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Sahand Communications in Mathematical Analysis

Print ISSN: 2322-5807
Online ISSN: 2423-3900
Volume: 19
Number: 3
Pages: 141-168

Sahand Commun. Math. Anal.
DOI: 10.22130/scma.2022.562857.1183

Sahand Communications in Mathematical Analysis (SCMA) Vol. 19 No. 3 (2022), 141-168
http://scma.maragheh.ac.ir
DOI: 10.22130/scma.2022.562857.1183

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Abstract. In this paper, we express and prove Bushell-Okrasiaski, Hardy and Minkowski type inequalities for two classes of pseudo-integrals. One of them, classes with pseudo-integrals where pseudo-operations are defined via a monotone and continuous generator function. The other one concerns the pseudo-integrals based on a semiring with an idempotent addition and a pseudo-multiplication generator. Those are important inequalities from both mathematical and application points of view. The established results are based on the classical Bushell-Okrasiaski, Hardy and Minkowski's inequalities for integrals. Also, some examples and applications are presented.

1. Introduction

The theory of fuzzy measures and fuzzy integrals was introduced by Sugeno [56] as a tool for modelling non-determinstic problems. Fuzzy integrals or Sugeno integrals have very interesting properties from a mathematical point of view which have been studied by many authors, studied by many authors including Pap [39], Ralescu and Adams [45], Wang and Klir [59] among others. Ralesco and Adams [45] studied several equivalent definitions of fuzzy integrals, while Pap [41] and Wang and Klir [59], provided an overview of fuzzy measure theory. The fuzzy integral for monotone functions was presented in [46]. In fact, fuzzy measures and fuzzy integrals are versatile operators which can be used in different areas. They have a broad use in information fusion, electronic auctions, decision making, and etc. Chen et al. [7] employed fuzzy integral and fuzzy measure to establish a public attitude analysis model.

2020 *Mathematics Subject Classification.* 03E72, 28E10, 26E50.
Key words and phrases. B-O type inequality, Convolution type inequality, Hardy type inequality, Minkowski's inequality, Fuzzy integral inequality, Pseudo-integrals.
Received: 02 September 2022, Accepted: 28 September 2022.

The integral inequalities are useful results in several theoretical and applied fields. For instance, integral inequalities play a major role in the development of a time scales calculus. $\ddot{O}$zkan et al. [38] obtained H$\ddot{o}$lderâ s inequality, Minkowski's inequality and Jensen's inequality on time scales. Some famous inequalities have been generalized to fuzzy integral. Román-Flores and Chalco-Cano [47] analyzed an interesting type of geometric inequalities for fuzzy integral with some applications to convex geometry. Román-Flores et al. [48, 49] studied a Jensen type inequality and a convolution type inequality for fuzzy integrals. Also, they have investigated a Chebyshev type inequality and a Stolarsky type inequality for fuzzy integrals [26, 50]. In [26], a fuzzy Chebyshev inequality for a special case was obtained which has been generalized by Ouyang et al. [36]. Furthermore, Chybyshev type inequalities for fuzzy integral were proposed in a rather general form by Mesiar and Ouyang [34]. In [51], RomÂ´an-Flores et al. proved a Hardy type inequality for fuzzy integrals. Recently, B. Daraby and L. Arabi Proved a related Fritz Carlson type inequality for Sugeno integrals [10] and continued fuzzification of some other inequalities (see [17, 18, 25]).

Pseudo-analysis is a generalization of the classical analysis, where instead of the field of real numbers a semiring is taken on a real interval $[a, b] \subset [-\infty, \infty]$ endowed with pseudo-addition $\oplus$ and with pseudo-multiplication $\odot$ ([1, 4, 10, 33, 40, 44, 56]). Based on this structure there where developed the concepts of $\oplus$-measure (pseudo-additive measure), pseudo-integral, pseudo-convolution, pseudo-Laplace transform and etc. (see [8–10, 12, 13, 15, 16, 19, 20, 23, 26, 27, 41–43, 46]).

The classical Bushell-Okrasinski [6] is a convolution type inequality. More precisely,

$$\int_0^x (x-t)^{s-1} g(t)^s dt \leq \left(\int_0^x g(t) \right)^s, \quad 0 \leq x \leq b, \tag{1.1}$$

holds for a continuous and increasing function $g : [0, 1] \to [0, \infty)$ and $s \geq 1$, $b \leq 1$. This inequality was used by Bushell and Okrasinski [6] in the study of solutions of Volterra integral equations (also see [32]). Later on Walter and Weckesser [58] studied some extensions of (1) and finally, after the change of variable $t = xs$, Malamud [31] analyzed the B-O inequality (1) in the following new form:

$$s \int_0^1 (1-t)^{s-1} g(t)^s dt \leq \left(\int_0^x g(t) \right)^s .$$

H. Román-Flores et al. [49] proved Bushell-Okrasinski type inequality for the Sugeno integrals at two cases in the following way:

Theorem 1.1 (Fuzzy B-O inequality: decreasing case)**.** *Let* $g : [0, 1] \to [0, \infty)$ *be a continuous and decreasing function. Then*

$$s \fint_0^1 (1-t)^{s-1} g(t)^s dt \geq \left(\fint_0^1 g(t)dt \right)^s ,$$

holds for all $s \geq 2$.

The following theorem establish an analogous result for the increasing case.

Theorem 1.2 (Fuzzy B-O inequality: increasing case)**.** *Let* $g : [0, 1] \to [0, \infty)$ *be a continuous and increasing function. Then*

$$s \fint_0^1 t^{s-1} g(t)^s dt \geq \left(\fint_0^1 g(t)dt \right)^s ,$$

holds for all $s \geq 2$.

The well-known Hardy inequality is a part of the classical mathematical analysis ([28]). The classical Hardy's integral inequality holds

$$\left(\frac{P}{P-1}\right)^P \int_0^\infty f^P(x)dx > \int_0^\infty \left(\frac{F}{x}\right)^P dx,$$

where $P > 1$ and $f : [0, \infty) \to [0, \infty)$ is an integrable function ($f \neq 0$) and $F(x) = \int_0^x f(t)dt$. Furthermore, for parameters a, b such that $0 < a < b < \infty$, the following inequality is also valid ([55]):

$$\left(\frac{P}{P-1}\right)^P \int_a^b f^P(x)dx > \int_a^b \left(\frac{F}{x}\right)^P dx,$$

where $0 < \int_0^\infty f^P(t)dt < \infty$. H. Román-Flores et al. have proved a Hardy type inequality for fuzzy integrals ([51]). The fuzzy Hardy's integral inequality holds

$$\left(\fint_0^1 f^P(x)dx\right)^{\frac{1}{P+1}} \geq \fint_0^1 \left(\frac{F}{x}\right)^P dx \tag{1.2}$$

where $P \geq 1$, $f : [0, 1] \to [0, \infty)$ is an integrable function and $F(x) = \fint_0^x f(t)dt$.

In this paper, we generalize their work for pseudo-integrals.

The classical Minkowski's inequality was published by Minkowski [35] in his famous book "Geometrie der Zahlen". A proof of Minkowski's inequality as well as several extensions, related results, and interesting geometrical interpretations can be found in [53, 54]. An extension of Minkowski's inequality, which is based on Hölder's inequality, is given in [59]. Applications of Minkowski's inequality have been studied by many authors. For example Özkan et al. [38] applied Minkowski's inequality, Hölder's inequality and Jensen's inequality on time scales. Lu et al. [30] used Minkowski's inequality for fast full search in motion estimation. The classical Minkowski's inequality [35] is as follows:

$$\left(\int_a^b (f(x)+g(x))^s dx\right)^{\frac{1}{s}} \leq \left(\int_a^b f(x)^s dx\right)^{\frac{1}{s}} + \left(\int_a^b g(x)^s dx\right)^{\frac{1}{s}} \tag{1.3}$$

where $1 \leq s < \infty$ and $f, g : [0,1] \to [0, \infty)$ are two non-negative functions.

