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$ps - ro$ β-Open (Closed) Fuzzy Sets and Related Fuzzy Function and Continuity

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Abstract. In this paper, the notion of $ps - ro$ β-open (closed) fuzzy sets on a fuzzy topological space(fts) has been introduced as a new tool to study fts, the properties of these sets are investigated and shown to be unrelated to the common concept of fuzzy β-open (closed) sets. Based on these fuzzy sets, $ps - ro$ fuzzy β-continuity and $ps - ro$ fuzzy β-open (closed) functions are proposed and they are also found to be different from the idea of fuzzy β-continuous and fuzzy β-open (closed) functions, respectively. Further, their characterizations and relationships with existing allied concepts are investigated.

1. Introduction and Preliminaries

After the generalization of the crisp set to the fuzzy set by L.A Zadeh [9], C.L Chang introduced and developed fuzzy topological space (in short, fts) [4]. Later, various concepts of fuzzy topology have been studied by different researchers.

The notion of fuzzy β-open (closed) [5] or fuzzy semi-preopen (closed) [7, 15] and related fuzzy function and fuzzy continuity were introduced and explored.

The introduction of *ps-ro* fuzzy topology in [1] initiated a new tool as well as new direction to study fts. Introducing *ps-ro* fuzzy semiopen (closed), *ps-ro* fuzzy preopen (closed) and related functions and continuity between two fts are explored in [12, 13]. Several fruitful researches

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are carried out related to different types of fuzzy continuous like functions in fuzzy topological spaces, such as, [6, 16], etc.

Now, we shall list some definitions and results that will be used subsequently:

Definition 1.1. On a nonempty set P, a fuzzy set U is a function from P into $[0,1] = I$. If g be a mapping between two sets P and Q, also U and V be fuzzy sets on P and Q, respectively, then $1-U$, $g(U)$ and $g^{-1}(V)$ are fuzzy sets respectively on P, Q and P and are given by $(1-U)(p) = 1-U(p)\ \forall\ p \in P$,

$$g(U)(t) = \begin{cases} \sup_{r\in g^{-1}(t)} U(r), & \text{if } g^{-1}(t) \neq \emptyset \\ 0, & \text{if otherwise.} \end{cases}$$

and $g^{-1}(V)(a) = V(g(a))\ \forall\ a \in P$. Any fuzzy sets U, V on P, U is a subset of V if $U(t) \leq V(t)\ \forall\ t \in P$ and is written as $U \leq V$ [9].

Definition 1.2. A fuzzy set x_r is called a fuzzy point whose value is $r \in (0,1]$ at x, otherwise, it is 0, also it is q-coincident to a fuzzy set U for $r + U(x) > 1$ and is denoted by $x_r qU$. A fuzzy set A is said to be quasi coincident (q-coincident, for short) with another fuzzy set B on P (written as AqB) if there exists $x \in P$ such that $A(x) + B(x) > 1$. If A and B are not q-coincident, we write $A \not{q} B$ [14].

Definition 1.3. A set B is called regular open if $int(clB) = B$, where B is a subset of a topological space [11].

Definition 1.4. A fts (U,σ) is a pair where σ is a collection of some fuzzy sets on U with $0,1 \in \sigma$ and finite intersection and arbitrary union of members of σ belongs to σ [4].

For a fts (U,σ), the collection $i_\alpha(\sigma) = \{P^\alpha : P \in \sigma : \alpha \in [0,1) = I_1\}$, where $P^\alpha = \{s \in U; P(s) > \alpha\}$, is a topology on U and named as strong α-level topology. P^α for being regular open in $(U, i_\alpha(\sigma))$, $\forall\ \alpha \in I_1$, the corresponding fuzzy open set P on fts (U,σ) is called pseudo regular open fuzzy set, the collection of which generates a fuzzy topology on U, named as $ps-ro$ fuzzy topology on U, the elements of which are termed as $ps-ro$ open fuzzy sets and as usual their complements are called $ps-ro$ closed fuzzy sets [1–3].

Definition 1.5. Fuzzy ps-interior of a fuzzy set P in the fts (U,σ), written as $ps-int(P)$ is the biggest $ps-ro$ open fuzzy set on U that is a subset of P and fuzzy ps-closure written as $ps-cl(P)$ is the tiniest $ps-ro$ closed fuzzy set that contains P [2, 3].

Definition 1.6. A fuzzy set P on a fts (U,σ) is known as:

(a) fuzzy regular open [8] if $P = int(cl(P))$.

(b) fuzzy β-open if $P \leq cl(int(cl(P)))$ and fuzzy β-closed [10] if $P \geq int(cl(int(P))$.
(c) $ps - ro$ semiopen if $P \leq ps - cl(ps - int(P))$ and $ps - ro$ semiclosed [12] if $P \geq ps - int(ps - cl(P))$.
(d) $ps-ro$ preopen if $P \leq ps-int(ps-cl(P))$ and $ps-ro$ preclosed [13] if $P \geq ps - cl(ps - int(P))$.

Let us use the symbols (o, U_σ), (β, U_σ), $ps - (r, U_\sigma)$, $ps - (o, U_\sigma)$, $ps - (s, U_\sigma)$ and $ps - (p, U_\sigma)$, respectively to denote the set of all fuzzy open, fuzzy β-open, pseudo regular open, $ps-ro$ open, $ps-ro$ semiopen and $ps - ro$ preopen fuzzy sets on a fts (U, σ) and also by (o^c, U_σ) (β^c, U_σ), $ps - (r^c, U_\sigma)$, $ps - (o^c, U_\sigma)$, $ps - (s^c, U_\sigma)$ and $ps - (p^c, U_\sigma)$, respectively for the set of all fuzzy closed, fuzzy β-closed, pseudo regular closed, $ps - ro$ closed, $ps - ro$ semiclosed and $ps - ro$ preclosed fuzzy sets on the fts (U, σ).

Definition 1.7. A function f between two fts (U, σ_1) and (V, σ_2) is called

(a) fuzzy β-continuous ([10]) if $f^{-1}(Q) \in (\beta, U_{\sigma_1})$ for any $Q \in (o, V_{\sigma_2})$.
(b) $ps - ro$ fuzzy continuous ([3]) if $f^{-1}(Q) \in ps - (\beta, U_{\sigma_1})$ for any $Q \in ps - (o, V_{\sigma_2})$.
(c) $ps-ro$ fuzzy semicontinuous ([12]) if $f^{-1}(Q) \in ps-(s, U_{\sigma_1})$ for any $Q \in ps - (o, V_{\sigma_2})$.
(d) $ps - ro$ fuzzy precontinuous ([13]) if $f^{-1}(Q) \in ps - (p, U_{\sigma_1})$ for any $Q \in ps - (o, V_{\sigma_2})$.
(e) $ps - ro$ fuzzy semiopen ([12]) if $f(Q) \in ps - (s, V_{\sigma_2})$ for any $Q \in ps - (o, U_{\sigma_1})$.
(f) $ps - ro$ fuzzy preopen ([13]) if $f(Q) \in ps - (p, V_{\sigma_2})$ for any $Q \in ps - (o, U_{\sigma_1})$.

2. $ps - ro$ β-Open (Closed) Fuzzy Sets

In this section, we introduce the notion of $ps-ro$ β-open (closed) fuzzy sets on a fuzzy topological space(fts) and study their several properties.

