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Rinku Deyand Binod Chandra Tripathy

Sahand Communications in Mathematical Analysis

Print ISSN: 2322-5807
Online ISSN: 2423-3900
Volume: 19
Number: 4
Pages: 25-38

Sahand Commun. Math. Anal.
DOI: 10.22130/scma.2022.538729.986

Sahand Communications in Mathematical Analysis (SCMA) Vol. 19 No. 4 (2022), 25-38
http://scma.maragheh.ac.ir
DOI: 10.22130/scma.2022.538729.986

On a Class of Sequences Related to p-Absolutely Summable Sequences in Metric Space Defined by Orlicz Functions

Rinku Dey[1*] and Binod Chandra Tripathy[2]

Abstract. In this article we have introduced the sequence space $m(\phi, d)$ and $m(M, \phi, d)$ of W. L. C. Sargent type in a metric space (X, d) on generalising the sequence space $m(\phi)$ and we have defined these sequence spaces using the Orlicz function M. We have investigated their different properties like solidness, symmetricity, monotone, sequence algebra, completeness etc. We have established some inclusion results involving the space $m(M, \phi, d)$ and some of the existing sequence spaces. We have provided suitable examples and discussed in detail, in order to justify the failure cases and the definitions we have introduced. The results established in this article generalized and unifies several existing results.

1. Introduction

Throughout the article $w(X)$, $\ell_\infty(X)$, $\ell^p(X)$, $c(X)$ and $c_0(X)$ denote the spaces of all, bounded, p-absolutely summable, convergent and null sequences respectively with elements in X, where (X, d) is a metric space and d is the distance function defined on X. The zero sequence is denoted by $\bar{\theta} = (\theta)$, where θ is the zero element of X. Different classes of sequences spaces have been studied by Hazarika and Esi [7], Tripathy et al. [18], Ercan [4], Shahraki and Ledari [16], Guler [5] in recent years.

The sequence space $m(\phi)$ was introduced by Sargent [15], who studied some of its properties and obtained its relationship with the space ℓ^p. Later on it was investigated from sequence space point of view and related with summability theory by Bilgin [1], Rath and Tripathy [13], Tripathy [17], Tripathy and Mahanta [22] and others.

2020 *Mathematics Subject Classification.* 40A05, 40A30, 40F05, 46E30.
Key words and phrases. Metric space, Orlicz function, p-absolutely summable sequences, Solid space, Symmetric space.
Received: 11 September 2021, Accepted: 06 March 2022.
* Corresponding author.

An Orlicz function is a function $M : [0, \infty] \to [0, \infty]$, which is continuous, non-decreasing and convex with $M(0) = 0$, $M(x) > 0$, for $x > 0$, and $M(x) \to \infty$, $x \to \infty$.

An Orlicz function is a function M is said to satisfy Δ_2-condition for all values of x, if there exists a constant $k > 0$, such that $M(kx) \leq kM(x)$ for all $x \geq 0$. The Δ_2-condition is equivalent to $M(lx) \leq lM(x)$ for all $x > 0$ and for $l > 1$.

An Orlicz function is a function M can always be represented in the following integral form: $M(x) = \int_0^x \eta(t)dt$, where η is known as the kernel of M, is right differentiable for $t \geq 0$, $\eta(0) = 0$, $\eta(t) > 0$, for $t > 0$, η is non-decreasing and $\eta(t) \to \infty$ as $t \to \infty$.

If the convexity of the Orlicz function is replaced by $M(x + y) \leq M(x) + M(y)$, then this function is called as *modulus function*, introduced by Nakano [10], and further studied by Ruckle [14] Maddox [9] and others.

Remark 1.1. An Orlicz function satisfies the inequality $M(\lambda x) \leq \lambda M(x)$, for all λ with $0 < \lambda < 1$.

2. Definitions and Background

Let P_s denote the class of all subsets of N, those do not contain more than s elements. Throughout $\{\phi_n\}$ represents a non-decreasing sequence of real numbers such that $n\phi_{n+1} \leq (n + 1)\phi_n$ for all $n \in N$.

The sequence space $m(\phi)$ introduced by Sargent [15] is defined as follows:

$$m(\phi) = \left\{ (x_k) \in w : ||x_k||_{m(\phi)} = \sup_{s\geq 1, \sigma \in P_s} \frac{1}{\phi_s} \sum_{k\in\sigma} |x_k| < \infty \right\}.$$

Lindenstrauss and Tzafriri [8] used the notion of Orlicz function and introduced the sequence space

$$\ell_M = \left\{ (x_k) \in w : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) < \infty, \text{for some} \rho > 0 \right\}.$$

The space ℓ_M with the norm

$$\|(x_k)\| = \inf \left\{ \rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) \leq 1 \right\},$$

becomes a Banach space, which is called an Orlicz sequence space. The space ℓ_M is closely related to the space ℓ_p, which is an Orlicz sequence space with $M(x) = x^p$, for $1 \leq p \leq \infty$.