Note we recall the following inequalities which are the fuzzy versions of Minkowski's inequality at two cases and appears in [2].

Theorem 1.3. *Let $f, g : [0,1] \to [0,\infty)$ be two real-valued functions and let μ be the Lebesgue measure on $\mathbb{R}$. If f, g are both continuous and strictly decreasing functions, then the inequality*

$$\left(⨍_0^1 (f+g)^s d\mu\right)^{\frac{1}{s}} \leq \left(⨍_0^1 f^s d\mu\right)^{\frac{1}{s}} + \left(⨍_0^1 g^s d\mu\right)^{\frac{1}{s}}$$

holds for all $1 \leq s < \infty$.

Theorem 1.4. *Let $f, g : [0,1] \to [0,\infty)$ be two real-valued functions and let μ be the Lebesgue measure on $\mathbb{R}$. If f, g are both continuous and strictly increasing functions, then the inequality*

$$\left(⨍_0^1 (f+g)^s d\mu\right)^{\frac{1}{s}} \leq \left(⨍_0^1 f^s d\mu\right)^{\frac{1}{s}} + \left(⨍_0^1 g^s d\mu\right)^{\frac{1}{s}} \tag{1.4}$$

holds for all $1 \leq s < \infty$.

The following theorem is pseudo version of Minkowski's inequality and appeared in [3].

Theorem 1.5. *Let $f, g : X \to [0,\infty)$ be two measurable functions and $s \in [1,\infty)$. If an additive generator $g : [a,b] \to [0,1]$ of the pseudo-addition $\oplus$ and the pseudo-multiplication $\odot$ are increasing, then for any $\sigma - \oplus$-measure m it holds:*

$$\left(\int_X^{\oplus} (f+g)^s d\mu\right)^{\frac{1}{s}} \leq \left(\int_X^{\oplus} f^s d\mu\right)^{\frac{1}{s}} + \left(\int_X^{\oplus} g^s d\mu\right)^{\frac{1}{s}}. \tag{1.5}$$

The following theorem shows the new classical version of Minkowski's inequality and appears in [5].

Theorem 1.6. *Let f and g be two positive functions satisfying*

$$0 < m \leq \frac{f(x)}{g(x)} \leq M, \quad \forall x \in [a,b],$$

we have

$$\left(\int_a^b f^s(x)dx\right)^{\frac{1}{s}}+\left(\int_a^b g^s(x)dx\right)^{\frac{1}{s}} \leq c\left(\int_a^b (f(x)+g(x))^s dx\right)^{\frac{1}{s}}, \tag{1.6}$$

where $1 \leq s < \infty$ *and* $c = \dfrac{M(m+1)+(M+1)}{(m+1)(M+1)}$.

In this paper we indicate and prove generalizations of the Bushell-Okrasinski, Hardy and Minkowski type inequality for pseudo-integrals that are appeared in [13, 14, 16].

2. Pseudo-Integral

Let $[a, b]$ be a closed (in some cases can be considered semiclosed) subinterval of $[-\infty, \infty]$. The full order on $[a, b]$ will be denoted by $\preceq$. A binary operation $\oplus$ on $[a, b]$ is pseudo-addition if it is commutative, non-decreasing(with respect to $\preceq$), associative and with a zero (neutral) element denoted by $\mathbf{0}$. Let $[a, b]_+ = \{x|x \in [a, b], \mathbf{0} \preceq x\}$. A binary operation $\odot$ on $[a, b]$ is Pseudo-multiplication if it is commutative, positively non-decreasing, i.e., $x \preceq y$ implies $x \odot z \preceq y \odot z$ for all $z \in [a, b]_+$, associative and with a unit element $1 \in [a, b]$, i.e., for each $x \in [a, b]$, $1 \odot x = x$. We assume also $\mathbf{0} \odot x = \mathbf{0}$ and that $\odot$ is distributive over $\oplus$, i.e.,

$$x \odot (y \oplus z) = (x \odot y) \oplus (x \odot z).$$

The structure $([a, b], \oplus, \odot)$ is a semiring ([9, 11, 12, 29, 37, 40]). In this paper we will consider semirings with following continuous operations:

Case I. The pseudo-addition is idempotent operation and the pseudo-multiplication is not.

(a) $x \oplus y = \sup(x, y)$, $\odot$ is arbitrary not idempotent pseudo-multiplication on the interval $[a, b]$. We have $\mathbf{0} = a$ and the idempotent operation sup induces a full order in the following way: $x \preceq y$ if and only if $\sup(x, y) = y$.

(b) $x \oplus y = \inf(x, y)$, $\odot$ is arbitrary not idempotent pseudo-multiplication on the interval $[a, b]$. We have $\mathbf{0} = b$ and the the idempotent operation inf induces a full order in the following way: $x \preceq y$ if and only if $\inf(x, y) = y$.

Case II. The pseudo-operations are defined by a monotone and continuous function $g : [a, b] \to [0, \infty]$ (additive generator of $\oplus$), i.e., pseudo-operations are given with

$$x \oplus y = g^{-1}(g(x) + g(y)), \qquad x \odot y = g^{-1}(g(x).g(y)).$$

If the zero element for the pseudo-addition is a, we will consider increasing generators.Then $g(a) = 0$ and $g(b) = \infty$. If the zero element for the pseudo-addition is b, we will consider decreasing generators. Then $g(b) = 0$ and $g(a) = \infty$.

If the generator g is increasing (respectively decreasing), the operation $\oplus$ induce the usual order (respectively opposite to the usual order) on the interval $[a, b]$ in the following way: $x \preceq y$ if and only if $g(x) \preceq g(y)$.

Case III. Both operation are idempotent. We have

(a) $x \oplus y = \sup(x, y)$, $x \odot y = \inf(x, y)$, on the interval $[a, b]$. We have $\mathbf{0} = a$ and $\mathbf{1} = b$. The idempotent operation *sup* induces a usual order ($x \prec y$ if and only if $\sup(x, y) = y$).

(b) $x \oplus y = \inf(x, y)$, $x \odot y = \sup(x, y)$, on the interval $[a, b]$. We have $\mathbf{0} = b$ and $\mathbf{1} = a$. The idempotent operation inf induces an order opposite to the usual order($x \preceq y$ if and only if $\inf(x, y) = y$).

3. Explicit Forms of Special Pseudo-Integrals

We shall consider the semiring $([a, b], \oplus, \odot)$ for three (with completely different behaviour) cases, namely I(a), II, and III(a). Observe that the cases I(b) and III(b) are linked to the cases I(a) and III(a) by duality. First case is when pseudo-operations are generated by a monotone and continuous function $g : [a, b] \to [0, \infty]$, case then the pseudo-integral for a measurable function $f : X \to [a, b]$ is given by,

$$\int_X^{\oplus} f \odot dm = g^{-1}\left(\int_X (g \circ f) d(g \circ m)\right), \tag{3.1}$$

Where the integral applied on the right side is the standard Lebesgue integral. In spacial case, when $X = [c, d]$, $\mathcal{A} = \mathcal{B}(X)$ and $m = g^{-1} \circ \lambda$, λ the standard Lebesgue measure on $[c, d]$, then we use notation

$$\int_{[c,d]}^{\oplus} f(x)dx = \int_X^{\oplus} f \odot dm.$$

By Case II,

$$\int_{[c,d]}^{\oplus} f(x)dx = g^{-1}\left(\int_c^d g(f(x))dx\right),$$

i.e., we have recovered the g-integral, (see[40, 44]).