Definition 2.1. On a fts (U, μ), a fuzzy set P is called $ps - ro$ β-open (resp. closed) fuzzy set if $P \leq ps - cl(ps - int(ps - cl(P)))$ (resp. $P \geq ps - int(ps - cl(ps - int(P)))$.

Remark 2.2. It follows directly that a $ps - ro$ semiopen (closed) is $ps - ro$ β-open (closed) but not the converse is shown below.

Example 2.3. Let $U = \{e, b, s, d\}$ and P, Q and R be fuzzy sets on U which given by $P(r) = 0.2\ \forall\ r \in U$; $Q(e) = 0.6, Q(b) = 0.6, Q(s) = 0.5, Q(d) = 0.6$ and $R(r) = 0.7\ \forall\ r \in U$. Then, (U, μ) is a fts where

$\mu = \{1, P, Q, R, 0\}$. The open sets in the topological space $(U, i_\alpha(\mu)), \forall \alpha \in [0, 1)$, are $\phi, U, P^\alpha, Q^\alpha$ and R^α where

$$P^\alpha = \begin{cases} U, & \text{if } \alpha < 0.2, \\ \phi, & \text{if } \alpha \geq 0.2, \end{cases}$$

$$R^\alpha = \begin{cases} U, & \text{if } \alpha < 0.7, \\ \phi, & \text{if } \alpha \geq 0.7, \end{cases}$$

and

$$Q^\alpha = \begin{cases} U, & \text{if } \alpha < 0.5, \\ \{e, b, d\}, & \text{if } 0.5 \leq \alpha < 0.6, \\ \phi, & \text{if } \alpha \geq 0.6. \end{cases}$$

Here, $int(cl(P^\alpha)) = P^\alpha$ and $int(cl(R^\alpha)) = R^\alpha$, $\forall\ \alpha \in I_1$. For $\alpha \in [0.5, 0.6)$, $int(cl(Q^\alpha)) = U$. So, P^α and R^α are regular open on $(U, i_\alpha(\mu))$, $\forall \alpha \in I_1$ and Q^α is not regular open on $(U, i_\alpha(\mu))$ for $\alpha \in [0.5, 0.6)$. Now, for $\alpha \in [0.5, 0.6)$, $Q \notin ps - (r, U_\mu)$ and hence, $\{1, P, R, 0\}$ is $ps - ro$ fuzzy topology on U. Let S be a fuzzy set on U given by $S(r) = 0.4\ \forall\ r \in U$, then, $S \leq ps - cl(ps - int(ps - cl(S)))$ and $1 - S \geq ps - int(ps - cl(ps - int(1 - S)))$. Thus, $S \in ps - (\beta, U_\mu)$ and $1 - S \in ps - (\beta^c, U_\mu)$, on the other hand, $S \geq ps - cl(ps - int(S))$ and $1 - S \leq ps - int(ps - cl(1 - S))$. This results in $S \notin ps - (s, U_\mu)$ and $1 - S \notin ps - (s^c, U_\mu)$.

Remark 2.4. $ps - ro$ preopen (closed) fuzzy set implies $ps - ro$ β-open (closed) fuzzy set but the converse is proved below.

Example 2.5. Let fuzzy sets P, Q and R on $U = \{e, b, s, d\}$ given as $P(r) = 0.4, Q(r) = 0.2\ \forall\ r \in\ U$, $R(e) = 0.7, R(b) = 0.7, R(s) = 0.7$ and $R(d) = 0.8$. Then (U, μ) is a fts where $\mu = \{1, P, Q, R, 0\}$. In the topological space $(U, i_\alpha(\mu)), \forall\ \alpha \in I_1$, the open sets are $\phi, U, P^\alpha, Q^\alpha$ and R^α where

$$P^\alpha = \begin{cases} U, & \text{if } \alpha < 0.4, \\ \phi, & \text{if } \alpha \geq 0.4, \end{cases}$$

$$Q^\alpha = \begin{cases} U, & \text{if } \alpha < 0.2, \\ \phi, & \text{if } \alpha \geq 0.2, \end{cases}$$

and

$$R^\alpha = \begin{cases} U, & \text{if } \alpha < 0.7, \\ \{d\}, & \text{if } 0.7 \leq \alpha < 0.8, \\ \phi, & \text{if } \alpha \geq 0.8. \end{cases}$$

Here, $R \notin ps - (r, U_\mu)$ for $\alpha \in [0.7, 0.8)$. So, on U, $\{1, P, Q, 0\}$ is the $ps - ro$ fuzzy topology. We define a fuzzy set V by $V(r) = 0.5\ \forall\ r \in\ U$

so that $V \in ps-(\beta, U_\mu)$, but $V \notin ps-(p, U_\mu)$, also, $1-V \in ps-(\beta^c, U_\mu)$ and $V \notin ps-(p^c, U_\mu)$.

We will now establish that fuzzy β-open (closed) and $ps-ro$ β-open (closed) fuzzy sets are independent notions.

Remark 2.6. In Example 2.3, $S \in ps-(\beta, U_\mu)$ and $1-S \in ps-(\beta^c, U_\mu)$, meanwhile $S \geq cl(int(cl(S)))$ and $(1-S) \leq int(cl(int(1-S)))$, thus, $S \notin ps-(\beta, U_\mu)$ and $1-S \notin ps-(\beta^c, U_\mu)$. According to Example 2.5, we take a fuzzy set F on U as $F(r) = 0.7\ \forall\ r \in U$, $F \geq ps-cl(ps-int(ps-cl(F)))$ and $(1-F) \leq ps-int(ps-cl(ps-int(1-F)))$. Thus, $F \notin ps-(\beta, U_\mu)$ and $1-F \notin ps-(\beta^c, U_\mu)$, on the other hand, $F \leq cl(int(cl(F)))$ and $(1-F) \geq int(cl(int(1-F)))$. Thus, $F \in (\beta, U_\mu)$ and $1-F \in (\beta^c, U_\mu)$. So, $ps-ro$ β-open (closed) and fuzzy β-open (closed) sets are two independent concepts.

The above relations can be put as a schematic diagram given below:

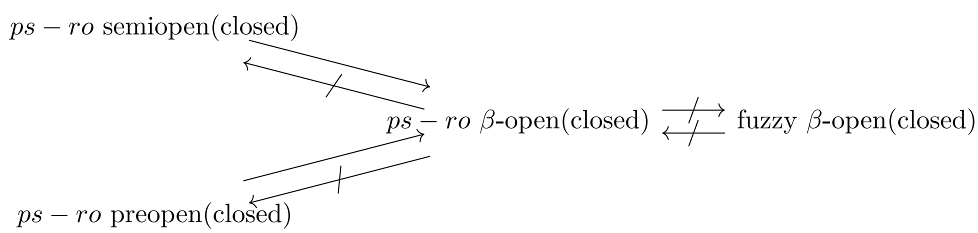

FIGURE 1. Relationships among various fuzzy sets

Remark 2.7. In [13], it is shown that every $ps-ro$ open fuzzy set is a $ps-ro$ semiopen fuzzy set , but the converse is not true. Hence, every $ps-ro$ open fuzzy set is a $ps-ro$ β-open fuzzy set, but the converse is not true.

