$\mathfrak{y}_\zeta(\varphi) = \frac{1}{\pi} \sum_{j=1}^\infty (-1)^{j-1} \varphi^{-\zeta j - 1} \frac{\Gamma(j\zeta + 1)}{j!} \sin(j\pi\zeta), \quad \varphi \in (0, \infty).$$

Remark 3.2. Note that $\{\mathcal{S}(\vartheta)\}_{\vartheta\geq 0}$ is uniformly bounded i.e there exists a constant $\varpi > 0$ such that $\|\mathcal{S}(\vartheta)\| \leq \varpi$ for all $\vartheta \geq 0$.

Remark 3.3. Direct computation, as per [30], yields

$$\|\mathfrak{T}_2(\vartheta)\| \leq C_{\zeta,\varpi}\vartheta^{\zeta-1}, \quad \vartheta > 0, \tag{3.2}$$

where $C_{\zeta,\varpi} = \dfrac{\zeta\varpi}{\Gamma(1+\zeta)}$.

Set

$$\mathcal{T}(\eta^-) = \{\eta(\varrho,\xi) : (\varrho,\xi) \in \Theta \times \chi, \eta(\varrho,\xi) \leq 0\}.$$

We always assume that $\eta : \Theta \times \chi \to (-\infty, \mathfrak{w}]$ is continuous. In addition, we present the following assumption:

($\mathcal{C}d_\xi$) The function $\vartheta \to \xi_\vartheta$ is continuous from $\mathcal{T}(\eta^-)$ into χ and there exists a continuous and bounded function $\Lambda^\beta : \mathcal{T}(\eta^-) \to (0,\infty)$ such that $\|\beta_\vartheta\|_\chi \leq \Lambda^\beta(\vartheta)\|\beta\|_\chi$ for every $\vartheta \in \mathcal{T}(\eta^-)$.

Remark 3.4. Consider the constants C_1^* and C_2^*, where

$$C_1^* = \sup_{\varrho\in\Theta} C_1(\varrho) \quad \text{and} \quad C_2^* = \sup_{\varrho\in\Theta} C_2(\varrho).$$

Lemma 3.5 ([23])**.** *If $\mathfrak{p} : \mathbb{R} \to \Xi$ is a function such that $\mathfrak{p}_0 = \beta$, then*

$$\|\mathfrak{p}_\varrho\|_{\mathcal{B}} \leq \left(C_2^* + \Lambda^\beta\right)\|\beta\|_{\mathcal{B}} + C_1^* \sup\{|\mathfrak{p}(\varepsilon)|; \varepsilon \in [0, \max\{0,\varrho\}]\},$$

for $\varrho \in \mathcal{T}(\eta^-) \cup \Theta$, where $\Lambda^\beta = \sup_{\vartheta\in\mathcal{T}(\eta^-)} \Lambda^\beta(\vartheta)$.

The hypotheses:

($\mathcal{C}d_{B_1}$) There exists $l_\Psi > 0$ such that for all $(\mathfrak{p}_p, \mathfrak{q}_p) \in \chi \times \Xi, p = 1,2$

$$\|\Psi(\vartheta,\mathfrak{p}_1,\mathfrak{q}_1) - \Psi(\vartheta,\mathfrak{p}_2,\mathfrak{q}_2)\|_\Xi \leq l_\Psi\left[\|\mathfrak{p}_1-\mathfrak{p}_2\|_\chi + \|\mathfrak{q}_1-\mathfrak{q}_2\|_\Xi\right].$$

($\mathcal{C}d_{B_2}$) The functions $\Phi_j : (\vartheta_j, \varrho_j] \times \chi \times \Xi \to \Xi, j = 1,\dots,\nu$, are continuous and there exists $l_\Phi > 0$ such that for all $(\mathfrak{p}_p, \mathfrak{q}_p) \in \chi \times \Xi, p = 1,2$

$$\|\Phi_j(\vartheta,\mathfrak{p}_1,\mathfrak{q}_1) - \Phi_j(\vartheta,\mathfrak{p}_2,\mathfrak{q}_2)\|_\Xi \leq l_\Phi\left[\|\mathfrak{p}_1-\mathfrak{p}_2\|_\chi + \|\mathfrak{q}_1-\mathfrak{q}_2\|_\Xi\right].$$

($\mathcal{C}d_{B_3}$) For each $\vartheta \in \Theta$, $a(\vartheta,\varrho)$ is measurable on $[0,\vartheta]$ and $a(\vartheta) = ess\sup\{|a(\vartheta,\varrho)|, 0 \leq \varrho \leq \vartheta\}$ is bounded on Θ. The map $\vartheta \to a_\vartheta$ is continuous from Θ to $L^\infty(\Theta,\mathbb{R})$, here, $a_\vartheta(\varrho) = a(\vartheta,\varrho)$.

Theorem 3.6. *Suppose that ($\mathcal{C}d_\xi$) and ($\mathcal{C}d_{B_1}$) – ($\mathcal{C}d_{B_3}$) hold, if*

$$\begin{aligned}\psi &= \varpi l_\Phi(C_1^*+1) + aC_{\zeta,\varpi}\frac{\mathfrak{w}^\zeta}{\zeta} l_\Psi(C_1^*+1)\\ &< 1,\end{aligned}$$

then there exists a unique solution to the problem (1.1).

Proof. Let $\tilde{\Xi} = \{\mathfrak{p} \in \mathcal{P}_{\mathfrak{C}}(\Xi) : \mathfrak{p}(0) = \beta(0) = 0\}$ and consider the operator $\mathfrak{S} : \tilde{\Xi} \to \tilde{\Xi}$ given by

$$\mathfrak{S}(\mathfrak{p})(\vartheta) = \begin{cases} \int_0^{\vartheta}\int_0^{\varrho} \mathfrak{T}_2(\vartheta-\varrho)\, a(\varrho,\kappa)\, \Psi\left(\kappa, \overline{\mathfrak{p}}_{\eta(\kappa,\overline{\mathfrak{p}}_\kappa)}, \overline{\mathfrak{p}}(\kappa)\right) d\kappa d\varrho, \\ \quad +\mathfrak{T}_1(\vartheta)\,\beta(0), \qquad \vartheta \in [0,\vartheta_1], \\ \Phi_j\left(\vartheta, \overline{\mathfrak{p}}_{\eta(\kappa,\overline{\mathfrak{p}}_\kappa)}, \overline{\mathfrak{p}}(\vartheta)\right), \qquad \vartheta \in (\vartheta_j, \varrho_j], j = 1,2,\ldots,\nu, \\ \int_{\varrho_j}^{\vartheta}\int_0^{\varrho} \mathfrak{T}_2(\vartheta-\varrho)\, a(\varrho,\kappa)\, \Psi\left(\kappa, \overline{\mathfrak{p}}_{\eta(\kappa,\overline{\mathfrak{p}}_\kappa)}, \overline{\mathfrak{p}}(\kappa)\right) d\kappa d\varrho \\ \quad +\mathfrak{T}_1(\vartheta-\varrho_j)\,\Phi_j\left(\vartheta, \overline{\mathfrak{p}}_{\eta(\kappa,\overline{\mathfrak{p}}_\kappa)}, \overline{\mathfrak{p}}(\vartheta)\right), \\ \qquad \vartheta \in (\varrho_j, \vartheta_{j+1}], \end{cases}$$

where $\mathfrak{p} : (-\infty, \mathfrak{w}] \to \Xi$, $\mathfrak{p}_0 = \beta$ and $\overline{\mathfrak{p}} = \mathfrak{p}$ on Θ. Let $\overline{\beta} : (-\infty, \mathfrak{w}] \longrightarrow \Xi$ be the extension of β to $(-\infty, \mathfrak{w}]$ such that $\overline{\beta}(\varepsilon) = \beta(0) = 0$ on Θ.