Second case is when the semiring is of the form $([a, b], \sup, \odot)$, case I(a) and III(a). We will consider complete sup-measure m only and $\mathcal{A} = 2^x$, i.e., for any system $(A_i)_i \in I$ of measurable sets,

$$m\left(\underset{i \in I}{\cup} A_i\right) = \sup_{i \in I} m(A_i)$$

Recall that if X is countable (especially, if X is finite) then any σ-sup-measure m is complete and, moreover, $m(A) = \sup_{x \in A} \psi(X)$, where

$\psi : X \to [a,b]$ is a density function given by $\psi(x) = m(\{x\})$. Then the pseudo-integral for a function $f : X \to [a,b]$ is given by

$$\int_X^{\oplus} f \odot dm = \sup_{x \in X} (f(x) \odot \psi(x)),$$

where function ψ defines sup-measure m.

Theorem 3.1. *Let m be a* sup*-measure on* $([0,\infty], \mathfrak{B}([0,\infty]))$*, where* $\mathfrak{B}([0,\infty])$ *is the Borel σ-algebra on* $[0,\infty]$*,* $m(A) = esssup_\mu(\psi(x)|x \in A)$*, where* $\psi : [0,\infty] \to [0,\infty]$ *is a continuous density. Then for any pseudo-addition $\oplus$ with a generator g there exists a family $\{m_\lambda\}$ of $\oplus_\lambda$-measure on* $([0,\infty), \mathfrak{B})$*, where $\oplus_\lambda$ is generated by g^λ (the function g of the power λ), $\lambda \in [0,\infty)$, such that* $\lim_{\lambda\to\infty} m_\lambda = m$.

For any continuous function $f : [0,\infty] \to [0,\infty]$ the integral $\int^{\oplus} f \odot dm$ can be obtained as a limit of g-integrals, [33].

Theorem 3.2. *Let* $([0,\infty], \sup, \odot)$ *be a semiring with $\odot$ generated by some increasing generator g, i.e., we have $x \odot y = g^{-1}(g(x)g(y))$ for every $x, y \in [a,b]$. Let m be the same as in Theorem 3.1. Then there exists a family $\{m_\lambda\}$ of $\oplus_\lambda$-measure, where $\oplus_\lambda$ is generated by g^λ, $\lambda \in [0,\infty)$, such that for every continuous function $f : [0,\infty] \to [0,\infty]$*

$$\begin{aligned}\int^{\sup} f \odot dm &= \lim_{\lambda\to\infty} \int^{\oplus_\lambda} f \odot dm_\lambda \\ &= \lim_{\lambda\to\infty} (g^\lambda)^{-1} \left(\int g^\lambda(f(x))dx \right)\end{aligned}$$

Now we recall generalization of the Jensen inequality for pseudo-integral that proved by E. Pap et al. on [44].

Theorem 3.3. Let $\Phi : [a,b] \to [a,b]$ be a convex and nondecreasing function. If a generator $g : [a,b] \to [a,b]$ of the pseudo-addition $\oplus$ and the pseudo-multiplication $\odot$ is a convex and increasing function, then for any measurable function $f : [0,1] \to [a,b]$ we have

$$\Phi\left(\int_{[0,1]}^{\oplus} f(x)dx \right) \le \int_{[0,1]}^{\oplus} \Phi(f(x))dx.$$

Theorem 3.4. Let $\Phi : [a,b] \to [a,b]$ be a convex and nondecreasing function, and the pseudo-multiplication $\odot$ is represented by a convex and increasing generator g. Let m be the same as in Theorem 3.1. Then for any continuous function $f : [0,1] \to [a,b]$ we have

$$\Phi\left(\int_{[0,1]}^{\sup} f \odot dm \right) \le \int_{[0,1]}^{\sup} \Phi(f) \odot dm.$$

The following Theorem shows that the Chebyshev's inequality for pseudo-integrals that is proved in [1].

Theorem 3.5. *Let $u, v : [0,1] \to [a,b]$ be two measurable functions and let a generator $g : [a,b] \to [0,\infty)$ of the pseudo-addition $\oplus$ and the pseudo-multiplication $\odot$ be an increasing function. If u and v are comonotone functions, then the inequality*

$$\int_{[0,1]}^{\oplus} (u \odot v)dx \geq \left(\int_{[0,1]}^{\oplus} udx\right) \odot \left(\int_{[0,1]}^{\oplus} vdx\right),$$

holds and the reserve inequality holds whenever u and v are countermonotone functions.

Theorem 3.6 ([40])**.** *For any measurable function f, f_1, f_2 and $\lambda \in \mathbb{R}$, we have*

(i) $\int_{[c,d]}^{\oplus} (f_1 \oplus f_2)dx = \int_{[c,d]}^{\oplus} f_1 dx \oplus \int_{[c,d]}^{\oplus} f_2 dx$,
(ii) $\int_{[c,d]}^{\oplus} (\lambda \otimes f)dx = \lambda \otimes \int_{[c,d]}^{\oplus} f dx$,
(iii) $f_1 \leq f_2 \Rightarrow \int_{[c,d]}^{\oplus} f_1 dx \leq \int_{[c,d]}^{\oplus} f_2 dx$.

Easily a straight calculus give the following Lemma:

Lemma 3.7. *Let f_1 and f_2 be integrable functions, $A \in \sum$ and $f_1 \leq f_2$, so we have:*

(i) $⨍_A f_1 dx \leq ⨍_A f_2 dx$,
(ii) $\int_{A}^{\oplus} f_1 dx \leq \int_{A}^{\oplus} f_2 dx$.

4. Bushell-Okrasiaski Type Inequality for Pseudo-Integrals

In this section, we prove two Bushell-Okrasiaski inequalities for pseudo-integrals.

Theorem 4.1 (Pseudo Bushell-Okrasiaski inequality: decreasing case)**.** *Let $f : [0,1] \to [a,b]$ be a continuous and decreasing function. If a generator $g : [a,b] \to [a,b]$ of the pseudo-addition $\oplus$ and the pseudo-multiplication $\odot$ is a convex and increasing function, then*

$$\int_{[0,1]}^{\oplus} (1-t)^{s-1} \odot f^s(t)dt \geq \frac{1}{s} \odot \left(\int_{[0,1]}^{\oplus} f(t)dt\right)^s,$$

holds for all $s \geq 2$.

Proof. By the definition of pseudo-integral and pseudo-operations we have

$$\int_{[0,1]}^{\oplus} (1-t)^{s-1} \odot f^s(t)dt = g^{-1}\left(\int_0^1 g\left[(1-t)^{s-1} \odot f^s(t)\right] dt\right)$$

$$= g^{-1}\left(\int_0^1 g\left[g^{-1}(g\left((1-t)^{s-1}\right) g(f^s(t))\right] dt\right)$$
$$= g^{-1}\left(\int_0^1 g\left((1-t)^{s-1}\right) g(f^s(t))dt\right).$$

By classic Chebyshev's integral inequality ([57]), we have;

$$\begin{aligned}
&g^{-1}\left(\int_0^1 g\left((1-t)^{s-1}\right) g(f^s(t))dt\right)\\
&\quad\geq g^{-1}\left[\left(\int_0^1 g\left((1-t)^{s-1}\right) dt\right)\left(\int_0^1 g(f^s(t))dt\right)\right]\\
&\quad= g^{-1}\left[gg^{-1}\left(\int_0^1 g\left((1-t)^{s-1}\right) dt\right) gg^{-1}\left(\int_0^1 g(f^s(t))dt\right)\right]\\
&\quad= g^{-1}\left[g\left(\int_{[0,1]}^{\oplus}(1-t)^{s-1}dt\right) g\left(\int_{[0,1]}^{\oplus} f^s(t)dt\right)\right]\\
&\quad= \left(\int_{[0,1]}^{\oplus}(1-t)^{s-1}dt\right)\odot\left(\int_{[0,1]}^{\oplus} f^s(t)dt\right).
\end{aligned}$$