Theorem 2.8. *In a fts (U, μ), $P \in ps-(\beta, U_\mu)$ if and only if $(1-P) \in ps-(\beta^c, U_\mu)$.*

Proof. Let $P \in ps-(\beta, U_\mu)$. Then, $P \leq ps-cl(ps-int(ps-cl(P)))$ and $(1-P) \geq ps-int(1-ps-int(ps-cl(P))) = ps-int(ps-cl(1-ps-cl(P)))$. So, $(1-P) \in ps-(\beta^c, U_\mu)$. Similarly, the converse follows. ☐

Theorem 2.9. *Let (U, μ) be a fts and P be a fuzzy set on U. Then,*

(a) *$P \in ps-(\beta, U_\mu)$ if $\exists\ Q \in ps-(p, U_\mu)$ such that $Q \leq P \leq ps-cl(Q)$.*
(b) *$P \in ps-(\beta^c, U_\mu)$ if $\exists\ Q \in ps-(p^c, U_\mu)$ such that $ps-int(Q) \leq P \leq Q$.*
(c) *$P \in ps-(\beta, U_\mu)$ if $\exists$ a $Q \in ps-(p, U_\mu)$ such that $Q \leq P \leq ps-cl(ps-int(Q))$.*

(d) *$P \in ps-(\beta^c, U_\mu)$ if $\exists$ a $Q \in ps-(p^c, U_\mu)$ such that $ps-int(ps-cl(Q)) \leq P \leq Q$.*

Proof. (a) Let $Q \in ps-(p, U_\mu)$ and $Q \leq P \leq ps-cl(Q)$. So, $Q \leq ps-int(ps-cl(Q))$. Consequently, $P \leq ps-cl(Q) \leq ps-cl(ps-int(ps-cl(Q)) \leq ps-cl(ps-int(ps-cl(P))$. Thus, $P \in ps-(\beta, U_\mu)$.

(b) It will be obtained similarly to (a) by taking the complement.

(c) Let $Q \in ps-(p, U_\mu)$ with $Q \leq P \leq ps-cl(ps-int(Q))$, So, $Q \leq ps-int(ps-cl(Q))$. Now, $P \leq ps-cl(ps-int(Q)) \leq ps-cl(ps-int(ps-int(ps-cl(Q))) = ps-cl(ps-int(ps-cl(Q))) \leq ps-cl(ps-int(ps-cl(P)))$. So, $P \in ps-(\beta, U_\mu)$.

(d) It will be obtained similarly to (c) by taking the complement. □

Theorem 2.10. *Let (U, μ) be a fts. Then the following holds:*

(a) *An arbitrary union of $ps-ro$ β-open fuzzy sets on U is a $ps-ro$ β-open fuzzy set on U.*

(b) *An arbitrary intersection of $ps-ro$ β-closed fuzzy sets on U is a $ps-ro$ β-closed fuzzy set on U.*

Proof. (a) We consider a collection $\{P_\alpha\}$ of $ps-ro$ β-open fuzzy sets on U. By Theorem 2.9, for each α, $\exists$ $Q_\alpha \in ps-(p, U_\mu)$ such that $Q_\alpha \leq P_\alpha \leq ps-cl(Q_\alpha)$. So, $\vee(Q_\alpha) \leq \vee(P_\alpha) \leq \vee(ps-cl(Q_\alpha)) \leq ps-cl(\vee Q_\alpha)$. This gives, $\vee(Q_\alpha) \leq \vee(P_\alpha) \leq ps-cl(\vee Q_\alpha)$. Thus, $\vee(P_\alpha) \in ps-(\beta, U_\mu)$ as $\vee(Q_\alpha) \in ps-(p, U_\mu)$.

(b) It will be obtained similarly to (a) by taking the complement. □

Theorem 2.11. *The product $P \times Q$ of $ps-ro$ β-open fuzzy sets P in U and Q in V is also $ps-ro$ β-open fuzzy set on the product space $U \times V$, where (U, μ_1) and (V, μ_2) are two fts.*

Proof. Since $P \in ps-(\beta, U_{\mu_1})$ and $Q \in ps-(\beta, V_{\mu_2})$, the other hand $P \leq ps-cl(ps-int(ps-cl(P)))$ and $Q \leq ps-cl(ps-int(ps-cl(Q)))$. Using Theorem 3.10 of [8], $ps-int(P \times Q) = ps-int(P) \times ps-int(Q)$ and $ps-cl(P \times Q) = ps-cl(P) \times ps-cl(Q)$. Thus, $P \times Q \leq ps-cl(ps-int(ps-cl(P))) \times ps-cl(ps-int(ps-cl(Q))) = ps-cl(ps-int(ps-cl(P \times Q)))$. Therefore, $P \times Q$ is also $ps-ro$ β-open fuzzy set on the product space $U \times V$. □

Theorem 2.12. *Let (U, μ) be a fts. If $P \in ps-(\beta, U_\mu)$ and $P \in ps-(s^c, U_\mu)$, then $P \in ps-(s, U_\mu)$.*

Proof. Since $P \in ps-(\beta, U_\mu)$, $P \in ps-(s^c, U_\mu)$, $P \leq ps-cl(ps-int(ps-cl(P)))$ and $P \geq ps-int(ps-cl(P))$. Thus, $ps-int(P) \geq$

$ps-int(ps-int(ps-cl(P)))=ps-int(ps-cl(P))$. So, $P \leq ps-cl(ps-int(ps-cl(P))) \leq ps-cl(ps-int(P))$. Therefore, $P \in ps-(s,U_\mu)$. □

Theorem 2.13. *Let (U,μ) be a fts. If $P \in ps-(\beta^c,U_\mu)$ and $P \in ps-(s,U_\mu)$, then $P \in ps-(s^c,U_\mu)$.*

Theorem 2.14. *$Q \in ps-(\beta,U_\mu)$ if and only if for every $x_\alpha \in Q$, $\exists$ $P \in ps-(\beta,U_\mu)$ such that $x_\alpha \in P \leq Q$.*

Proof. First, let $Q \in ps-(\beta,U_\mu)$. If $P=Q$ for every $x_\alpha \in Q$, then the result is obtained. Conversely, let for every $x_\alpha \in Q$, $\exists$ $P \in ps-(\beta,U_\mu)$ such that $x_\alpha \in P \leq Q$. Since, $x_\alpha \in P \leq Q$, by taking union over such x_α, we have, $Q \leq \vee_{x_\alpha \in Q} P \leq Q$. This gives $Q = \vee_{x_\alpha \in Q} P$. Thus, by Theorem 2.10, we conclude $Q \in ps-(\beta,U_\mu)$. □

Definition 2.15. Corresponding to a fuzzy set P on a fts (U,μ), fuzzy $ps-\beta$ closure of P (denoted by $ps-\beta\ cl(P)$) is the tiniest $ps-ro$ β-closed fuzzy set on U that contains P and fuzzy $ps-\beta$ interior of P (denoted by $ps-\beta\ int(P)$) is the biggest $ps-ro$ β-open fuzzy set on U contained on P.