Now, we shall show that operator $\mathfrak{S}$ has a fixed point which is, in turn, a mild solution to the problem (1.1). Let $\mathfrak{p}, \mathfrak{p}^* \in \tilde{\Xi}$, we have

Case 1. For each $\vartheta \in [0, \vartheta_1]$, we get

$$\begin{aligned} &\|\mathfrak{S}(\mathfrak{p})(\vartheta) - \mathfrak{S}(\mathfrak{p}^*)(\vartheta)\| \\ &\quad \le \int_0^{\vartheta}\int_0^{\varrho} \Big\| \mathfrak{T}_2(\vartheta-\varrho)\, a(\varrho,\kappa) \Big[\Psi\left(\kappa, \overline{\mathfrak{p}}_{\eta(\kappa,\overline{\mathfrak{p}}_\kappa)}, \overline{\mathfrak{p}}(\kappa)\right) \\ &\qquad - \Psi\left(\kappa, \overline{\mathfrak{p}}^*_{\eta(\kappa,\overline{\mathfrak{p}}^*_\kappa)}, \overline{\mathfrak{p}}^*(\kappa)\right) \Big] \Big\| d\kappa d\varrho \\ &\quad \le a C_{\zeta,\varpi} \int_0^{\vartheta}\int_0^{\varrho} (\vartheta-\varrho)^{\zeta-1}\, l_\Psi \left[\|\overline{\mathfrak{p}}_\kappa - \overline{\mathfrak{p}}^*_\kappa\| + \|\overline{\mathfrak{p}}(\kappa) - \overline{\mathfrak{p}}^*(\kappa)\| \right] d\kappa d\varrho \\ &\quad \le a C_{\zeta,\varpi} \frac{\mathfrak{w}^\zeta}{\zeta} l_\Psi (C_1^* + 1) \|\mathfrak{p} - \mathfrak{p}^*\|. \end{aligned}$$

Case 2. For each $\vartheta \in (\vartheta_j, \varrho_j]$, $j = 1, \ldots, \nu$, we find

$$\begin{aligned} &\|\mathfrak{S}(\mathfrak{p})(\vartheta) - \mathfrak{S}(\mathfrak{p}^*)(\vartheta)\| \\ &\quad = \left\| \Phi_j\left(\vartheta, \overline{\mathfrak{p}}_{\eta(\kappa,\overline{\mathfrak{p}}_\kappa)}, \overline{\mathfrak{p}}(\vartheta)\right) - \Phi_j\left(\vartheta, \overline{\mathfrak{p}}^*_{\eta(\kappa,\overline{\mathfrak{p}}^*_\kappa)}, \overline{\mathfrak{p}}^*(\vartheta)\right) \right\| \\ &\quad \le l_\Phi \left[\|\overline{\mathfrak{p}}_\kappa - \overline{\mathfrak{p}}^*_\kappa\| + \|\overline{\mathfrak{p}}(\kappa) - \overline{\mathfrak{p}}^*(\kappa)\| \right] \\ &\quad \le l_\Phi (C_1^* + 1) \|\mathfrak{p} - \mathfrak{p}^*\|. \end{aligned}$$

Case 3. For each $\vartheta \in (\varrho_j, \vartheta_{j+1}]$, $j = 1, \ldots, \nu$, we obtain

$$\begin{aligned} &\|\mathfrak{S}(\mathfrak{p})(\vartheta) - \mathfrak{S}(\mathfrak{p}^*)(\vartheta)\| \\ &\quad \le \left\| \mathfrak{T}_1(\vartheta-\varrho_j) \left[\Phi_j\left(\vartheta, \overline{\mathfrak{p}}_{\eta(\kappa,\overline{\mathfrak{p}}_\kappa)}, \overline{\mathfrak{p}}(\vartheta)\right) - \Phi_j\left(\vartheta, \overline{\mathfrak{p}}^*_{\eta(\kappa,\overline{\mathfrak{p}}^*_\kappa)}, \overline{\mathfrak{p}}^*(\vartheta)\right) \right] \right\| \end{aligned}$$

$$+\int_{\varrho_j}^{\vartheta}\int_0^{\varrho}\left\|\mathfrak{T}_2(\vartheta-\varrho)\,a(\varrho,\kappa)\left[\Psi\left(\kappa,\overline{\mathfrak{p}}_\kappa,\overline{\mathfrak{p}}(\kappa)\right)-\Psi\left(\kappa,\overline{\mathfrak{p}}^*_\kappa,\overline{\mathfrak{p}}^*(\kappa)\right)\right]\right\|d\kappa d\varrho$$

$$\leq \varpi l_\Phi\left(C_1^*+1\right)\|\mathfrak{p}-\mathfrak{p}^*\|+aC_{\zeta,\varpi}\frac{\mathfrak{w}^\zeta}{\zeta}l_\Psi\left(C_1^*+1\right)\|\mathfrak{p}-\mathfrak{p}^*\|$$

$$\leq\left[\varpi l_\Phi\left(C_1^*+1\right)+aC_{\zeta,\varpi}\frac{\mathfrak{w}^\zeta}{\zeta}l_\Psi\left(C_1^*+1\right)\right]\|\mathfrak{p}-\mathfrak{p}^*\|.$$

From the above simulation, we conclude that

$$\left\|\mathfrak{S}(\mathfrak{p})-\mathfrak{S}(\mathfrak{p}^*)\right\|_{PC}\leq\psi\left\|\mathfrak{p}-\mathfrak{p}^*\right\|_{PC},$$

which implies that $\mathfrak{S}$ is a contraction map and there exists a unique fixed point which is the mild solution of the system (1.1). This completes the proof of the theorem.

In order to get the second main result based on Krasnoselskii's fixed point Theorem, we give the following assumptions.

($\mathcal{C}d_{B_4}$) The semigroup $\mathcal{S}(\vartheta)$ is compact for $\vartheta>0$.

($\mathcal{C}d_{B_5}$) $\Psi:\Theta\times\chi\times\Xi\longrightarrow\Xi$ satisfies the Carathéodory conditions.

($\mathcal{C}d_{B_6}$) There exists a continuous function $\theta:\Theta\to(0,+\infty)$ and a continuous nondecreasing function $\upsilon:\mathbb{R}^+\to(0,+\infty)$ such that

$$\|\Psi(\vartheta,\mathfrak{p},y)\|\leq\theta(\vartheta)\,\upsilon\left(\|\mathfrak{p}\|_\chi+\|y\|_\Xi\right),\quad(\vartheta,\mathfrak{p},y)\in\Theta\times\chi\times\Xi.$$

($\mathcal{C}d_{B_7}$) The functions $\vartheta\to\Phi_j(\vartheta,0,0)$ are bounded with

$$\Phi^*=\max_{j=1,\ldots,\nu}\|\Phi_j(\vartheta,0,0)\|_\Xi.$$

Theorem 3.7. *Assume* ($\mathcal{C}d_\xi$) *and* ($\mathcal{C}d_{B_1}$)–($\mathcal{C}d_{B_7}$) *hold. If* $\varpi l_\Phi\left(C_1^*+1\right)<1$*, then there exists a mild solution to the problem (1.1).*