By using the Theorem 3.5,

$$\int_{[0,1]}^{\oplus}(1-t)^{s-1}\odot f^s(t)dt \geq \left(\int_{[0,1]}^{\oplus}(1-t)^{s-1}dt\right)\odot\left(\int_{[0,1]}^{\oplus} f(t)dt\right)^s, \tag{4.1}$$

in the other hand by using the classic Jensen inequality ([52]), we can show that

$$\begin{aligned}
\int_{[0,1]}^{\oplus}(1-t)^{s-1}dt &= g^{-1}\left(\int_0^1 g\left((1-t)^{s-1}\right) dt\right)\\
&\geq g^{-1}\left(g\int_0^1 (1-t)^{s-1}dt\right)\\
&= \int_0^1 (1-t)^{s-1}dt\\
&= \frac{1}{s},
\end{aligned} \tag{4.2}$$

so by (4.1) and (4.2) we obtain that:

$$\int_{[0,1]}^{\oplus}(1-t)^{s-1}\odot f^s(t)dt \geq \frac{1}{s}\odot\left(\int_{[0,1]}^{\oplus} f(t)dt\right)^s.$$

Thereby, the theorem is proved. $\square$

Example 4.2. Let $g(x) = e^x$. The corresponding pseudo-operations are $x \oplus y = \ln(e^x + e^y)$ and $x \odot y = x + y$, the Theorem 4.1 reduces on the following inequality,

$$\ln\left(\int_0^1 e^{(1-t)^{s-1}+f^s(t)}dt\right) \geq \frac{1}{s} + \left(\ln(\int_0^1 e^{f(t)}dt)\right)^s.$$

Theorem 4.3 (Pseudo Bushell-Okrasiaski inequality: increasing case)**.** *Let $f : [0,1] \to]a,b[$ be a continuous and increasing function. If a generator $g : [a,b] \to [a,b]$ of the pseudo-addition $\oplus$ and the pseudo-multiplication $\odot$ is a convex and increasing function, then*

$$\int_{[0,1]}^{\oplus} t^{s-1} \odot f^s(t)dt \geq \frac{1}{s} \odot \left(\int_{[0,1]}^{\oplus} f(t)dt\right)^s,$$

holds for all $s \geq 2$.

Proof. The proof is similar to the Theorem 4.1. □

Theorem 4.4 (Pseudo Bushell-Okrasiaski inequality: decreasing case)**.** *Let $f : [0,1] \to [a,b]$ be a continuous and decreasing function, and $\odot$ is represented by a convex and increasing multiplication generator g. If m is the same as in Theorem 3.1, then*

$$\int_{[0,1]}^{\sup} (1-t)^{s-1} \odot f^s(t) \odot dm \geq \frac{1}{s} \odot \left(\int_{[0,1]}^{\sup} f(t)dt\right)^s,$$

holds for all $s \geq 2$.

Proof. By Theorem 3.2 we have:

$$\begin{aligned}\int_{[0,1]}^{\sup} (1-t)^{s-1} \odot f^s(t) \odot dm &= \lim_{\lambda\to\infty} \int_{[0,1]}^{\oplus_\lambda} (1-t)^{s-1} \odot f^s(t) \odot dm_\lambda \\ &= \lim_{\lambda\to\infty} \left(g^\lambda\right)^{-1} \left(\int_0^1 g^\lambda((1-t)^{s-1} \odot f^s(t))dt\right).\end{aligned}$$

Using the Theorem 3.2 so we have

(4.3)

$$\begin{aligned}&\int_{[0,1]}^{\sup} (1-t)^{s-1} \odot f^s(t) \odot dm \\ &\quad\geq \lim_{\lambda\to\infty} \left[\left(g^\lambda\right)^{-1}\left(\int_0^1 g^\lambda\left((1-t)^{s-1}\right)dt\right) \odot \left(g^\lambda\right)^{-1}\left(\int_0^1 g^\lambda(f^s(t))dt\right)\right] \\ &\quad= \left[\lim_{\lambda\to\infty}\left(g^\lambda\right)^{-1}\int_0^1 g^\lambda\left((1-t)^{s-1}\right)dt\right) \odot \left(\lim_{\lambda\to\infty}\left(g^\lambda\right)^{-1}\int_0^1 g^\lambda((f^s(t))dt\right] \\ &\quad= \left(\int_{[0,1]}^{\sup} (1-t)^{s-1} \odot dm\right) \odot \left(\int_{[0,1]}^{\sup} f^s(t) \odot dm\right).\end{aligned}$$

Appling the Theorem 3.5, we obtain that:
(4.4)

$$\int_{[0,1]}^{\sup} (1-t)^{s-1} \odot f^s(t) \odot dm \geq \left(\int_{[0,1]}^{\sup} (1-t)^{s-1} \odot dm \right) \odot \left(\int_{[0,1]}^{\sup} f(t) \odot dm \right)^s .$$

Also we have:

$$\begin{aligned} \int_{[0,1]}^{\sup} (1-t)^{s-1} \odot dm &= \lim_{\lambda \to \infty} \left(\int_{[0,1]}^{\oplus_\lambda} (1-t)^{s-1} \odot dm_\lambda \right) \\ &= \lim_{\lambda \to \infty} \left(g^\lambda \right)^{-1} \left(\int_0^1 g^\lambda \left((1-t)^{s-1} \right) dt \right) \\ &\geq \lim_{\lambda \to \infty} \left(g^\lambda \right)^{-1} \left(g^\lambda \int_0^1 \left((1-t)^{s-1} \right) dt \right) \\ &= \lim_{\lambda \to \infty} \int_0^1 \left((1-t)^{s-1} \right) dt \\ &= \frac{1}{s} \end{aligned}$$

from (4.3) and (4.4) we have;

$$\int_{[0,1]}^{\sup} (1-t)^{s-1} \odot f^s(t) \odot dm \geq \frac{1}{s} \odot \left(\int_{[0,1]}^{\sup} f(t) dt \right)^s . \qquad \square$$

Example 4.5. Let $g^\lambda = e^{\lambda x}$ and $\psi(x)$ be the same as in Theorem 3.1, then

$$x \odot_\lambda y = x + y, \qquad \lim_{\lambda \to \infty} \left(\frac{1}{\lambda} \ln \left(e^{\lambda x} + e^{\lambda y} \right) \right) = \max(x, y).$$

Therefore B-O type inequality from Theorem 4.4 reduces on

$$\sup_{x \in [0,1]} \left[\left((1-x)^{s-1} + f^s(x) \right) + \psi(x) \right] \geq \frac{1}{s} + \left[\sup_{x \in [0,1]} \left(f(x) + \psi(x) \right) \right]^s .$$

Theorem 4.6. *(Pseudo Bushell-Okrasiaski inequality: increasing case) Let $f : [0,1] \to [a,b]$ be a continuous and increasing function, and $\odot$ is represented by a convex and increasing multiplication generator g. If m be the same as in Theorem 3.1, then*

$$\int_{[0,1]}^{\sup} t^{s-1} \odot f^s(t) \odot dm \geq \frac{1}{s} \odot \left(\int_{[0,1]}^{\sup} f(t) dt \right)^s ,$$

holds for all $s \geq 2$.

Proof. The proof is similar to Theorem 4.4. $\square$

Note that third important case $\oplus$ =max and $\odot$ =min has been studied in [49] and the pseudo-integrals in such a case yields the Sugeno integral.

5. Hardy Type Inequalities for Pseudo Integrals

Our purpose in this section is to prove a Hardy type inequality for pseudo-integrals.

Unfortunately, the following example shows that, the Hardy's integral inequality is not valid for the pseudo-integrals.