Theorem 2.16. *Let A and P be two fuzzy sets on a fts (U,μ). Then,*

(a) *$ps-\beta cl(P) \leq ps-cl(P)$ and $ps-int(P) \leq ps-\beta int(P)$.*
(b) *$ps-\beta cl(P) \in ps-(\beta^c,U_\mu)$ and $ps-\beta int(P) \in ps-(\beta,U_\mu)$.*
(c) *$P \in ps-(\beta^c,U_\mu)$ if only if $P = ps-\beta cl(P)$ and $P \in ps-(\beta,U_\mu)$.*
(d) *If $A \leq P$, then $ps-\beta cl(A) \leq ps-\beta cl(P)$ and $ps-\beta int(A) \leq ps-\beta int(P)$.*
(e) *$ps-\beta cl(1-P) = 1-(ps-\beta int(P))$ and $ps-\beta int(1-P) = 1-(ps-\beta cl(P))$.*

Proof. The proofs are straightforward, hence omitted. □

Definition 2.17. On a fts (U,μ), a fuzzy set P is called

(a) $ps-ro$ fuzzy β-nbd of x_α, if $\exists$ $Q \in ps-(\beta,U_\mu)$ with $x_\alpha \in Q \leq P$.
(b) $ps-ro$ fuzzy $\beta-q-nbd$ of a fuzzy point x_α, if $\exists$ $Q \in ps-(\beta,U_\mu)$ with $x_\alpha q Q \leq P$.

Theorem 2.18. *Let P be a fuzzy set on a fts (U,μ) and let $x_\alpha \in U$. Then $x_\alpha \in ps-\beta cl(P)$ if only if for every $ps-ro$ fuzzy $\beta-q-nbd$ Q of x_α, QqP.*

Proof. First, let $\exists$ a $ps-ro$ fuzzy $\beta-q-nbd$ Q of x_α such that $Q\ /qP$. Then, $\exists$ $Q_1 \in ps-(\beta,U_\mu)$ with $x_\alpha q Q_1 \leq Q$ and $Q_1 \not{q} P$. Now, $(1-Q_1) \in ps-(\beta^c,U_\mu)$ containing P, $ps-\beta cl(P) \leq (1-Q_1)$. Since $x_\alpha \notin (1-Q_1)$, therefore $x_\alpha \notin ps-\beta cl(P)$, which is a contradiction. Thus, for every $ps-ro$ fuzzy $\beta-q-nbd$ Q of x_α, QqP.

Conversely, let, $x_\alpha \notin ps-\beta cl(P)$. Then $\exists\ Q \in ps-(\beta^c, U_\mu)$ with $x_\alpha \notin Q$ and $P \leq Q$. Thus, $(1-Q) \in ps-(\beta, U_\mu)$ such that $x_\alpha q(1-Q)$ and $P \not{q}(1-Q)$, which is a contradiction. Hence, $x_\alpha \in ps-\beta cl(P)$. □

Theorem 2.19. *In a fts (U,μ), let P be a fuzzy set and $Q \in ps-(\beta, U_\mu)$. If $P \not{q} Q$, then $ps-\beta cl(P) \not{q} Q$.*

Proof. Let, $ps-\beta cl(P) q Q$. Then $\exists$ an $x \in U$ with $ps-\beta cl(P)(x)+Q(x) > 1$. We put $ps-\beta-cl(P)(x)=\alpha$, then $\alpha + Q(x) > 1$. So, Q is $ps-ro$ fuzzy $\beta-q-nbd$ Q of x_α with $P \not{q} Q$. Thus, $x_\alpha \notin ps-\beta cl(P)$, which is a contradiction.. Hence, $ps-\beta cl(P) \not{q} Q$. □

3. $ps-ro$ Fuzzy β-Continuous Function

Here, we introduce a fuzzy continuity-like concept called $ps-ro$ fuzzy β-continuity and its characteristic.

Definition 3.1. A mapping g between fts (U,μ_1) and (V,μ_2) is called $ps-ro$ fuzzy β-continuous if $g^{-1}(P) \in ps-(\beta, U_{\mu_1})$ for every $P \in ps-(o, V_{\mu_2})$.

Obviously, a $ps-ro$ fuzzy semicontinuous is $ps-ro$ fuzzy β-continuous, but not the converse is given below.

Example 3.2. Consider $U=\{e,b,s,d\}$, $V=\{i,m,z,w\}$ and P, Q, R be the fuzzy sets on U that given as $P(r)=0.1$ and $Q(r)=0.7\ \forall\ r \in U$; and $R(e)=0.6, R(b)=0.6, R(s)=0.5$ and $R(d)=0.5$. Then, (U,μ_1) is a fts where $U=\{0,1,P,Q,R\}$. In the corresponding strong α-level topological space $(U, i_\alpha(\mu_1))\ \forall\ \alpha \in [0,1)$, the open sets are U, ϕ P^α, Q^α and R^α where P^α and Q^α are regular open on $(U, i_\alpha(\mu_1))\ \forall\ \alpha \in [0,1)$, but R^α is not regular open for $0.5 \leq \alpha < 0.6$. Hence, $\{1,P,Q,0\}$ is $ps-ro$ fuzzy topology on U.

We define fuzzy sets D, E and F on V given by $D(r)=0.8$ and $E(r)=0.4\ \forall\ r \in V$, $F(i)=0.5$,$F(m)=0.6$, $F(z)=0.5$ and $F(w)=0.6$. Then, $\mu_2=\{1,D,E,F,0\}$ is fuzzy topology on V. In the topological space $(V, i_\alpha(\mu_2))\ \forall\ \alpha \in [0,1)$, the open sets are U, ϕ, D^α, E^α and F^α where D^α and E^α are regular open on $(V, i_\alpha(\mu_2))\ \forall\ \alpha \in [0,1)$, but F^α is not regular open for $0.5 \leq \alpha < 0.6$. Hence, $\{0,1,D,E\}$ is $ps-ro$ fuzzy topology on V. Let g be a function between fts (U,μ_1) and (V,μ_2) by $g(e)=i$, $g(b)=m$, $g(s)=z$ and $g(d)=w$. Here, $D, E \in ps-(o, V_{\mu_2})$. Now, $f^{-1}(D)(r)=0.8\ \forall\ r \in U$ and $g^{-1}(D) \leq ps-cl(ps-int(ps-cl(g^{-1}(D))))$. Again, $g^{-1}(E)(r)=0.4\ \forall\ r \in U$ and $g^{-1}(E) \leq ps-cl(ps-int(ps-cl(g^{-1}(E))))$. Thus, $g^{-1}(D), g^{-1}(E) \in ps-(\beta, U_{\mu_1})$. Also $g^{-1}(0), f^{-1}(1) \in ps-(\beta, U_{\mu_1})$. So, g is $ps-ro$ fuzzy β-continuous. But, $g^{-1}(E) \geq ps-cl(ps-int(g^{-1}(E)))$, thus $g^{-1}(E) \notin ps-(s, U_{\mu_1})$ whereas $E \in ps-(o, V_{\mu_2})$, that shows g is not $ps-ro$ fuzzy semicontinuous.

Remark 3.3. Evidently, $ps-ro$ fuzzy precontinuity implies $ps-ro$ fuzzy β-continuity, but the converse is not true. In Example 3.2 g is $ps-ro$ fuzzy β-continuous, but $f^{-1}(S) \geq ps-int(ps-cl(f^{-1}(S)))$, thus $g^{-1}(S) \notin ps-(p, U_{\mu_1})$ for $S \in ps-(o, V_{\mu_2})$. So, g is not $ps-ro$ fuzzy precontinuous.