Proof. Let $\mathfrak{S}$ be the operator considered as in the proof of Theorem 3.6. We introduce the decomposition $\mathfrak{S}\mathfrak{p}(\vartheta)=\mathfrak{S}^1\mathfrak{p}(\vartheta)+\mathfrak{S}^2\mathfrak{p}(\vartheta)$, where

$$\mathfrak{S}^1(\mathfrak{p})(\vartheta)=\begin{cases}\displaystyle\int_{\varrho_j}^{\vartheta}\int_0^{\varrho}\mathfrak{T}_2(\vartheta-\varrho)\,a(\varrho,\kappa)\,\Psi\left(\kappa,\overline{\mathfrak{p}}_{\eta(\kappa,\overline{\mathfrak{p}}_\kappa)},\overline{\mathfrak{p}}(\kappa)\right)d\kappa d\varrho\\ +\mathfrak{T}_1(\vartheta-\varrho_j)\,\Phi_j\left(\vartheta,\overline{\mathfrak{p}}_{\eta(\kappa,\overline{\mathfrak{p}}_\kappa)},\overline{\mathfrak{p}}(\vartheta)\right), \\ \hfill\text{if } \vartheta\in(\varrho_j,\vartheta_{j+1}],\ j\geq1,\\[1em] \displaystyle\int_0^{\vartheta}\int_0^{\varrho}\mathfrak{T}_2(\vartheta-\varrho)\,a(\varrho,\kappa)\,\Psi\left(\kappa,\overline{\mathfrak{p}}_{\eta(\kappa,\overline{\mathfrak{p}}_\kappa)},\overline{\mathfrak{p}}(\kappa)\right)d\kappa d\varrho\\ +\mathfrak{T}_1(\vartheta)\,\beta(0), \qquad\qquad \text{if } \vartheta\in[0,\vartheta_1],\\[1em] 0, \qquad\qquad \text{if } \vartheta\in(\vartheta_j,\varrho_j],\ j\geq1,\end{cases}$$

and

$$\mathfrak{S}^2(\mathfrak{p})(\vartheta)=\begin{cases}\Phi_j\left(\vartheta,\overline{\mathfrak{p}}_{\eta(\kappa,\overline{\mathfrak{p}}_\kappa)},\overline{\mathfrak{p}}(\vartheta)\right), & \text{if } \vartheta\in(\vartheta_j,\varrho_j],\, j\geq 1,\\ 0, & \text{if } \vartheta\in(\varrho_j,\vartheta_{j+1}],\, j\geq 1,\\ 0, & \text{if } \vartheta\in[0,\vartheta_1].\end{cases}$$

Choose

$$\mu\geq\frac{\varpi\left[l_\Phi\left(C_2^*+\Lambda^\beta\right)\|\beta\|_{\mathcal{B}}+\Phi^*\right]}{1-\varpi l_\Phi\left(C_1^*+1\right)}+\frac{aC_{\zeta,\varpi}\frac{\mathfrak{w}^\zeta}{\zeta}\upsilon\left(\left(C_2^*+\Lambda^\beta\right)\|\beta\|_{\mathcal{B}}+\left(C_1^*+1\right)\mu\right)\|\theta\|_{L^1[0,\vartheta_1]}}{1-\varpi l_\Phi\left(C_1^*+1\right)},$$

and define the set

$$B_\mu=\left\{\mathfrak{p}\in\tilde{\Xi}:\|\mathfrak{p}\|_{PC}\leq\mu\right\},$$

then B_μ is a bounded closed-convex subset in $\tilde{\Xi}$.

In order to use Krasnoselskii's fixed-point theorem, we divide our proof into three steps.

Step 1: For any $\mathfrak{p}\in B_\mu$, we prove that $\mathfrak{S}^1\mathfrak{p}+\mathfrak{S}^2\mathfrak{p}\in B_\mu$.

Case 1. For all $\vartheta\in[0,\vartheta_1]$, we obtain

$$\begin{aligned}&\left\|\left(\mathfrak{S}^1\mathfrak{p}+\mathfrak{S}^2\mathfrak{p}\right)(\vartheta)\right\|\\ &\leq\|\mathfrak{T}_1(\vartheta)\beta(0)\|+\int_0^\vartheta\int_0^\varrho\left\|\mathfrak{T}_2(\vartheta-\varrho)a(\varrho,\kappa)\Psi\left(\kappa,\overline{\mathfrak{p}}_{\eta(\kappa,\overline{\mathfrak{p}}_\kappa)},\overline{\mathfrak{p}}(\kappa)\right)\right\|d\kappa d\varrho\\ &\leq\varpi\|\beta\|_\chi+aC_{\zeta,\varpi}\int_0^\vartheta\int_0^\varrho(\vartheta-\varrho)^{\zeta-1}\theta(\kappa)\upsilon\left(\left\|\overline{\mathfrak{p}}_{\eta(\kappa,\overline{\mathfrak{p}}_\kappa)}\right\|_\chi+\|\overline{\mathfrak{p}}\|\right)d\kappa d\varrho\\ &\leq\varpi\|\beta\|_\chi+aC_{\zeta,\varpi}\int_0^\vartheta\int_0^\varrho(\vartheta-\varrho)^{\zeta-1}\theta(\kappa)\\ &\quad\times\upsilon\left(\left(C_2^*+\Lambda^\beta\right)\|\beta\|_{\mathcal{B}}+C_1^*\mu+\mu\right)d\kappa d\varrho\\ &\leq\varpi\|\beta\|_\chi+aC_{\zeta,\varpi}\frac{\mathfrak{w}^\zeta}{\zeta}\upsilon\left(\left(C_2^*+\Lambda^\beta\right)\|\beta\|_{\mathcal{B}}+\left(C_1^*+1\right)\mu\right)\|\theta\|_{L^1[0,\vartheta_1]}\\ &\leq\mu.\end{aligned}$$

Case 2. For all $\vartheta\in[\vartheta_j,\varrho_j)$, $j=1,2,\ldots,\nu$, we have

$$\begin{aligned}&\left\|\left(\mathfrak{S}^1\mathfrak{p}+\mathfrak{S}^2\mathfrak{p}\right)(\vartheta)\right\|\\ &\quad\leq\left\|\Phi_j\left(\vartheta,\overline{\mathfrak{p}}_{\eta(\kappa,\overline{\mathfrak{p}}_\kappa)},\overline{\mathfrak{p}}(\vartheta)\right)\right\|\\ &\quad\leq\left\|\Phi_j\left(\vartheta,\overline{\mathfrak{p}}_{\eta(\kappa,\overline{\mathfrak{p}}_\kappa)},\overline{\mathfrak{p}}(\vartheta)\right)-\Phi_j(\vartheta,0,0)\right\|+\|\Phi_j(\vartheta,0,0)\|\\ &\quad\leq l_\Phi\left(\left\|\overline{\mathfrak{p}}_{\eta(\kappa,\overline{\mathfrak{p}}_\kappa)}\right\|_\chi+\|\overline{\mathfrak{p}}\|\right)+\Phi^*\end{aligned}$$

$$\leq l_{\Phi}\left[\left(C_2^*+\Lambda^{\beta}\right)\|\beta\|_{\chi}+(C_1^*+1)\mu\right]+\Phi^*$$
$$\leq \mu.$$