Example 5.1. Let $f(x) = k$ where $k > 1$ and $P \geq 1$. If $g : [0, 1] \to [0, 1]$ is defined as follows

$$g(x) = x.$$

Then by using Case II, we have

$$\begin{aligned}
\left(\int_{[0,1]}^{\oplus} f^P(x)dx\right)^{\frac{1}{P+1}} &= \left(\int_{[0,1]}^{\oplus} k^P dx\right)^{\frac{1}{P+1}} \\
&= \left(g^{-1}\int_0^1 g\left(k^P\right)dx\right)^{\frac{1}{P+1}} \\
&= \left(g^{-1}\int_0^1 k^P dx\right)^{\frac{1}{P+1}} \\
&= \left(g^{-1}\left(k^P\right)\right)^{\frac{1}{P+1}} \\
&= \left(k^P\right)^{\frac{1}{P+1}} \\
&= k^{\frac{P}{P+1}}.
\end{aligned}$$

Since

$$F(x) = \int_{[0,x]}^{\oplus} f(t)dt,$$

then by Case II we obtain that

$$\begin{aligned}
F(x) &= g^{-1}\int_0^x g(f(t))dt \\
&= g^{-1}\int_0^x g(k)dt \\
&= g^{-1}\int_0^x kdt \\
&= g^{-1}(kx) \\
&= kx.
\end{aligned}$$

It follows that

$$\frac{F(x)}{x} = k.$$

So by using (1.2) we have

$$\begin{aligned}\int_{[0,1]}^{\oplus} \left(\frac{F}{x}\right)^P dx &= g^{-1} \int_0^1 g\left(\frac{F}{x}\right)^P dx \\ &= g^{-1} \int_0^1 g(k^P) dx \\ &= g^{-1} \int_0^1 k^P dx \\ &= g^{-1}\left(k^P\right) \\ &= k^P.\end{aligned}$$

Consequently, (1.2) is not valid for pseudo-integrals.

In order to prove Theorems 5.4 and 5.6 we need some Lemmas.

Lemma 5.2. *If* $f : [0,1] \to [0,1]$ *is a* μ*-measurable function and* $g : [0,1] \to [0,1]$ *is a continuous and decreasing function, then*

$$\int_{[0,1]}^{\oplus} f^P d\mu \geq \left(\int_{[0,1]}^{\oplus} f d\mu\right)^P \tag{5.1}$$

holds for all $P \geq 1$.

Proof. By induction: For $P = 2$, inequality (5.1) is valid by Theorem 3.5.

For $P - 1$, we suppose that the Lemma is valid as follows

$$\int_{[0,1]}^{\oplus} f^{P-1} d\mu \geq \left(\int_{[0,1]}^{\oplus} f d\mu\right)^{P-1}.$$

Hence for P we have

$$\begin{aligned}\int_{[0,1]}^{\oplus} f^P d\mu &= \int_{[0,1]}^{\oplus} f \ldots f d\mu \\ &\geq \int_{[0,1]}^{\oplus} (f^{P-1}) f d\mu.\end{aligned}$$

So from case $P = 2$, we get

$$\int_{[0,1]}^{\oplus} f^P d\mu \geq \left(\int_{[0,1]}^{\oplus} f d\mu\right)^P.$$

Thereby, the Lemma is proved. □

Lemma 5.3. *Let $f : [0,1] \to [0,1]$ be a continuouse function. If m be the same as in Theorem 3.1 and $g : [0,1] \to [0,1]$ is a continuous and decreasing function, then*

$$\int_{[0,1]}^{\sup} f^P dm \geq \left(\int_{[0,1]}^{\sup} f dm\right)^P$$

holds for all $P \geq 1$.

Proof. Using the same arguments in Lemma 5.2. proof is easy. □

Theorem 5.4 (Pseudo Hardy's inequality)**.** *Let $f : [0,1] \to [0,1]$ be a μ-measurable and $g : [0,1] \to [0,1]$ be a continuous and decreasing function. If*

$$F(x) = \int_{[0,x]}^{\oplus} f(t)dt$$

where $x \in [0,1]$, then the inequality

$$\left(\frac{P}{P-1}\right)^P \int_{[0,1]}^{\oplus} f^P(x)dx > \int_{[0,1]}^{\oplus} \left(\frac{F}{x}\right)^P dx \tag{5.2}$$

holds for all $P > 1$.

Proof. By using Lemma 5.2? we have

$$\begin{aligned}
\int_{[0,1]}^{\oplus} \left(\frac{F}{x}\right)^P dx &= \int_{[0,1]}^{\oplus} \left(\frac{\int_{[0,x]}^{\oplus} f(t)dt}{x}\right)^P dx \\
&= \int_{[0,1]}^{\oplus} \frac{\left(\int_{[0,x]}^{\oplus} f(t)dt\right)^P}{x^P} dx \\
&\leq \int_{[0,1]}^{\oplus} \frac{\int_{[0,x]}^{\oplus} f^P(t)dt}{x^P} dx.
\end{aligned}$$

Thus, by Case II, we have

$$\begin{aligned}
\int_{[0,1]}^{\oplus} \left(\frac{F}{x}\right)^P dx &\leq \int_{[0,1]}^{\oplus} \frac{\int_{[0,x]}^{\oplus} f^P(t)dt}{x^P} dx \\
&= \int_{[0,1]}^{\oplus} \int_{[0,x]}^{\oplus} \left(\frac{f^P(t)}{x^P}\right) dtdx \\
&= g^{-1} \int_0^1 g \int_{[0,x]}^{\oplus} \left(\frac{f^P(t)}{x^P}\right) dtdx \\
&= g^{-1} \int_0^1 g \left(g^{-1} \int_0^x g \left(\frac{f^P(t)}{x^P}\right) dt\right) dx
\end{aligned}$$

$$= g^{-1}\int_0^1\int_0^x g\left(\frac{f(t)}{x}\right)^P dtdx$$
$$= g^{-1}\int_0^1\int_0^x g(f(t))^P g\left(\frac{1}{x^P}\right) dtdx$$
$$= g^{-1}\left(\int_0^1 g\left(\frac{1}{x^P}\right)dx\right)\left(\int_0^x g(f(t))^P dt\right).$$

Since $\frac{1}{x^P} > 1$ and g is a decreasing function, we have $g\left(\frac{1}{x^P}\right) < g(1)$, It follows that

$$\int_{[0,1]}^{\oplus}\left(\frac{F}{x}\right)^P dx \leq g^{-1}\left(\int_0^1 g\left(\frac{1}{x^P}\right)dx\right)\left(\int_0^x g(f(t))^P dt\right)$$
$$< g^{-1}\left(\int_0^1 g(1)dx\right)\left(\int_0^x g(f(t))^P dt\right)$$
$$= g^{-1}\left(gg^{-1}\int_0^1 g(1)dx\right)\left(gg^{-1}\int_0^x g(f(t))^P dt\right)$$
$$= \left(\int_{[0,1]}^{\oplus} 1dx\right)\odot\left(\int_{[0,x]}^{\oplus} g(f(t))^P dt\right).$$

By using Theorem(3.6 (ii)), we have

$$\int_{[0,1]}^{\oplus}\left(\frac{F}{x}\right)^P dx < \left(\int_{[0,1]}^{\oplus} 1dx\right)\odot\left(\int_{[0,x]}^{\oplus} g(f(t))^P dt\right)$$
$$< \left(\int_{[0,x]}^{\oplus} g(f(t))^P dt\right)$$
$$< \left(\int_{[0,1]}^{\oplus} g(f(x))^P dx\right)$$
$$< \left(\frac{P}{P-1}\right)^P \int_{[0,1]}^{\oplus} f^P(x)dx.$$