Now we shall establish that *ps-ro* fuzzy β-continuity and fuzzy β-continuity do not imply each other.

Example 3.4. Let $U = \{e, b, s, d\}$, $V = \{i, m, z, w\}$ and P, Q, R be the fuzzy sets on U so that $P(e) = 0.3$, $P(b) = 0.3$, $P(s) = 0.2$, $P(d) = 0.2$; $Q(r) = 0.7 \ \forall \ r \in U$, $R(e) = 0.7$, $R(b) = 0.6$, $R(s) = 0.7$ and $R(d) = 0.7$. Let S, E, F be the fuzzy sets on V so that $S(r) = 0.2$, $E(r) = 0.4 \ \forall \ r \in V$, $F(i) = 0.5$, $F(m) = 0.5$, $F(z) = 0.6$ and $F(w) = 0.6$. Then, (U, μ_1) and (V, μ_2) are two fts where $\mu_1 = \{0, 1, P, Q, R\}$ and $\mu_2 = \{0, 1, S, E, F\}$. Here, $P \notin ps-(r, U_{\mu_1})$ and $R \notin ps-(r, V_{\mu_2})$, respectively for $\alpha \in [0.2, 0.3)$ and $\alpha \in [0.6, 0.7)$. So, On U, $\{1, Q, 0\}$ is the $ps-ro$ fuzzy topology. Also, $F \notin ps-(r, V_{\mu_2})$ for $\alpha \in [0.5, 0.6)$. So, the $ps-ro$ fuzzy topology on V is $\{1, S, E, 0\}$.

Let g be a function from (U, μ_1) to (V, μ_2) by $g(e) = i$, $g(b) = m$, $g(s) = z$ and $g(d) = w$. Here, $S, E \in ps-(o, V_{\mu_2})$ and $S, E, F \in (o, V_{\mu_2})$. It can be verified that the pre-images of any fuzzy open set on V is a fuzzy β-open set on U, thus g is fuzzy β-continuous.

Next, we will demonstrate that $ps-ro$ fuzzy β-continuity and fuzzy β-continuity do not imply each other.

Remark 3.5. In Example 3.2, $g^{-1}(E)(r) = 0.4 \ \forall \ r \in U$ and $g^{-1}(E) \geq cl(int(cl(g^{-1}(E))))$, therefore $g^{-1}(E) \notin (o, U_{\mu_1})$ and $E \in (o, V_{\mu_2})$. So, g is $ps-ro$ fuzzy β-continuous, but fails to be fuzzy β-continuous. In Example 3.4, g is fuzzy β-continuous, but fails to be $ps-ro$ fuzzy β-continuous. Hence, $ps-ro$ fuzzy β-continuity and fuzzy β-continuity are independent notions.

The above relations may be shown in the form of a schematic diagram as follows:

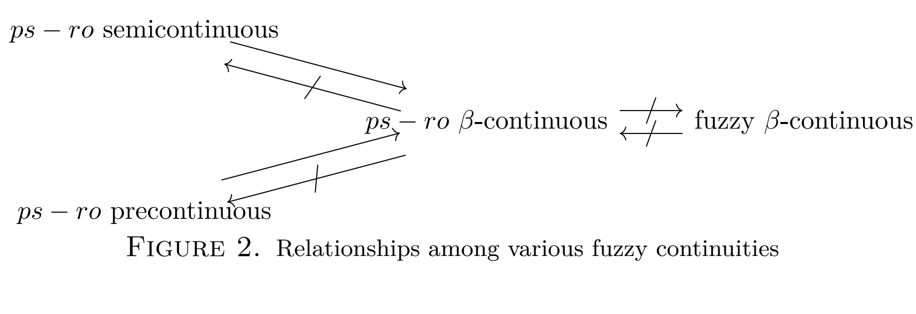

FIGURE 2. Relationships among various fuzzy continuities

Remark 3.6. In [13], it is shown that every $ps-ro$ fuzzy continuous function is $ps-ro$ semicontinuous , but the converse is not true. Hence, every $ps-ro$ fuzzy continuous function is $ps-ro$ fuzzy β-continuous , but the converse is not true.

Theorem 3.7. *For a function g between fts (U,μ_1) and (V,μ_2), the following are equivalent.*

(a) *g is $ps-ro$ fuzzy β-continuous.*
(b) *$g^{-1}(E) \in ps-(\beta^c, U_{\mu_1})\ \forall\ B \in ps-(o^c, V_{\mu_2})$.*
(c) *For $x_\alpha \in U$ and any $Q \in ps-(o, V_{\mu_2})$ such that $g(x_\alpha) \in Q$, $\exists$ $P \in ps-(\beta, U_{\mu_1})$ such that $x_\alpha \in P$ and $g(P) \leq Q$.*
(d) *For $x_\alpha \in U$, the preimage of every $ps-ro$ fuzzy $\beta-nbd$ Q of $g(x_\alpha)$ is a $ps-ro$ fuzzy $\beta-nbd$ of x_α.*
(e) *For $x_\alpha \in U$ and every $ps-ro$ fuzzy $\beta-nbd$ Q of $g(x_\alpha)$, $\exists$ a $ps-ro$ fuzzy $\beta-nbd$ R of x_α such that $g(R) \leq Q$.*
(f) *For $x_\alpha \in U$ and $Q \in ps-(o, V_{\mu_2})$ such that $g(x_\alpha)qQ$, $\exists$ $P \in ps-(\beta, U_{\mu_1})$ satisfying $x_\alpha qP$ and $g(P) \leq Q$.*
(g) *For $x_\alpha \in U$, the preimage of every $ps-ro$ fuzzy $\beta-q-nbd$ Q of $g(x_\alpha)$ is a $ps-ro$ fuzzy $\beta-q-nbd$ of x_α.*
(h) *For $x_\alpha \in U$ and every $ps-ro$ fuzzy $\beta-q-nbd$ Q of $g(x_\alpha)$, $\exists$ a $ps-ro$ fuzzy $\beta-q-nbd$ R of x_α such that $g(R) \leq Q$.*
(i) *$g(ps-\beta cl(P)) \leq ps-cl(g(P))$, for every fuzzy set P on U.*
(j) *$ps-\beta cl(g^{-1}(Q)) \leq g^{-1}(ps-cl(Q))$, $\forall$ fuzzy set Q on V.*

Proof. $(a) \Rightarrow (b)$: Let $(1-Q) \in ps-(o^c, V_{\mu_2 1})$. As g is $ps-ro$ fuzzy β-continuous and $g^{-1}(Q) \in ps-(\beta, U_{\mu_1})$ which implies that $1-g^{-1}(Q) = g^{-1}(1-Q) \in ps-(\beta^c, U_{\mu_1})$. So,

$$g^{-1}(E) \in ps-(\beta^c, U_{\mu_1}), \quad \forall B \in ps-(o^c, V_{\mu_2}).$$

$(a) \Rightarrow (c)$: Let $x_\alpha \in U$ and $Q \in ps-(o, V_{\mu_2})$ such that $g(x_\alpha) \in Q$. Let $P = g^{-1}(Q)$. Then, $P \in ps-(\beta, U_{\mu_1})$ with $x_\alpha \in P$ and $g(P) \leq Q$.