Case 3. For all $\vartheta\in(\varrho_j,\vartheta_{j+1}]$, $j=1,2,\ldots,\nu$, we obtain

$$\begin{aligned}
&\left\|\left(\mathfrak{S}^1\mathfrak{p}+\mathfrak{S}^2\mathfrak{p}\right)(\vartheta)\right\|\\
&\leq\left\|\mathfrak{T}_1(\vartheta-\varrho_j)\,\Phi_j\left(\vartheta,\overline{\mathfrak{p}}_{\eta(\kappa,\overline{\mathfrak{p}}_\kappa)},\overline{\mathfrak{p}}(\vartheta)\right)\right\|\\
&\quad+\int_{\varrho_j}^{\vartheta}\int_0^{\varrho}\left\|\mathfrak{T}_2(\vartheta-\varrho)\,a(\varrho,\kappa)\,\Psi\left(\kappa,\overline{\mathfrak{p}}_{\eta(\kappa,\overline{\mathfrak{p}}_\kappa)},\overline{\mathfrak{p}}(\kappa)\right)\right\|d\kappa d\varrho,\\
&\leq\varpi\left[l_{\Phi}\left(\left(C_2^*+\Lambda^{\beta}\right)\|\beta\|_{\mathcal{B}}+(C_1^*+1)\,\mu\right)+\Phi^*\right]\\
&\quad+aC_{\zeta,\varpi}\frac{\mathfrak{w}^{\zeta}}{\zeta}\upsilon\left(\left(C_2^*+\Lambda^{\beta}\right)\|\beta\|_{\mathcal{B}}+(C_1^*+1)\,\mu\right)\|\theta\|_{L^1[0,\vartheta_1]}\\
&\leq\mu.
\end{aligned}$$

Step 2: Next, we will prove that $\mathfrak{S}^1$ is compact and continuous. Therefore, 3 claims will be given.

Claim 1: $\mathfrak{S}^1$ is continuous.

Consider the sequence $\{\mathfrak{p}^k\}_{k\in\mathbb{N}}$, where $\mathfrak{p}^k\to\mathfrak{p}$ in B_μ if $k\to\infty$.

For $\vartheta\in[0,\vartheta_1]$, we get

$$\begin{aligned}
&\left\|\mathfrak{S}^1\left(\mathfrak{p}^k\right)(\vartheta)-\mathfrak{S}^1(\mathfrak{p})(\vartheta)\right\|\\
&\leq\int_0^{\vartheta}\int_0^{\varrho}\|\mathfrak{T}_2(\vartheta-\varrho)\|\,\|a(\varrho,\kappa)\|\left\|\Psi\left(\kappa,\overline{\mathfrak{p}}^k_{\eta\left(\kappa,\overline{\mathfrak{p}}^k_\kappa\right)},\overline{\mathfrak{p}}^k(\kappa)\right)\right.\\
&\qquad\left.-\Psi\left(\kappa,\overline{\mathfrak{p}}_{\eta(\kappa,\overline{\mathfrak{p}}_\kappa)},\overline{\mathfrak{p}}(\kappa)\right)\right\|d\kappa d\varrho\\
&\leq aC_{\zeta,\varpi}\int_0^{\vartheta}\int_0^{\varrho}(\vartheta-\varrho)^{\zeta-1}\left\|\Psi\left(\kappa,\overline{\mathfrak{p}}^k_{\eta\left(\kappa,\overline{\mathfrak{p}}^k_\kappa\right)},\overline{\mathfrak{p}}^k(\kappa)\right)\right.\\
&\qquad\left.-\Psi\left(\kappa,\overline{\mathfrak{p}}_{\eta(\kappa,\overline{\mathfrak{p}}_\kappa)},\overline{\mathfrak{p}}(\kappa)\right)\right\|d\kappa d\varrho.
\end{aligned}$$

For each $\vartheta\in[\vartheta_j,\varrho_j)$, $j=1,2,\ldots,\nu$, we have

$$\left\|\mathfrak{S}^1\left(\mathfrak{p}^k\right)(\vartheta)-\mathfrak{S}^1(\mathfrak{p})(\vartheta)\right\|=0.$$

Further, for each $\vartheta\in(\varrho_j,\vartheta_{j+1}]$, $j=1,2,\ldots,\nu$, we obtain

$$\begin{aligned}
&\left\|\mathfrak{S}^1\left(\mathfrak{p}^k\right)(\vartheta)-\mathfrak{S}^1(\mathfrak{p})(\vartheta)\right\|\\
&\leq\|\mathfrak{T}_1(\vartheta-\varrho_j)\|\left\|\Phi_j\left(\vartheta,\overline{\mathfrak{p}}^k_{\eta\left(\vartheta,\overline{\mathfrak{p}}^k_\vartheta\right)},\overline{\mathfrak{p}}^k(\vartheta)\right)-\Phi_j\left(\vartheta,\overline{\mathfrak{p}}_{\eta(\vartheta,\overline{\mathfrak{p}}_\vartheta)},\overline{\mathfrak{p}}(\vartheta)\right)\right\|\\
&\quad+\int_{\varrho_j}^{\vartheta}\int_0^{\varrho}\|\mathfrak{T}_2(\vartheta-\varrho)\|\,\|a(\varrho,\kappa)\|\left\|\Psi\left(\kappa,\overline{\mathfrak{p}}^k_{\eta\left(\kappa,\overline{\mathfrak{p}}^k_\kappa\right)},\overline{\mathfrak{p}}^k(\kappa)\right)\right.
\end{aligned}$$

$$- \Psi\left(\kappa, \overline{\mathfrak{p}}_{\eta(\kappa,\overline{\mathfrak{p}}_\kappa)}, \overline{\mathfrak{p}}(\kappa)\right)\Big\| d\kappa d\varrho$$
$$\leq \varpi \left\| \Phi_j\left(\vartheta, \overline{\mathfrak{p}}^k_{\eta\left(\vartheta,\overline{\mathfrak{p}}^k_\vartheta\right)}, \overline{\mathfrak{p}}^k(\vartheta)\right) - \Phi_j\left(\vartheta, \overline{\mathfrak{p}}_{\eta(\vartheta,\overline{\mathfrak{p}}_\vartheta)}, \overline{\mathfrak{p}}(\vartheta)\right)\right\|$$
$$+ aC_{\zeta,\varpi}\int_{\varrho_j}^{\vartheta}\int_0^{\varrho} (\vartheta-\varrho)^{\zeta-1}\Big\|\Psi\left(\kappa, \overline{\mathfrak{p}}^k_{\eta\left(\kappa,\overline{\mathfrak{p}}^k_\kappa\right)}, \overline{\mathfrak{p}}^k(\kappa)\right)$$
$$- \Psi\left(\kappa, \overline{\mathfrak{p}}_{\eta(\kappa,\overline{\mathfrak{p}}_\kappa)}, \overline{\mathfrak{p}}(\kappa)\right)\Big\| d\kappa d\varrho.$$

By the continuity of the function Φ_j and Ψ, we have

$$\left\|\mathfrak{S}^1\left(\mathfrak{p}^k\right)(\vartheta) - \mathfrak{S}^1(\mathfrak{p})(\vartheta)\right\| \to 0 \quad \text{as } k \to \infty.$$

Thus, $\mathfrak{S}^1$ is continuous.

Claim 2: $\mathfrak{S}^1$ maps a bounded set into bounded sets of B_μ. By Step 1 $\left\|\mathfrak{S}^1\mathfrak{p}\right\| \leq \mu$.

Claim 3: $\mathfrak{S}^1$ maps a bounded set into equicontinuous sets of B_μ.