Which complete the proof. □

Example 5.5. Let $f(x) = \frac{1}{2}$ and $g : [0,1] \to [0,\infty]$ define as follows $g(x) = \frac{1}{x^2}$. By using Case II, we have

$$\int_{[0,1]}^{\oplus} f^P(x)dx = g^{-1}\int_0^1 g\left(f^P(x)\right)dx$$
$$= g^{-1}\int_0^1 g\left(\left(\frac{1}{2}\right)^P\right)dx$$

$$
\begin{aligned}
&= g^{-1} \int_0^1 \frac{1}{\left(\frac{1}{2^P}\right)^2} dx \\
&= g^{-1} \left(2^{2P}\right) \\
&= \frac{1}{\sqrt{2^{2p}}} \\
&= \frac{1}{2^P}.
\end{aligned}
$$

Then a straightforward calculus shows that

$$
\begin{aligned}
F(x) &= \int_{[0,x]}^{\oplus} f(t)dt \\
&= g^{-1} \int_0^x g\left(\frac{1}{2}\right) dt \\
&= g^{-1} \int_0^x 4dt \\
&= g^{-1}(4x) \\
&= \frac{1}{\sqrt{4x}} \\
&= \frac{1}{2\sqrt{x}}.
\end{aligned}
$$

It follows that,

$$
\frac{F(x)}{x} = \frac{1}{2x\sqrt{x}} = \frac{1}{2}x^{-\frac{3}{2}}
$$

On the other hand,

$$
\begin{aligned}
\int_{[0,1]}^{\oplus} \left(\frac{F}{x}\right)^p dx &= g^{-1} \int_0^1 g\left(\frac{F}{x}\right)^P dx \\
&= g^{-1} \int_0^1 g\left(\frac{1}{2}x^{-\frac{3}{2}}\right)^P dx \\
&= g^{-1} \int_0^1 \frac{2^{2P}}{x^{-3P}} dx \\
&= g^{-1} \left(\frac{2^{2P}}{3P+1}\right) \\
&= \frac{1}{\sqrt{\frac{2^{2P}}{3P+1}}}.
\end{aligned}
$$

This shows that the Hardy's inequality is valid for pseudo-integral.

Now, we generalize the Hardy type inequality by the semiring $([a,b], \max, \odot)$, where $\odot$ is generated.

Theorem 5.6. *Let $f:[0,1]\to[0,1]$ be a μ-measurable, $g:[0,1]\to[0,1]$ be a continuous and decreasing function and m be the same as in Theorem 3.1. If $\odot$ is represented by a decreasing multiplicative generator g and*

$$F(x)=\int_{[0,x]}^{\sup} f dm$$

where $x\in[0,1]$, then the inequality

$$\left(\frac{P}{P-1}\right)^{P}\int_{[0,1]}^{\sup} f^{P} dm > \int_{[0,1]}^{\sup}\left(\frac{F}{x}\right)^{P} dm \tag{5.3}$$

holds for all $P>1$.

Proof. By using Lemma 5.3 and Theorem 3.2 we have

$$\begin{aligned}
&\int_{[0,1]}^{\sup}\left(\frac{F}{x}\right)^{P} dm\\
&=\lim_{\lambda\to\infty}\int_{[0,1]}^{\oplus_\lambda}\left(\frac{F}{x}\right)^{P} dm_\lambda\\
&=\lim_{\lambda\to\infty}\left(g^{\lambda}\right)^{-1}\int_0^1 g^{\lambda}\left(\frac{F}{x}\right)^{P} dx\\
&=\lim_{\lambda\to\infty}\left(g^{\lambda}\right)^{-1}\int_0^1 g^{\lambda}\left(F(x)\right)^{P} g^{\lambda}\left(\frac{1}{x^{p}}\right) dx\\
&=\lim_{\lambda\to\infty}\left(g^{\lambda}\right)^{-1}\int_0^1 g^{\lambda}\left(\int_{[0,x]}^{\sup} f(t)dm\right)^{P} g^{\lambda}\left(\frac{1}{x^{p}}\right) dx\\
&=\lim_{\lambda\to\infty}\left(g^{\lambda}\right)^{-1}\int_0^1 g^{\lambda}\left(\lim_{\lambda\to\infty}\int_{[0,x]}^{\oplus_\lambda} f(t)dm_\lambda\right)^{P} g^{\lambda}\left(\frac{1}{x^{P}}\right) dx\\
&\le\lim_{\lambda\to\infty}\left(g^{\lambda}\right)^{-1}\int_0^1 g^{\lambda}\left(\lim_{\lambda\to\infty}\int_{[0,x]}^{\oplus_\lambda} f^{P}(t)dm_\lambda\right) g^{\lambda}\left(\frac{1}{x^{P}}\right) dx\\
&=\lim_{\lambda\to\infty}\left(g^{\lambda}\right)^{-1}\int_0^1 g^{\lambda}\left(\lim_{\lambda\to\infty}\left(g^{\lambda}\right)^{-1}\int_0^x g^{\lambda}\left(f^{P}(t)\right)dt\right) g^{\lambda}\left(\frac{1}{x^{P}}\right) dx\\
&=\lim_{\lambda\to\infty}\lim_{\lambda\to\infty}\left(g^{\lambda}\right)^{-1}\int_0^1\int_0^x g^{\lambda}\left(g^{\lambda}\right)^{-1} g^{\lambda}\left(f^{P}(t)\right) g^{\lambda}\left(\frac{1}{x^{P}}\right) dtdx.
\end{aligned}$$

Thus, we conclude

$$\int_{[0,1]}^{\sup}\left(\frac{F}{x}\right)^{P} dm$$

$$\leq \lim_{\lambda\to\infty}\lim_{\lambda\to\infty}\left(g^{\lambda}\right)^{-1}\int_0^1\int_0^x g^{\lambda}\left(f^P(t)\right)g^{\lambda}\left(\frac{1}{x^P}\right)dtdx$$
$$=\left(\lim_{\lambda\to\infty}\left(g^{\lambda}\right)^{-1}\int_0^x g^{\lambda}\left(f^P(t)\right)dt\right)\left(\lim_{\lambda\to\infty}\left(g^{\lambda}\right)^{-1}\int_0^1 g^{\lambda}\left(\frac{1}{x^P}\right)dx\right).$$

Since $\frac{1}{x^P}>1, g$ is a decreasing function and $\lambda\in(0,\infty)$, so we have

$$g^{\lambda}\left(\frac{1}{x^P}\right)<g^{\lambda}(1),$$

then

$$\int_{[0,1]}^{\sup}\left(\frac{F}{x}\right)^P dm$$
$$\leq\left(\lim_{\lambda\to\infty}\left(g^{\lambda}\right)^{-1}\int_0^x g^{\lambda}\left(f^P(t)\right)dt\right)\left(\lim_{\lambda\to\infty}\left(g^{\lambda}\right)^{-1}\int_0^1 g^{\lambda}\left(\frac{1}{x^P}\right)dx\right)$$
$$<\left(\lim_{\lambda\to\infty}\left(g^{\lambda}\right)^{-1}\int_0^x g^{\lambda}\left(f^P(t)\right)dt\right)\left(\lim_{\lambda\to\infty}\left(g^{\lambda}\right)^{-1}\int_0^1 g^{\lambda}(1)dx\right)$$
$$<\left(\lim_{\lambda\to\infty}\int_{[0,x]}^{\oplus_\lambda}\left(f^P(t)\right)dm\right)\left(\lim_{\lambda\to\infty}\int_{[0,1]}^{\oplus_\lambda}(1)dm\right)$$
$$<\left(\int_{[0,x]}^{\sup}f^P(t)dm\right)$$
$$<\left(\int_{[0,1]}^{\sup}f^P(x)dm\right)$$
$$<\left(\frac{P}{P-1}\right)^P\int_{[0,1]}^{\sup}f^P(x)dm.$$

Which complete the proof. □

Example 5.7. Let $f:[0,1]\to[0,1]$ be a μ-measurable, and $g^{\lambda}(x)=x^{-\lambda}$. So

$$x\oplus y=\left(x^{-\lambda}+y^{-\lambda}\right)^{-\lambda},\qquad x\odot y=xy.$$

Therefore Relation (5.3) reduces on the following inequality:

$$\sup\left(\left(\frac{F}{x}\right)^P+\psi(x)\right)<\left(\frac{P}{P-1}\right)^P\sup\left(f^p(x)+\psi(x)\right).$$

where ψ is the same as in Theorem 3.1.