$(c) \Rightarrow (a)$: Let $Q \in ps-(\beta, V_{\mu_2})$ and $x_\alpha \in g^{-1}(Q)$. So, $g(x_\alpha) \in Q$. Then, $\exists$ $P \in ps-(\beta, U_{\mu_1})$ with $x_\alpha \in P$ and $g(P) \leq Q$. Thus, $x_\alpha \in P \leq g^{-1}(Q)$. By Theorem 2.14, $g^{-1}(Q) \in ps-(\beta, U_{\mu_1})$. Thus, g is fuzzy β-continuous.

$(a) \Rightarrow (d)$: Let $x_\alpha \in U$ and Q be a $ps-ro$ fuzzy $\beta-nbd$ of $g(x_\alpha)$. Then, $\exists$ $R \in ps-(\beta, V_{\mu_2})$ such that $g(x_\alpha) \in R \leq Q$. Now, $g^{-1}(Q) \in ps-(\beta, U_{\mu_1})$ and $x_\alpha \in g^{-1}(R) \leq g^{-1}(Q)$. Thus, $g^{-1}(Q)$ is a $ps-ro$ fuzzy $\beta-nbd$ of x_α.

$(d) \Rightarrow (e)$: Let $x_\alpha \in U$ and Q be a $ps-ro$ fuzzy $\beta-nbd$ of $g(x_\alpha)$. Then $R = g^{-1}(Q)$ is a $ps-ro$ fuzzy $\beta-nbd$ of x_α and

$$g(R) = g\left(g^{-1}(Q)\right) \leq Q.$$

$(e) \Rightarrow (c)$: Let $x_\alpha \in U$ and $Q \in ps-(\beta, U_{\mu_1})$ such that $g(x_\alpha) \in Q$. Then, Q is a $ps-ro$ fuzzy $\beta - nbd$ of $g(x_\alpha)$. So, $\exists$ a $ps-ro$ fuzzy $\beta - nbd$ R of x_α in U such that $x_\alpha \in R$ and $g(R) \leq Q$. Therefore, $\exists$ $P \in ps-(\beta, U_{\mu_1})$ such that $x_\alpha \in P \leq R$ and $g(P) \leq g(R) \leq Q$.

$(a) \Rightarrow (f)$: Let $x_\alpha \in U$ and $Q \in ps-(o, V_{\mu_2})$ with $g(x_\alpha)qQ$. Also, let $P = g^{-1}(Q)$. Then, $P \in ps-(\beta, U_{\mu_1})$ with $x_\alpha qP$ and

$$g(P) = g\left((g)^{-1}(Q)\right) \leq Q.$$

$(f) \Rightarrow (a)$: Let $Q \in ps-(\beta, V_{\mu_2})$ and $x_\alpha \in g^{-1}(Q)$. So, $g(x_\alpha) \in Q$. We take a fuzzy point $(1-x_\alpha)$, where $(1-x_\alpha)(x) = 1-x_\alpha(x)$. So, $g(x_\alpha)qQ$. Then by (f), $\exists$ $P \in ps-(\beta, U_{\mu_1})$ with $(1-x_\alpha)qP$ and $g(P) \leq Q$. Now, $(1-x_\alpha)qP$ and $(1-x_\alpha)(x)+P(x) = 1-\alpha+P(x) > 1$. So, $P(x) > \alpha$. i.e. $x_\alpha \in P$. Thus, $x_\alpha \in P \leq f^{-1}(Q)$. By Theorem 2.14, $g^{-1}(Q)$ is a $ps-ro$ β-open fuzzy set on U.

$(f) \Rightarrow (g)$: Let $x_\alpha \in U$ and Q be a $ps-ro$ fuzzy $\beta - nbd$ of $g(x_\alpha)$. Then $\exists$ $Q_1 \in ps-(\beta, V_{\mu_2})$ with $g(x_\alpha)qQ_1 \leq Q$. Therefore $\exists$ $P \in ps-(\beta, U_{\mu_1})$ such that $x_\alpha qP$ and $g(P) \leq Q$. So, $x_\alpha qP \leq g^{-1}(Q_1) \leq g^{-1}(Q)$. Thus, $g^{-1}(Q)$ is a $ps-ro$ fuzzy $\beta-q-nbd$ of x_α.

$(g) \Rightarrow (h)$: Let $x_\alpha \in U$ and Q be a $ps-ro$ fuzzy $\beta-q-nbd$ of $g(x_\alpha)$. Then $R = g^{-1}(Q)$ is a $ps-ro$ fuzzy $\beta-q-nbd$ of x_α and

$$g(R) = g\left(g^{-1}(Q)\right) \leq Q.$$

$(h) \Rightarrow (f)$: Let $x_\alpha \in U$ and $Q \in ps-(\beta, V_{\mu_2})$ such that $g(x_\alpha)qQ$. Then Q be a $ps-ro$ fuzzy $\beta-q-nbd$ of $g(x_\alpha)$. So $\exists$ a $ps-ro$ fuzzy $\beta-q-nbd$ R of x_α such that $g(R) \leq Q$. Now, since R is a $ps-ro$ fuzzy $\beta-q-nbd$ of x_α, so $\exists$ $P \in ps-(\beta, U_{\mu_1})$ with $x_\alpha qP \leq R$. Thus, $x_\alpha qP$ and $g(P) \leq g(R) \leq Q$.

$(b) \Rightarrow (i)$: Let P be a fuzzy set on U. Then $ps-cl(g(P)) \in ps-(o^c, V_{\mu_2})$ and $g^{-1}(ps-cl(g(P))) \in ps-(\beta^c, U_{\mu_1})$ so $g^{-1}(ps-cl(g(P))) = ps-\beta cl(g^{-1}(ps-cl(g(P))))$. Since, $P \leq g^{-1}(g(P))$, we have,

$$\begin{aligned} ps-\beta cl(P) &\leq ps-\beta cl(g^{-1}(g(P))) \\ &\leq ps-\beta cl\left(g^{-1}\left(ps-cl\left(g(P)\right)\right)\right) \\ &= g^{-1}\left(ps-cl\left(g(P)\right)\right). \end{aligned}$$

Thus, $g(ps-\beta cl(P)) \leq ps-cl(g(P))$.

$(i) \Rightarrow (j)$: Let Q be a fuzzy set on V. Then, $g(ps-\beta cl(g^{-1}(Q))) \leq ps-cl(g(g^{-1}(Q)))$. So, $ps-\beta cl(g^{-1}(Q)) \leq g^{-1}(ps-cl(Q))$.

$(j) \Rightarrow (a)$: Let $Q \in ps-(o, V_{\mu_2})$. Then, $ps-\beta cl(g^{-1}(1-Q)) \leq g^{-1}(ps-cl(1-Q)) = g^{-1}(1-Q)$. By Theorem 2.16 we have,

$g^{-1}(1-Q) \geq ps-\beta cl(g^{-1}(1-Q) = 1-(ps-\beta int(g^{-1}(Q)))$.
Thus, $g^{-1}(1-Q) \in ps-(\beta^c, U_{\mu_1})$. □

Lemma 3.8. ([8]) Let $h : X \to (U \times V)$ be the graph of a mapping $r : (U, \mu_1) \to (V, \mu_2)$ given by $h(x) = (x, r(x))$. For any fuzzy sets P and Q on U and V, respectively, then $h^{-1}(P \times Q) = P \wedge r^{-1}(Q)$.