Case 1. For each $\vartheta \in [0, \vartheta_1], 0 \leq \kappa_2 \leq \kappa_1 \leq \vartheta_1$ and $\mathfrak{p} \in B_\mu$. Then, we have

$$\left\|\mathfrak{S}^1(\mathfrak{p})(\kappa_1) - \mathfrak{S}^1(\mathfrak{p})(\kappa_2)\right\| \leq \sigma_1 + \sigma_2 + \sigma_3,$$

where

$$\sigma_1 = \|\mathfrak{T}_1(\kappa_1) - \mathfrak{T}_1(\kappa_2)\|\,\|\beta(0)\|,$$
$$\sigma_2 = \left\|\int_0^{\kappa_2}\int_0^{\varrho}[\mathfrak{T}_2(\kappa_1-\varrho) - \mathfrak{T}_2(\kappa_2-\varrho)]a(\varrho,\kappa)\,\Psi\left(\kappa, \overline{\mathfrak{p}}_{\eta(\kappa,\overline{\mathfrak{p}}_\kappa)}, \overline{\mathfrak{p}}(\kappa)\right)d\kappa d\varrho\right\|,$$
$$\sigma_3 = \left\|\int_{\kappa_2}^{\kappa_1}\int_0^{\varrho}\mathfrak{T}_2(\kappa_1-\varrho)\,a(\varrho,\kappa)\,\Psi\left(\kappa, \overline{\mathfrak{p}}_{\eta(\kappa,\overline{\mathfrak{p}}_\kappa)}, \overline{\mathfrak{p}}(\kappa)\right)d\kappa d\varrho\right\|.$$

σ_1 tends to zero as $\kappa_2 \to \kappa_1$ since $\mathcal{S}(\vartheta)$ is a uniformly continuous operator.

For σ_2, using (3.2) and $(\mathcal{C}d_{B_2})$, we have

$$\sigma_2 \leq av\left(\left(C_2^* + \Lambda^\beta\right)\|\beta\|_\chi + (C_1^*+1)\,\mu\right)\|\theta\|_{L^1}$$
$$\times \int_0^{\kappa_2}[\mathfrak{T}_2(\kappa_1-\varrho) - \mathfrak{T}_2(\kappa_2-\varrho)]d\varrho$$
$$\leq av\left(\left(C_2^* + \Lambda^\beta\right)\|\beta\|_\chi + (C_1^*+1)\,\mu\right)\|\theta\|_{L^1}$$
$$\times \int_0^{\kappa_2}\left[\zeta\int_0^{\infty}\varphi(\kappa_1-\varrho)^{\zeta-1}\rho_\zeta(\varphi)\,\mathcal{S}\left((\kappa_1-\varrho)^\zeta\varphi\right)d\varphi\right.$$
$$\left.-\zeta\int_0^{\infty}\varphi(\kappa_2-\varrho)^{\zeta-1}\rho_\zeta(\varphi)\,\mathcal{S}\left((\kappa_2-\varrho)^\zeta\varphi\right)d\varphi\right]d\varrho$$
$$\leq av\left(\left(C_2^* + \Lambda^\beta\right)\|\beta\|_\chi + (C_1^*+1)\,\mu\right)\|\theta\|_{L^1}$$

$$\times\left[\zeta\int_0^{\kappa_2}\int_0^{\infty}\varphi\left\|[(\kappa_1-\varrho)^{\zeta-1}-(\kappa_2-\varrho)^{\zeta-1}]\rho_\zeta(\varphi)\mathcal{S}\left((\kappa_1-\varrho)^\zeta\varphi\right)\right\|\right.$$
$$\left.+\zeta\int_0^{\kappa_2}\int_0^{\infty}\varphi(\kappa_2-\varrho)^{\zeta-1}\rho_\zeta(\varphi)\left\|\mathcal{S}\left((\kappa_1-\varrho)^\zeta\varphi\right)-\mathcal{S}\left((\kappa_2-\varrho)^\zeta\varphi\right)\right\|\right]$$
$$\le av\left(\left(C_2^*+\Lambda^\beta\right)\|\beta\|_{\mathcal{B}}+(C_1^*+1)\mu\right)\|\theta\|_{L^1}$$
$$\times\left[C_{\zeta,\varpi}\int_0^{\kappa_2}\left|(\kappa_1-\varrho)^{\zeta-1}-(\kappa_2-\varrho)^{\zeta-1}\right|d\varrho\right.$$
$$+\zeta\int_0^{\kappa_2}\int_0^{\infty}\varphi(\kappa_2-\varrho)^{\zeta-1}\rho_\zeta(\varphi)\left\|\mathcal{S}\left((\kappa_1-\varrho)^\zeta\varphi\right)\right.$$
$$\left.\left.-\mathcal{S}\left((\kappa_2-\varrho)^\zeta\varphi\right)\right\|d\varphi d\varrho\right].$$

The right-hand side of the above inequality tends to zero as $\kappa_2\to\kappa_1$.

From $(\mathcal{C}d_{B_6})$, we have

$$\sigma_3\le aC_{\zeta,\varpi}\int_{\kappa_2}^{\kappa_1}\int_0^{\varrho}(\kappa_1-\varrho)^{\zeta-1}\left\|\Psi\left(\kappa,\overline{\mathfrak{p}}_{\eta(\kappa,\overline{\mathfrak{p}}_\kappa)},\overline{\mathfrak{p}}(\kappa)\right)\right\|d\kappa d\varrho$$
$$\le aC_{\zeta,\varpi}\upsilon\left(\left(C_2^*+\Lambda^\beta\right)\|\beta\|_{\mathcal{B}}+(C_1^*+1)\mu\right)\|\theta\|_{L^1}\int_{\kappa_2}^{\kappa_1}(\kappa_1-\varrho)^{\zeta-1}d\varrho.$$

As $\kappa_2\to\kappa_1$, σ_3 tends to zero.

Case 2. For each $\vartheta\in[\vartheta_j,\varrho_j), j=1,2,\ldots,\nu,\vartheta_j\le\kappa_2\le\kappa_1\le\varrho_j$ and $\mathfrak{p}\in B_\mu$. Then, we have

$$\left\|\mathfrak{S}^1(\mathfrak{p})(\kappa_1)-\mathfrak{S}^1(\mathfrak{p})(\kappa_2)\right\|=0.$$

Case 3. For each $\vartheta\in(\varrho_j,\vartheta_{j+1}], j=1,2,\ldots,\nu,\varrho_j\le\kappa_2\le\kappa_1\le\vartheta_{j+1}$, and $\mathfrak{p}\in B_\mu$. Then, we have

$$\left\|\mathfrak{S}^1(\mathfrak{p})(\kappa_1)-\mathfrak{S}^1(\mathfrak{p})(\kappa_2)\right\|$$
$$\le\|\mathfrak{T}_1(\kappa_1-\varrho_j)-\mathfrak{T}_1(\kappa_2-\varrho_j)\|\left\|\Phi_j\left(\vartheta,\overline{\mathfrak{p}}_{\eta(\vartheta,\overline{\mathfrak{p}}_\vartheta)},\overline{\mathfrak{p}}(\vartheta)\right)\right\|+\sigma_1+\sigma_2+\sigma_3.$$

By the uniform continuity of $\mathcal{S}(\vartheta)$, we have

$$\lim_{\kappa_2\to\kappa_1}\|\mathfrak{T}_1(\kappa_1-\varrho_j)-\mathfrak{T}_1(\kappa_2-\varrho_j)\|=0,\quad j=1,\ldots,\nu.$$

Consequently

$$\lim_{\kappa_2\to\kappa_1}\left\|\mathfrak{S}^1(\mathfrak{p})(\kappa_1)-\mathfrak{S}^1(\mathfrak{p})(\kappa_2)\right\|=0.$$

Thus, $\mathfrak{S}^1(B_\mu)$ is equicontinuous.

Claim 4: The set $\{\mathfrak{S}^1(\mathfrak{p})(\vartheta):\mathfrak{p}\in B_\mu\}$ is relatively compact in $\tilde{\Xi}$.