6. Minkowsky Type Inequality for Pseudo Integrals

Our purpose in this section is to prove the Minkowski's inequality derived from 1.3 for the pseudo-integrals.

Theorem 6.1 (Pseudo Minkowski's inequality, decreasing case)**.** *Let $f, h : [0,1] \to [0,1]$ be continuous and strictly decreasing functions and μ be the Lebesgue measure on $\mathbb{R}$. If the pseudo-operations are defined by a continuous and decreasing $g : [0,1] \to [0,\infty]$ and functions satisfying*

$$0 < m \leq \frac{f(x)}{h(x)} \leq M, \quad \forall x \in [0,1]$$

then the inequality

(6.1)

$$\left(\int_{[0,1]}^{\oplus} f(x)^s d\mu\right)^{\frac{1}{s}} + \left(\int_{[0,1]}^{\oplus} h(x)^s d\mu\right)^{\frac{1}{s}} \leq 2c\left(\int_{[0,1]}^{\oplus} (f(x)+h(x))^s d\mu\right)^{\frac{1}{s}},$$

holds, where $1 \leq s < \infty$, *and* $c = \dfrac{M(m+1)+(M+1)}{(m+1)(M+1)}$.

Proof. Since $\frac{f(x)}{g(x)} \leq M$, $f(x) \leq M(f(x)+g(x)) - Mf(x)$. Therefore

$$(M+1)^s f(x)^s \leq M^s (f(x)+g(x))^s$$

and so,

$$f(x)^s \leq \frac{M^s}{(M+1)^s}(f(x)+g(x))^s.$$

Now from Theorem 3.6 (iii),

$$\left(\int_{[0,1]}^{\oplus} f(x)^s d\mu\right)^{\frac{1}{s}} \leq \left(\int_{[0,1]}^{\oplus} (\frac{M}{M+1})^s (f(x)+g(x))^s d\mu\right)^{\frac{1}{s}} \tag{6.2}$$

Since $\dfrac{M}{M+1} < 1$, from Relation (6.2), we have

$$\left(\int_{[0,1]}^{\oplus} f(x)^s d\mu\right)^{\frac{1}{s}} \leq \left(\int_{[0,1]}^{\oplus} (f(x)+g(x))^s d\mu\right)^{\frac{1}{s}}. \tag{6.3}$$

On the other hand, since $mg(x) \leq f(x)$, Hence

$$g(x) \leq \frac{1}{m}(f(x)+g(x)) - \frac{1}{m}g(x).$$

Therefore,

$$\left(\frac{1}{m}+1\right)^s g(x)^s \leq \left(\frac{1}{m}\right)^s (f(x)+g(x))^s$$

and so, from Lemma 3.7 (ii),

$$\left(\int_{[0,1]}^{\oplus} g(x)^s d\mu\right)^{\frac{1}{s}} \le \left(\int_{[0,1]}^{\oplus} \left(\frac{1}{m+1}\right)^s (f(x)+g(x))^s d\mu\right)^{\frac{1}{s}}. \tag{6.4}$$

Since $\dfrac{1}{m+1} < 1$, from Relation (6.4) we have

$$\left(\int_{[0,1]}^{\oplus} g(x)^s d\mu\right)^{\frac{1}{s}} \le \left(\int_{[0,1]}^{\oplus} (f(x)+g(x))^s d\mu\right)^{\frac{1}{s}}. \tag{6.5}$$

Now by adding the Inequalities (6.3) and (6.5) we have

$$\begin{aligned}\left(\int_{[0,1]}^{\oplus} f(x)^s d\mu\right)^{\frac{1}{s}} + \left(\int_{[0,1]}^{\oplus} g(x)^s d\mu\right)^{\frac{1}{s}} &\le 2\left(\int_{[0,1]}^{\oplus} (f(x)+g(x))^s d\mu\right)^{\frac{1}{s}} \\ &\le 2c\left(\int_{[0,1]}^{\oplus} (f(x)+g(x))^s d\mu\right)^{\frac{1}{s}}.\end{aligned}$$

The proof is now complete. □

Example 6.2. Let $f, h : [0,1] \to [0,1]$ be two real valued functions as $f(x) = -x + \frac{1}{2}$, $h(x) = -x + \frac{3}{2}$ and μ be the Lebesgue measure on $\mathbb{R}$. Let $s = 1, g(x) = -x$, A straightforward calculus shows that $0 < \frac{4}{3} \le \frac{f}{g} \le 2$. Since

$$\begin{aligned}\text{(i)} \int_{[0,1]}^{\oplus} f(x) d\mu &= g^{-1} \int_0^1 g(f(x)) d\mu \\ &= g^{-1} \int_0^1 -\left(-x + \frac{1}{2}\right) d\mu \\ &= g^{-1} \int_0^1 \left(x - \frac{1}{2}\right) d\mu \\ &= g^{-1} \left(\frac{1}{2}x^2 - \frac{1}{2}x \Big|_0^1\right) \\ &= g^{-1}(0) \\ &= 0,\end{aligned}$$

$$\begin{aligned}\text{(ii)} \int_{[0,1]}^{\oplus} h(x) d\mu &= g^{-1} \int_0^1 g(h(x) d\mu \\ &= g^{-1} \int_0^1 -\left(-x + \frac{3}{2}\right) d\mu\end{aligned}$$

$$
\begin{aligned}
&= g^{-1} \int_0^1 \left(x - \frac{3}{2} \right) d\mu \\
&= g^{-1} \left(\frac{1}{2} x^2 - \frac{3}{2} x \Big|_0^1 \right) \\
&= g^{-1}(-1) \\
&= 1
\end{aligned}
$$

and

$$
\begin{aligned}
\text{(iii)} \int_{[0,1]}^{\oplus} ((f+h)(x))\, d\mu &= g^{-1} \int_0^1 g\left((f+h)(x)\right) d\mu \\
&= g^{-1} \int_0^1 g(-2x+2) d\mu \\
&= g^{-1} \int_0^1 (2x-2) d\mu \\
&= g^{-1} \left(x^2 - 2x \big|_0^1 \right) \\
&= g^{-1}(-1) \\
&= 1.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
1 &= 0 + 1 \\
&= \left(\int_{[0,1]}^{\oplus} f d\mu \right) + \left(\int_{[0,1]}^{\oplus} g d\mu \right) \\
&\leq 2c \left(\int_{[0,1]}^{\oplus} (f+g) d\mu \right) \\
&\leq 2 \times c \times 1 \\
&= 2c.
\end{aligned}
$$

Theorem 6.3 (Pseudo Minkowski inequality, increasing case)**.** *Let* $f, h : [0,1] \to [0,1]$ *be continuous and strictly increasing functions and* μ *be the Lebesgue measure on* $\mathbb{R}$*. If the pseudo-operations are defined by a continuous and increasing* $g : [0,1] \to [0,1]$ *and functions satisfying*

$$
0 < m \leq \frac{f(x)}{h(x)} \leq M, \quad \forall x \in [0,1]
$$

then the inequality

(6.6)

$$\left(\int_{[0,1]}^{\oplus} f(x)^s d\mu\right)^{\frac{1}{s}} + \left(\int_{[0,1]}^{\oplus} h(x)^s d\mu\right)^{\frac{1}{s}} \leq 2c\left(\int_{[0,1]}^{\oplus} (f(x)+h(x))^s d\mu\right)^{\frac{1}{s}},$$

holds, where $1 \leq s < \infty$ *and* $c = \dfrac{M(m+1)+(M+1)}{(m+1)(M+1)}$.