Theorem 3.9. *Let $r : (U, \mu_1) \to (Y, \mu_2)$ be a mapping and $h : U \to (U \times Y)$ be the graph of r. If h is $ps-ro$ fuzzy β-continuous, then r is also $ps-ro$ fuzzy β-continuous.*

Proof. Let $V \in ps-(o, Y_{\mu_2})$, by Lemma 3.8, we have, $r^{-1}(V) = 1 \wedge r^{-1}(V) = h^{-1}(1 \times V)$. Now, $(1 \times V)$ is $ps-ro$ open fuzzy set on $(U \times Y)$ and since h is $ps-ro$ fuzzy β-continuous function and $h^{-1}(1 \times V) \in ps-(\beta, U_{\mu_1})$. So, $r^{-1}(V) \in ps-(\beta, U_{\mu_1})$. Therefore, r is $ps-ro$ fuzzy β-continuous. □

Theorem 3.10. *Let (U, μ_1), (V, μ_2),(W, μ_3) be three fts and $f : U \to V$ and $g : V \to W$ be two functions. $g \circ f$ be $ps-ro$ fuzzy β-continuous if f is $ps-ro$ fuzzy β-continuous and g be $ps-ro$ fuzzy continuous.*

Proof. Let $Q \in ps-(o, W_{\mu_3})$. As, g is $ps-ro$ fuzzy continuous, $g^{-1}(Q) \in ps-(o, V_{\mu_2})$ and $(g \circ f)^{-1}(Q) \in ps-(\beta, U_{\mu_1})$. Therefore, $g \circ f$ is a $ps-ro$ fuzzy β-continuous. □

Theorem 3.11. *A bijective function $g : (U, \mu_1) \to (V, \mu_2)$, where (U, μ_1) and (V, μ_2) are two fts, is $ps-ro$ fuzzy β-continuous if only if for every fuzzy set P on U, $ps-int(g(P)) \leq g(ps-\beta int(P))$.*

Proof. Let g be $ps-ro$ fuzzy β-continuous and P be any fuzzy set on U, $ps-int(g(P)) \in ps-(o, V_{\mu_2})$ and $g^{-1}(ps-int(g(P))) \in ps-(\beta, U_{\mu_1})$. Assuming g is bijective, $g^{-1}(ps-int(g(P))) \leq ps-\beta int(g^{-1}(g(P))) = ps-\beta int(P)$ and $g(g^{-1}(ps-int(g(P))) = psint(g(P)) \leq g(ps-\beta int(P))$.

Conversely, let $Q \in ps-(o, V_{\mu_2})$. Assuming g is surjective, $Q = ps-int(Q) = ps-int(g(g^{-1}(Q)))$. Now, $ps-int(g(g^{-1}(Q))) \leq g(ps-\beta int(g^{-1}(Q))$. As g is injective, therefore

$$g^{-1}(Q) \leq g^{-1}(g(ps-\beta int(g^{-1}(Q))) = ps-\beta int(g^{-1}(Q)).$$

Again, $ps-\beta int(g^{-1}(Q)) \leq g^{-1}(Q)$, $g^{-1}(Q) = ps-\beta int(g^{-1}(Q))$ showing that $g^{-1}(Q) \in ps-(\beta, U_{\mu_1})$. Thus, g is $ps-ro$ fuzzy β-continuous. □

4. $ps-ro$ Fuzzy β-Open (Closed) Function

In this section, we begin the idea of $ps-ro$ fuzzy β-open (closed) function and study their roles between fts.

Definition 4.1. A function g between two fts (U, μ_1) and (V, μ_2) is called $ps-ro$ fuzzy β-open (closed) function if $g(P) \in ps-(\beta, V_{\mu_2})$ (resp. $ps-(\beta^c, V_{\mu_2})$) for every $P \in ps-(o, U_{\mu_1})$ (resp. $ps-(o^c, U_{\mu_1})$).

Clearly, $ps-ro$ fuzzy semiopen (closed) function is $ps-ro$ fuzzy β-open (closed) function and also $ps-ro$ fuzzy preopen (closed) function is $ps-ro$ fuzzy β-open (closed) function, but the converse is not true, since a $ps-ro$ β-open (closed) fuzzy set does not imply both $ps-ro$ semiopen (closed) and $ps-ro$ preopen (closed) fuzzy sets.

We shall see that $ps-ro$ fuzzy β-open function does not imply a fuzzy β-open function.

Example 4.2. Consider $U = \{e, b, s, d\}$, $V = \{i, m, z, w\}$ and fuzzy sets P, Q and R on U given as $P(e) = 0.3, P(b) = 0.3, P(s) = 0.3, P(d) = 0.4$, $Q(r) = 0.4$ and $R(r) = 0.8 \; \forall \; r \in U$. Then, $\mu_1 = \{1, P, Q, R, 0\}$ is a fuzzy topology on U. Again let S, E, F be the fuzzy sets on V given as $S(r) = 0.1 \; \forall \; r \in V$; $E(i) = 0.5, E(m) = 0.5, E(z) = 0.6, E(w) = 0.5$ and $F(r) = 0.7 \; \forall \; r \in V$. Clearly, $\mu_2 = \{0, 1, S, E, F\}$ is a fuzzy topology on V. Here, $P \notin ps-(r, U_{\mu_1})$ for $\alpha \in [.3, .4)$. Hence, $\{1, Q, R, 0\}$ is $ps-ro$ fuzzy topology on U. Also, $E \notin ps-(r, V_{\mu_2})$ for $\alpha \in [.5, .6)$. Hence, $\{1, S, F, 0\}$ is $ps-ro$ fuzzy topology on V.

Let g be a function from (U, μ_1) to (V, μ_2) given by $g(a) = i$, $g(b) = m$, $g(s) = z$ and $g(d) = w$. Here, $Q, R \in ps-(o, U_{\mu_1})$. Now, $g(Q)(r) = 0.4$ $\forall \; r \in U$ and $g(Q) \leq ps-cl(ps-int(ps-cl(g(Q))))$. Again, $g(R)(r) = 0.8$ $\forall \; r \in U$ and $g(R) \leq ps-cl(ps-int(ps-cl(g(R))))$. Thus, $g(Q), g(R) \in ps-(\beta, V_{\mu_2})$. Also, $g(0), g(1) \in ps-(\beta, V_{\mu_2})$. Thus, g is $ps-ro$ fuzzy β-open function. But, $g(Q) \geq cl(int(cl(g(Q))))$, thus $g(Q) \notin (\beta, V_{\mu_2})$ for $Q \in (o, U_{\mu_1})$. Therefore, g is not a fuzzy β-open function.

Next, we shall see that the fuzzy β-open function does not imply $ps-ro$ fuzzy β-open function.

Example 4.3. Consider $U = \{e, b, s, d\}$, $V = \{i, m, z, w\}$ and fuzzy sets P, Q and R on U given as $P(e) = 0.7, P(b) = 0.7, P(s) = 0.6, P(d) = 0.7$, $Q(r) = 0.1 \; \forall \; r \in U$, $R(e) = 0.5, R(b) = 0.5, R(s) = 0.5$ and $R(d) = 0.4$. Also, let S, E, F be the fuzzy sets on V given as $S(i) = 0.2, S(m) = 0.2, S(z) = 0.3, S(w) = 0.2$, $E(r) = 0.6 \; \forall \; r \in V$ and $F(r) = 0.2 \; \forall \; r \in V$. Clearly, $\mu_1 = \{0, 1, P, Q, R\}$ and $\mu_2 = \{0, 1, S, E, F\}$ are fuzzy topologies on U and V, respectively.