Case 1. For the interval $\vartheta\in[0,\vartheta_1]$. Let $0<\vartheta\le\vartheta_1$ be fixed and let λ

be a real number satisfying $0 < \lambda < \vartheta$. For arbitrary $\psi > 0$, we define

$$\begin{aligned}\mathfrak{S}^1_{\lambda,\psi}(\mathfrak{p})(\vartheta) &= \mathfrak{T}_1(\vartheta)\beta(0) + \zeta\int_0^{\vartheta-\lambda}(\vartheta-\varrho)^{\zeta-1}\int_\psi^\infty \varphi\rho_\zeta(\varphi)\mathcal{S}\left((\vartheta-\varrho)^\zeta\varphi\right)\\ &\quad\times\int_0^\varrho a(\varrho,\kappa)\Psi\left(\kappa,\overline{\mathfrak{p}}_{\eta(\kappa,\overline{\mathfrak{p}}_\kappa)},\overline{\mathfrak{p}}(\kappa)\right)d\kappa d\varphi d\varrho\\ &= \zeta\mathcal{S}\left(\lambda^\zeta\psi\right)\int_0^{\vartheta-\lambda}(\vartheta-\varrho)^{\zeta-1}\int_\psi^\infty\varphi\rho_\zeta(\varphi)\mathcal{S}\left((\vartheta-\varrho)^\zeta\varphi-\lambda^\zeta\psi\right)\\ &\quad\times\int_0^\varrho a(\varrho,\kappa)\Psi\left(\kappa,\overline{\mathfrak{p}}_{\eta(\kappa,\overline{\mathfrak{p}}_\kappa)},\overline{\mathfrak{p}}(\kappa)\right)d\kappa d\varphi d\varrho.\end{aligned}$$

Since $\mathcal{S}(\vartheta)$ is a compact operator, the set

$$H_{\lambda,\psi} = \left\{\mathfrak{S}^1_{\lambda,\psi}(\mathfrak{p})(\vartheta) : \mathfrak{p}\in P_1(B_\mu)\right\}$$

is relatively compact. Moreover,

$$\begin{aligned}&\left\|\mathfrak{S}^1(\mathfrak{p})(\vartheta)-\mathfrak{S}^1_{\lambda,\psi}(\mathfrak{p})(\vartheta)\right\|\\ &\quad\le\zeta\int_0^{\vartheta-\lambda}(\vartheta-\varrho)^{\zeta-1}\int_0^\psi\varphi\rho_\zeta(\varphi)\left\|\mathcal{S}\left((\vartheta-\varrho)^\zeta\varphi\right)\right\|\\ &\qquad\int_0^\varrho\|a(\varrho,\kappa)\|\left\|\Psi\left(\kappa,\overline{\mathfrak{p}}_{\eta(\kappa,\overline{\mathfrak{p}}_\kappa)},\overline{\mathfrak{p}}(\kappa)\right)\right\|d\kappa d\varphi d\varrho\\ &\qquad+\zeta\int_{\vartheta-\lambda}^\vartheta(\vartheta-\varrho)^{\zeta-1}\int_0^\infty\varphi\rho_\zeta(\varphi)\left\|\mathcal{S}\left((\vartheta-\varrho)^\zeta\varphi\right)\right\|\\ &\qquad\int_0^\varrho\|a(\varrho,\kappa)\|\left\|\Psi\left(\kappa,\overline{\mathfrak{p}}_{\eta(\kappa,\overline{\mathfrak{p}}_\kappa)},\overline{\mathfrak{p}}(\kappa)\right)\right\|d\kappa d\varphi d\varrho\\ &\quad\le\mathfrak{w}^\zeta\varpi a\upsilon\left(\left(C_2^*+\Lambda^\beta\right)\|\beta\|_\chi+(C_1^*+1)\mu\right)\|\theta\|_{L^1}\int_0^\psi\varphi\rho_\zeta(\varphi)d\varphi\\ &\qquad+\frac{\lambda^\zeta\varpi a}{\Gamma(1+\zeta)}\upsilon\left(\left(C_2^*+\Lambda^\beta\right)\|\beta\|_\chi+(C_1^*+1)\mu\right)\|\theta\|_{L^1}.\end{aligned}$$

Case 3. For the interval $\vartheta\in(\varrho_j,\vartheta_{j+1}], j=1,2,\dots,\nu$. Let $\varrho_j<\vartheta\le\vartheta_{j+1}$ be fixed and let λ be a real number satisfying $0<\lambda<\vartheta$. For arbitrary $\psi>0$, we define

$$\begin{aligned}&\mathfrak{S}^1_{\lambda,\psi}(\mathfrak{p})(\vartheta)\\ &\quad=\mathfrak{T}_1(\vartheta-\varrho_j)\Phi_j\left(\vartheta,\overline{\mathfrak{p}}_{\eta(\vartheta,\overline{\mathfrak{p}}_\vartheta)},\overline{\mathfrak{p}}(\vartheta)\right)\\ &\qquad+\zeta\int_0^{\vartheta-\lambda}(\vartheta-\varrho)^{\zeta-1}\int_\psi^\infty\varphi\rho_\zeta(\varphi)\mathcal{S}\left((\vartheta-\varrho)^\zeta\varphi\right)\\ &\qquad\times\int_0^\varrho a(\varrho,\kappa)\Psi\left(\kappa,\overline{\mathfrak{p}}_{\eta(\kappa,\overline{\mathfrak{p}}_\kappa)},\overline{\mathfrak{p}}(\kappa)\right)d\kappa d\varphi d\varrho\end{aligned}$$

$$
\begin{aligned}
&= \mathfrak{T}_1 (\vartheta - \varrho_j) \Phi_j \left(\vartheta, \overline{\mathfrak{p}}_{\eta(\vartheta, \overline{\mathfrak{p}}_\vartheta)}, \overline{\mathfrak{p}} (\vartheta)\right) \\
&\quad + \zeta \mathcal{S} \left(\lambda^\zeta \psi\right) \int_0^{\vartheta - \lambda} (\vartheta - \varrho)^{\zeta - 1} \int_\psi^\infty \varphi \rho_\zeta (\varphi) \\
&\quad \times \mathcal{S} \left((\vartheta - \varrho)^\zeta \varphi - \lambda^\zeta \psi\right) \int_0^\varrho a (\varrho, \kappa) \Psi \left(\kappa, \overline{\mathfrak{p}}_{\eta(\kappa, \overline{\mathfrak{p}}_\kappa)}, \overline{\mathfrak{p}} (\kappa)\right) d\kappa d\varphi d\varrho.
\end{aligned}
$$

Since $\mathcal{S} (\vartheta)$ is a compact operator, the set

$$H_{\lambda,\psi} = \left\{\mathfrak{S}^1_{\lambda,\psi} (\mathfrak{p}) (\vartheta) : \mathfrak{p} \in P_1 (B_\mu)\right\}$$

is relatively compact. As a consequence of Claims 1 to 4 together with the Arzelá-Ascoli theorem, we can conclude that P_1 is completely continuous.