Proof. Using the same argument in Theorem 6.1. proof is obvious. □

Now we generalize the Minkowski type inequality by the semiring $([0,1], \max, \odot)$, where $\odot$ is generated.

Theorem 6.4. *Let* $f, h : [0,1] \to [0,1]$ *be continuous and strictly decreasing functions and let* m *be the same as in Theorem 3.1. If* $\odot$ *is represented by an decreasing multiplicative generator* g *and functions satisfying*

$$0 < m \leq \frac{f(x)}{h(x)} \leq M, \quad \forall x \in [0,1]$$

then the inequality

(6.7)

$$\left(\int_{[0,1]}^{\sup} f^s \odot dm\right)^{\frac{1}{s}} + \left(\int_{[0,1]}^{\sup} h^s \odot dm\right)^{\frac{1}{s}} \leq 2c\left(\int_{[0,1]}^{\sup} (f+h)^s \odot dm\right)^{\frac{1}{s}},$$

holds, where $1 \leq s < \infty$ *and* $c = \dfrac{M(m+1)+(M+1)}{(m+1)(M+1)}$.

Proof. Since $\frac{f(x)}{g(x)} \leq M, f \leq M(f(x)+g(x)) - Mf(x)$. Therefore

$$(M+1)^s f(x)^s \leq M^s (f(x)+g(x))^s$$

and so,

$$f(x)^s \leq \frac{M^s}{(M+1)^s}(f(x)+g(x))^s.$$

Now,

$$\left(\int_{[0,1]}^{\oplus_\lambda} f(x)^s \odot dm\right)^{\frac{1}{s}} \leq \left(\int_{[0,1]}^{\oplus_\lambda} (\frac{M}{M+1})^s (f(x)+g(x))^s \odot dm\right)^{\frac{1}{s}}.$$

Since $\dfrac{M}{M+1} < 1$, so

$$\left(\int_{[0,1]}^{\oplus_\lambda} f(x)^s \odot dm\right)^{\frac{1}{s}} \leq \left(\int_{[0,1]}^{\oplus_\lambda} (f(x)+g(x))^s \odot dm\right)^{\frac{1}{s}}.$$

It follows that

$$\left(\lim_{\lambda\to\infty}\int_{[0,1]}^{\oplus_\lambda} f(x)^s \odot dm\right)^{\frac{1}{s}} \leq \left(\lim_{\lambda\to\infty}\int_{[0,1]}^{\oplus_\lambda} (f(x)+g(x))^s \odot dm\right)^{\frac{1}{s}}.$$

Finally,

$$\left(\int_{[0,1]}^{\sup} f(x)^s \odot dm\right)^{\frac{1}{s}} \leq \left(\int_{[0,1]}^{\sup} (f(x)+g(x))^s \odot dm\right)^{\frac{1}{s}}. \tag{6.8}$$

On the other hand, since $mg(x) \leq f(x)$, hence

$$g(x) \leq \frac{1}{m}(f(x)+g(x)) - \frac{1}{m}g(x).$$

Therefore,

$$\left(\frac{1}{m}+1\right)^s g(x)^s \leq \left(\frac{1}{m}\right)^s (f(x)+g(x))^s$$

and so,

$$\left(\int_{[0,1]}^{\oplus_\lambda} g(x)^s \odot dm\right)^{\frac{1}{s}} \leq \left(\int_{[0,1]}^{\oplus_\lambda} \left(\frac{1}{m+1}\right)^s (f(x)+g(x))^s \odot dm\right)^{\frac{1}{s}}.$$

Since $\frac{1}{m+1} < 1$, so

$$\left(\int_{[0,1]}^{\oplus_\lambda} g(x)^s \odot dm\right)^{\frac{1}{s}} \leq \left(\int_{[0,1]}^{\oplus_\lambda} (f(x)+g(x))^s \odot dm\right)^{\frac{1}{s}}.$$

It follows that

$$\left(\lim_{\lambda\to\infty}\int_{[0,1]}^{\oplus_\lambda} g(x)^s \odot dm\right)^{\frac{1}{s}} \leq \left(\lim_{\lambda\to\infty}\int_{[0,1]}^{\oplus_\lambda} (f(x)+g(x))^s \odot dm\right)^{\frac{1}{s}}.$$

Finally,

$$\left(\int_{[0,1]}^{\sup} g(x)^s \odot dm\right)^{\frac{1}{s}} \leq \left(\int_{[0,1]}^{\sup} (f(x)+g(x))^s \odot dm\right)^{\frac{1}{s}} \tag{6.9}$$

Now by adding the Inequalities (6.8) and (6.9):

$$\left(\int_{[0,1]}^{\sup} f(x)^s \odot dm\right)^{\frac{1}{s}} + \left(\int_{[0,1]}^{\sup} g(x)^s \odot dm\right)^{\frac{1}{s}}$$
$$\leq 2\left(\int_{[0,1]}^{\sup} (f(x)+g(x))^s \odot dm\right)^{\frac{1}{s}}$$

$$\leq 2c\left(\int_{[0,1]}^{\sup}(f(x)+g(x))^s\odot dm\right)^{\frac{1}{s}}.$$

The proof is now complete. □

Example 6.5. Let $f,h:[0,1]\to[0,\infty)$ be a μ-measurable, and $g^{\lambda}(x)=x^{-\lambda}$. So

$$x\oplus y=(x^{-\lambda}+y^{-\lambda})^{-\lambda},\qquad x\odot y=xy.$$

Therefore Relation (6.7) reduces on the following inequality:

$$\sup\left((f(x)^s)^{\frac{1}{s}}+\psi(x)\right)+\sup\left((h(x)^s)^{\frac{1}{s}}+\psi(x)\right)$$
$$\leq 2c\sup\left((f+h)^s(x)+\psi(x)\right).$$

where ψ is defined same as in Theorem 3.1.

Theorem 6.6. *Let $f,h:[0,1]\to[0,\infty)$ be continuous and strictly increasing functions and let m be the same as in Theorem 3.1. If $\odot$ is represented by an increasing multiplicative generator g and functions satisfying*

$$0<m\leq\frac{f}{h}\leq M,\quad\forall x\in[0,1]$$

then the inequality

(6.10)

$$\left(\int_{[0,1]}^{\sup}f^s\odot dm\right)^{\frac{1}{s}}+\left(\int_{[0,1]}^{\sup}h^s\odot dm\right)^{\frac{1}{s}}\leq 2c\left(\int_{[0,1]}^{\sup}(f+h)^s\odot dm\right)^{\frac{1}{s}},$$

holds, where $1\leq s<\infty$ and $c=\dfrac{M(m+1)+(M+1)}{(m+1)(M+1)}$.

Proof. The proof is similar to Theorem 6.4. □

Note that third important case $\oplus$ =max and $\odot$ =min has been studied in [2] and the pseudo-integrals in such a case yields the Sugeno integral.

7. Conclusion

We have proved the Bushell-Okrasiaski, Hardy and Minkowski type inequalities for the pseudo-integral for two characteristic cases: generated and max-plus i.e. The first class is including the pseudo-integral based on a function reduces on the $g-$integral, where pseudo-addition and pseudo-multiplication are defined by a monotone and continuous function g. The second class is including the pseudo-integral based on the semiring $([a,b],\max,\odot)$ is given by $\sup-$measure, where $x\odot y$ is generated by $g^{-1}(g(x)g(y))$. For further investigation we continue to explore other integral inequalities for fuzzy integrals.

Open Problem

Dose the Bushell-Okrasiaski, Hardy and Minkowski type inequalities hold for the Chaquet integral?

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