In $(U, i_\alpha, (\mu_1)) \; \forall \; \alpha \in I_1$, the open sets are $\phi, U, P^\alpha, Q^\alpha$ and R^α where Q^α is regular open on $(U, i_\alpha, (\mu_1)) \; \forall \; \alpha \in I_1$, but P^α and R^α are not regular open for $\alpha \in [.6, .7)$ and $\alpha \in [.4, .5)$, respectively. Hence, $\{1, Q, 0\}$ is $ps-ro$ fuzzy topology on U. Similarly, $\{1, E, F, 0\}$ is $ps-ro$ fuzzy topology on V.

We define a function g from (U, μ_1) to (V, μ_2) by $g(e) = i, g(b) = m, g(s) = z$ and $g(d) = w$. Here, $Q \in ps - (o, U_{\mu_1})$, but $g(Q) \notin ps - (\beta, V_{\mu_2})$. Hence, g is not $ps - ro$ fuzzy β-open function.

Here it can be verified that the image of each fuzzy open set on U is a fuzzy β-open set on V, hence g is a fuzzy β-open function.

Remark 4.4. In Example 4.2, g is $ps - ro$ fuzzy β-open function, but g is not a fuzzy β-open function. In Example 4.3, g is a fuzzy β-open function, but g is not $ps - ro$ fuzzy β-open function. Thus, the idea of $ps-ro$ fuzzy β-open function and fuzzy β-open function are independent. Similarly, it can be proved that $ps-ro$ fuzzy β-closed and fuzzy β-open functions are independent.

Theorem 4.5. *Let $g : (U, \mu_1) \to (V, \mu_2)$ be a $ps - ro$ fuzzy β-open function. If $Q \in ps - (o^c, U_{\mu_1})$ which contains $g^{-1}(Q)$, then $\exists\ C \in ps - (\beta^c, V_{\mu_2})$ such that $Q \leq C$ and $g^{-1}(C) \leq P$.*

Proof. Put $C = 1-g(1-P)$. Now, $g^{-1}(Q) \leq P$. So, $1-g^{-1}(Q) \geq 1-P$. $g^{-1}(1 - Q) \geq 1 - P$. i.e. $g(1 - P) \leq 1 - Q$. Assuming that g is $ps-ro$ fuzzy β-open, $g(1-P) \in ps-(\beta, V_{\mu_2})$, so $C \in ps-(\beta^c, V_{\mu_2})$ and $g^{-1}(C) = 1 - g^{-1}(g(1-P)) \leq 1 - (1-P) = P$. Thus, $g^{-1}(C) \leq P$. □

Theorem 4.6. *Let $g : (U, \mu_1) \to (V, \mu_2)$ be a $ps - ro$ fuzzy β-closed. If $Q \in ps - (o, U_{\mu_1})$ which contains $g^{-1}(Q)$, then $\exists\ C \in ps - (\beta, V_{\mu_2})$ such that $Q \leq C$ and $g^{-1}(C) \leq P$.*

Proof. The proof is similar to Theorem 4.5. □

Theorem 4.7. *A function $h : (U, \mu_1) \to (V, \mu_2)$ is $ps - ro$ fuzzy β-open if only if $h(ps - int(P)) \leq ps - \beta int(h(P))$, for any fuzzy set P on U.*

Proof. Let h be $ps - ro$ fuzzy β-open and P be a fuzzy set on U. Then $ps-int(P) \in ps-(o, U_{\mu_1})$. Also, $ps-int(P) \leq P$. Now, $h(ps-int(P)) \in ps - (\beta, V_{\mu_2})$. So, $h(ps - int(P)) = ps - \beta int(h(ps - int(P))) \leq ps - \beta int(h(P))$. Thus, $h(ps - int(P)) \leq ps - \beta int(h(P))$.

Conversely, let $P \in ps - (o, U_{\mu_1})$, then, $h(P) = h(ps - int(P)) \leq ps - \beta int(h(P))$. Thus, $h(P) \in ps - (\beta, V_{\mu_2})$, so h is $ps - ro$ fuzzy β-open. □

Theorem 4.8. *A function $h : (U, \mu_1) \to (V, \mu_2)$ is $ps-ro$ fuzzy β-closed if only if $ps - \beta cl(h(P)) \leq h(ps - cl(P))$, for any fuzzy set P on U.*

Proof. Let h be $ps-ro$ fuzzy β-closed and P be a fuzzy set on U. Then $ps - cl(P) \in ps - (o^c, U_{\mu_1})$. Also, $P \leq ps - cl(P)$

Now, $h(ps - cl(P)) \in ps - (\beta^c, V_{\mu_2})$. So, $ps - \beta cl(h(ps - cl(P))) \leq h(ps - cl(P))$.

Conversely, let $P \in ps - (o^c, U_{\mu_1})$. Then, $ps - \beta cl(h(ps - cl(P))) \leq h(ps - cl(P)) = h(P)$. Also, $h(P) \leq ps - cl(h(P))$. Thus, $h(P) \in ps - (o^c, V_{\mu_2})$, so h is $ps - ro$ fuzzy β-closed. □

Theorem 4.9. *If a function $h:(U,\mu_1)\to(V,\mu_2)$ is $ps-ro$ fuzzy open (closed) and another function $g:(V,\mu_2)\to(W,\mu_3)$ is $ps-ro$ fuzzy β-open (closed), then $g\circ h$ is $ps-ro$ fuzzy β-open (closed).*

Proof. Here, we know that for each fuzzy set P of U, $g\circ h(P)=g(h(P))$. So, for a $ps-ro$ open (closed) fuzzy set P in U, $h(P)$ is $ps-ro$ open (closed) fuzzy set on V and $g(h(P))$ is $ps-ro$ fuzzy β-open (closed) set on W. Hence, $g\circ h$ is $ps-ro$ fuzzy β-open (closed). □

Theorem 4.10. *Let $h:(U,\mu_1)\to(V,\mu_2)$ and $g:(V,\mu_2)\to(W,\mu_3)$ be two functions. If h is $ps-ro$ fuzzy continuous and surjective and $g\circ f$ is $ps-ro$ fuzzy β-closed, then g is $ps-ro$ fuzzy β-closed.*

Proof. For $P\in ps-(o^c,V_{\mu_2})$ we have, $h^{-1}(P)\in ps-(o^c,U_{\mu_1})$. Also, $g\circ h$ is $ps-ro$ fuzzy β-closed and h is surjective, therefore $(g\circ h)(h^{-1}(P))=g(h(h^{-1}(P)))=g(P)\in ps-(\beta^c,W_{\mu_3})$. Hence, g is $ps-ro$ fuzzy β-closed. □

Conclusion

The article introduces the notion $ps-ro$ β-open (closed) fuzzy sets, related fuzzy function and fuzzy continuity on a fts are used as a new tool to study fts. In a similar line, other such notions may be explored.

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