Step 3: We shall show that $\mathfrak{S}^2$ is a contraction map. Let $\mathfrak{p}_1, \mathfrak{p}_2 \in B_\mu$, we have

Case 1. For $\vartheta \in [0, \vartheta_1]$, we have

$$\left\|\left(\mathfrak{S}^2 \mathfrak{p}_1\right) (\vartheta) - \left(\mathfrak{S}^2 \mathfrak{p}_2\right) (\vartheta)\right\| = 0.$$

Case 2. Let $\mathfrak{p}_1, \mathfrak{p}_2 \in B_\mu$ and for $\vartheta \in [\vartheta_j, \varrho_j), j = 1, 2, \ldots, \nu$, we find

$$\left\|\left(\mathfrak{S}^2 \mathfrak{p}_1\right) (\vartheta) - \left(\mathfrak{S}^2 \mathfrak{p}_2\right) (\vartheta)\right\| \leq l_\Phi (C_1^* + 1) \|\mathfrak{p}_2 - \mathfrak{p}_1\|.$$

Case 3. Let $\mathfrak{p}_1, \mathfrak{p}_2 \in B_\mu$ and for $\vartheta \in (\varrho_j, \vartheta_{j+1}], j = 1, 2, \ldots, \nu$, we obtain

$$\left\|\left(\mathfrak{S}^2 \mathfrak{p}_1\right) (\vartheta) - \left(\mathfrak{S}^2 \mathfrak{p}_2\right) (\vartheta)\right\| = 0,$$

which implies that $\mathfrak{S}^2$ is a contraction mapping. As a consequence of Krasnoselskii's fixed point Theorem, we deduce that $P_1 + P_2$ has a fixed point, which is a solution to the problem (1.1).

4. An Example

We consider the following problem:

(4.1)

$$
\begin{cases}
\frac{\partial^\zeta_\vartheta}{\partial \vartheta^\zeta} \mathfrak{q} (\vartheta, \gamma) + \frac{\partial^2}{\partial \gamma^2} \mathfrak{q} (\vartheta, \gamma) = \int_{-\infty}^\vartheta a_1 (\varrho - \vartheta) \mathfrak{p} (\varrho - \eta_1 (\vartheta) \eta_2 (|\mathfrak{p} (\vartheta)|), \gamma) d\varrho \\
\quad + \int_0^\vartheta \frac{\varrho^2}{2} \cos |\mathfrak{q} (\varrho, \gamma)| d\varrho, \qquad (\vartheta, \mathfrak{p}) \in \mathbb{N} \in \cup_{j=1}^n [\varrho_j, \vartheta_{j+1}] \times [0, \pi], \\
\mathfrak{q} (\vartheta, 0) = \mathfrak{q} (\vartheta, \pi) = 0, \qquad \vartheta \in [0, \mathfrak{w}], \\
\mathfrak{q} (\kappa, \gamma) = \mathfrak{q}_0 (\varepsilon, \gamma), \qquad \varepsilon \in (-\infty, 0], \mathfrak{p} \in [0, \pi], \\
\mathfrak{q} (\vartheta, \gamma) = H_j (\vartheta, \mathfrak{p} (\vartheta - \eta_1 (\vartheta) \eta_2 (|\mathfrak{p} (\vartheta)|), \gamma), \gamma (\vartheta)), \\
\qquad (\vartheta, \mathfrak{p}) \in (\vartheta_j, \varrho_j] \times [0, \pi],
\end{cases}
$$

where $0 < \zeta < 1, 0 = \vartheta_0 = \varrho_0 < \vartheta_1 \le \varrho_1 \le \vartheta_2 < \cdots < \vartheta_{\nu-1} \le \varrho_\nu \le \vartheta_\nu \le \vartheta_{\nu+1} = \mathfrak{w}$ and $a_1 : \mathbb{R} \to \mathbb{R}, \eta_j : [0, +\infty) \to [0, +\infty), j = 1, 2$ are continuous functions.

Set $\Xi = L^2([0, \pi])$ and consider $\mathcal{Z} : D(\mathcal{Z}) \subset \Xi \to \Xi$ by $\mathcal{Z}\tau = \tau''$ with

$$D(\mathcal{Z}) = \Big\{\tau \in \Xi : \tau, \tau' \text{ are absolutely continuous}, \tau'' \in \Xi, \tau(0) = \tau(\pi) = 0\Big\}.$$

Thus

$$\mathcal{Z}\tau = \sum_{n=1}^{\infty} n^2 (\tau, \tau_n) \tau_n, \quad \tau \in D(\mathcal{Z}),$$

where $\tau_n(\mathfrak{p}) = \sqrt{\frac{2}{\pi}} \sin(n\mathfrak{p})$, $n \in \mathbb{N}$ is the orthogonal set of eigenvectors of $\mathcal{Z}$. It is well known that $\mathcal{Z}$ is the infinitesimal generator of an analytic semigroup $\{\mathcal{S}(\vartheta)\}_{\vartheta \ge 0}$ in Ξ and is given by

$$\mathcal{S}(\vartheta)\tau = \sum_{n=1}^{\infty} e^{-n^2\vartheta} (\tau, \tau_n) \tau_n, \quad \text{for all } \tau \in \Xi, \text{ and every } \vartheta > 0.$$

Consequently, $\{\mathcal{S}(\vartheta)\}_{\vartheta \ge 0}$ is a uniformly bounded compact semigroup on Ξ. We choose $\chi = \mathfrak{C}_0 \times L^2(\Upsilon, \Xi)$ as phase space.

Set

$$\begin{aligned}
&\mathfrak{p}(\vartheta)(\gamma) = \mathfrak{q}(\vartheta, \gamma), \\
&\beta(\varepsilon)(\gamma) = \mathfrak{q}_0(\varepsilon, \gamma), \\
&\Psi(\vartheta, \xi, \mathfrak{p}(\vartheta))(\gamma) = \int_{-\infty}^{0} a_1(\varrho)\, \xi(\varrho, \rho)\, d\varrho + \frac{\vartheta^2}{2} \cos|\mathfrak{p}(\vartheta)(\gamma)|, \\
&\Phi_j(\vartheta, \xi, \mathfrak{p}(\vartheta))(\gamma) = H_j(\vartheta, \mathfrak{p}(\vartheta - \eta_1(\vartheta)\eta_2(\|\mathfrak{p}(\vartheta)\|), \gamma), \gamma(\vartheta)) \\
&\eta(\vartheta, \xi) = \vartheta - \eta_1(\vartheta)\eta_2(\|\xi(0)\|).
\end{aligned}$$

Then the problem (4.1) can be rewritten as the abstract problem (1.1). The following result is a direct consequence of Theorem 3.7.

Proposition 4.1. *Let $\xi \in \chi$ such that (H_ξ) holds, and let $\vartheta \to \xi_\vartheta$ be continuous on $\mathcal{T}(\eta^-)$. Then there exists a mild solution of* (4.1).

References

1. S. Abbas, M. Benchohra and G.M. N'Guérékata, *Topics in Fractional Differential Equations*, Springer, New York, 2012.
2. S. Abbas, M. Benchohra and G.M. N'Guérékata, *Advanced Fractional Differential and Integral Equations*, Nava Science Publishers, New York, 2015.
3. R.S. Adiguzel, U. Aksoy, E. Karapinar and I.M. Erhan, *On the solution of a boundary value problem associated with a fractional differential equation*, Math. Meth. Appl. Sci., (2020), pp. 1-12.

4. R.S. Adiguzel, U. Aksoy, E. Karapinar and I.M. Erhan, *Uniqueness of solution for higherorder nonlinear fractional differential equations with multi-point and integral boundary conditions*, RACSAM, (2021), pp. 115-155.
5. H. Afshari and E. Karapınar, *A discussion on the existence of positive solutions of the boundary value problems via ψ-Hilfer fractional derivative on b-metric spaces*, Adv. Difference Equ., 616 (2020), pp. 1-11.
6. H. Afshari, S. Kalantari and E. Karapinar, *Solution of fractional differential equations via coupled fixed point*, Electron. J. Differ. Equ., 2015 (2015), pp. 1-12.
7. B. Alqahtani, H. Aydi, E. Karapınar and V. Rakocevic, *A Solution for Volterra Fractional Integral Equations by Hybrid Contractions*, Mathematics, 7 (2019).
8. E. Karapinar, A. Fulga, M. Rashid, L. Shahid and H. Aydi, *Large Contractions on QuasiMetric Spaces with an Application to Nonlinear Fractional Differential-Equations*, Mathematics, 7 (2019).
9. E. Karapinar, H.D. Binh, N.H. Luc and N.H. Can, *On continuity of the fractional derivative of the time-fractional semilinear pseudo-parabolic systems*, Adv. Difference Equ., 70 (2021), pp.1-24.
10. K. Aissani and M. Benchohra, *Global existence results for fractional integro-differential equations with state-dependent delay*, An. Ştiinţ. Univ. Al. I. Cuza Iasi. Mat. (N.S.)., 62 (2016), pp. 411-422.
11. K. Aissani and M. Benchohra, *Existence results for fractional integro-differential equations with state-dependent delay*, Adv. Dyn. Syst. Appl., 9 (2014), pp. 17-30.
12. R.P. Agarwal, B. De Andrade and G. Siracusa, *On fractional integro-difierential equations with state-dependent delay*, Comput. Math. Appl., 62 (2011), pp. 1143-1149.
13. J. M. Appell, A.S. Kalitvin, and P.P. Zabrejko, *Partial Integral Operators and Integrodifferential Equations*, 230, Marcel and Dekker, Inc., New York, 2000.
14. D. Baleanu, K. Diethelm, E. Scalas and J.J. Trujillo, *Fractional Calculus Models and Numerical Methods*, World Scientific Publishing, New York, 2012.
15. M. Benchohra, J. Henderson and S.K. Ntouyas, *Impulsive Differential Equations and Inclusions*, Hindawi Publishing Corporation, 2, New York, 2006.
16. K.M. Case and P.F. Zweifel, *Linear Transport Theory*, Addison-Wesley, Reading, MA 1967.
17. S. Chandrasekher, *Radiative Transfer*, Dover Publications, New York, 1960.

18. M.A. Darwish and S.K. Ntouyas, *Semilinear functional differential equations of fractional order with state-dependent delay*, Electron. J. Differential Equations, 2009 (2009), pp. 1-10.
19. L. Debnath and D. Bhatta, *Integral Transforms and Their Applications(Second Edition)*, CRC Press, 2007.
20. K. Diethelm, *The Analysis of Fractional Differential Equations*, Springer, Berlin, 2010.
21. J.K. Hale and J. Kato, *Phase space for retarded equations with infinite delay*, Funk. Ekvacioj, 21 (1978), pp. 11-41.
22. E. Hernández and D. O'Regan, *On a new class of abstract impulsive differential equations*, Proc. Amer. Math. Soc., 141 (2013), pp. 1641-1649.
23. E. Hernández, A. Prokopczyk and L. Ladeira, *A note on partial functional differential equations with state-dependent delay*, Nonlinear Anal. RWA, 7 (2006), pp. 510-519.
24. R. Hilfer, *Applications of fractional calculus in physics*, Singapore, World Scientific, 2000.
25. Y. Hino, S. Murakami and T. Naito, *Functional Differential Equations with Unbounded Delay*, Springer-Verlag, Berlin, 1991.
26. V. Kavitha, P-Z. Wang and R. Murugesu, *Existence results for neutral functional fractional differential equations with state dependent-delay*, Malaya J. Math., 1 (2012), pp. 50-61.
27. V. Lakshmikantham, D.D. Bainov and P.S. Simeonov, *Theory of Impulsive Differential Equations*, World Scientific, NJ, 1989.
28. J. E. Lazreg, S. Abbas, M. Benchohra, and E. Karapinar, *Impulsive Caputo-Fabrizio fractional differential equations in b-metric spaces*, Open Math., 19 (2021), pp. 363-372.
29. P. Li and C.J. Xu , *Mild solution of fractional order differential equations with not instantaneous impulses*, Open Math., 13 (2015), pp. 436-443.
30. F. Mainardi, P. Paradisi and R. Gorenflo, *Probability distributions generated by fractional diffusion equations*, Dordrecht, The Netherlands, 2000.
31. M. Meghnafi, M. Benchohra and K. Aissani, *Impulsive fractional evolution equations with state-dependent delay*, Nonlinear Stud., 22 (2015), pp. 659-671.
32. D.N. Pandey, S. Das and N. Sukavanam, *Existence of solution for a second-order neutral differential equation with state dependent delay and non-instantaneous impulses*, Int. J. Nonlin. Sci., 18 (2014), pp. 145-155.
33. I. Podlubny, *Fractional Differential Equations*, Academic Press, San Diego, 1999.

34. A. Salim, M. Benchohra, J.R. Graef and J.E. Lazreg, *Boundary value problem for fractional generalized Hilfer-type fractional derivative with non-instantaneous impulses*, Fractal Fract., 5 (2021), pp. 1-21.
35. A. Salim, M. Benchohra, E. Karapinar and J.E. Lazreg, *Existence and Ulam stability for impulsive generalized Hilfer-type fractional differential equations*, Adv. Differ. Equ., 2020 (2020), pp. 1-21.
36. A. Salim, M. Benchohra, J.E. Lazreg and J. Henderson, *Nonlinear implicit generalized Hilfer-type fractional differential equations with non-instantaneous impulses in Banach spaces*, Advances in the Theory of Nonlinear Analysis and its Application, 4 (2020), pp. 332-348.
37. A. Salim, M. Benchohra, J.E. Lazreg and G. N'Guérékata, *Boundary value problem for nonlinear implicit generalized Hilfer-type fractional differential equations with impulses*, Abstr. Appl. Anal., 2021 (2021), pp. 1-17.
38. A. Salim, M. Benchohra, J.E. Lazreg, J.J. Nieto and Y. Zhou, *Nonlocal initial value problem for hybrid generalized Hilfer-type fractional implicit differential equations*, Nonauton. Dyn. Syst., 8 (2021), pp. 87-100.
39. S.G. Samko, A A. Kilbas and O.I. Marichev, *Fractional Integrals and Derivatives. Theory and Applications*, Gordon and Breach, Yverdon, 1993.
40. Y. Zhou, *Fractional Evolution Equations and Inclusions : Analysis and Control*, Elsevier Science, 2016.

[1] Laboratory of Mathematics, University of Sidi Bel-Abbès, PO Box 89, 22000, Sidi Bel-Abbès, Algeria.
Email address: benkhettou_na@yahoo.fr

[2] Laboratory of Mathematics, University of Sidi Bel-Abbès, PO Box 89, 22000, Sidi Bel-Abbès, Algeria. and Faculty of Technology, Hassiba Benbouali University, P.O. Box 151 Chlef 02000, Algeria.
Email address: salim.abdelkrim@yahoo.com, a.salim@univ-chlef.dz

[3] University of Bechar, PO Box 417, 08000, Bechar, Algeria.
Email address: aissani_k@yahoo.fr

[4] Laboratory of Mathematics, University of Sidi Bel-Abbès, PO Box 89, 22000, Sidi Bel-Abbès, Algeria.
Email address: benchohra@yahoo.com

[5] Division of Applied Mathematics, Thu Dau Mot University, Thu Dau Mot City 820000, Binh Duong Province, Vietnam and Department of Mathematics, Çankaya University, 06790, Etimesgut, Ankara, Turkey.
Email address: erdalkarapinar@tdmu.edu.vn