In the later stage different Orlicz sequence spaces were introduced and studied by Hazarika [6], Parashar and Choudhary [12], Et. et al. [3], Esi and Et [2], Nath and Tripathy [11], Tripathy and Dutta [19], Tripathy and Goswami [20], Tripathy and Hazarika [21] and many others.

The concept of metric space has been applied by various authors to introduce new classes of sequences, those generalize and unify different notions of real and complex sequence space.

On generalizing the notion of $m(\phi)$ space due to Sargent [15], we introduce the following classes of sequences in this article.

$$m(\phi, d) = \left\{ (x_k) \in w : \sup_{s \geq 1, \sigma \in P_s} \frac{1}{\phi_s} \sum_{k \in \sigma} d(x_k, \theta) < \infty \right\}.$$

$$\ell_\infty(M, d) = \left\{ (x_k) \in w : \sup_{k \geq 1} M\left(\frac{d(x_k, \theta)}{\rho} \right) < \infty, \text{ for some } \rho > 0 \right\}.$$

$$\ell_1(M, d) = \left\{ (x_k) \in w : \sum_{k=1}^{\infty} M\left(\frac{d(x_k, \theta)}{\rho} \right) < \infty, \text{ for some } \rho > 0 \right\}.$$

$$c(M, d) = \left\{ (x_k) \in w : \lim_{k \to \infty} M\left(\frac{d(x_k, L)}{\rho} \right) = 0, \text{ for some } \rho > 0 \text{ and} L \in X \right\}.$$

$$c_0(M, d) = \left\{ (x_k) \in w : \lim_{k \to \infty} M\left(\frac{d(x_k, \theta)}{\rho} \right) = 0, \text{for some } \rho > 0 \right\}.$$

$$m(M, \phi, d) = \left\{ (x_k) \in w : \sup_{s \geq 1, \sigma \in P_s} \frac{1}{\phi_s} \sum_{k \in \sigma} M\left(\frac{d(x_k, \theta)}{\rho} \right) < \infty, \text{for some } \rho > 0 \right\}.$$

Definition 2.1. A sequence space E is said to be solid (or normal) if $(\alpha_k x_k) \in E$, whenever $(x_k) \in E$, for all sequence (α_k) of scalars such that $|\alpha_k| \leq 1$ for all $k \in N$.

Definition 2.2. A sequence space E is said to be symmetric if $(x_k) \in E$ implies $(x_{\pi(k)}) \in E$, where $\pi(k)$ is a permutation of the elements of N.

Definition 2.3. A sequence space E is said to be monotone if E contains the canonical pre images of all its step spaces.

The following Lemma will be used for establishing the result of this article.

Lemma 2.4. *A sequence space E is solid implies E is monotone.*

3. Main Results

In this section we prove some results involving the sequence spaces $m(\phi, d)$, $\ell_\infty(M, d)$, $\ell_1(M, d)$ and $m(M, \phi, d)$.

Theorem 3.1. *The space $m(\phi, d)$ is a linear space.*

Proof. Given

$$m(\phi, d) = \left\{ (x_k) \in w : \sup_{s \geq 1, \sigma \in P_s} \frac{1}{\phi_s} \sum_{k \in \sigma} d(x_k, \theta) < \infty \right\}.$$

Let $(x_k), (y_k) \in m(\phi, d)$ and $\alpha, \beta \in C$.

Then,

$$\sup_{s\geq 1, \sigma\in P_s} \frac{1}{\phi_s} \sum_{k\in\sigma} d(x_k, \theta) < \infty,$$

and

$$\sup_{s\geq 1, \sigma\in P_s} \frac{1}{\phi_s} \sum_{k\in\sigma} d(y_k, \theta) < \infty.$$

Now,

$$\begin{aligned}\sum_{k\in\sigma} d(\alpha x_k + \beta y_k, \theta) &\leq \sum_{k\in\sigma} d(\alpha x_k, \theta) + \sum_{k\in\sigma} d(\beta y_k, \theta) \\ &\leq |\alpha| \sum_{k\in\sigma} d(x_k, \theta) + |\beta| \sum_{k\in\sigma} d(x_k, \theta).\end{aligned}$$

Then

$$\begin{aligned}\sup_{s\geq 1, \sigma\in P_s} \frac{1}{\phi_s} \sum_{k\in\sigma} d(\alpha x_k + \beta y_k, \theta) \leq |\alpha| &\sup_{s\geq 1, \sigma\in P_s} \frac{1}{\phi_s} \sum_{k\in\sigma} d(x_k, \theta) \\ &+ |\beta| \sup_{s\geq 1, \sigma\in P_s} \frac{1}{\phi_s} \sum_{k\in\sigma} d(x_k, \theta).\end{aligned}$$

Therefore, $(\alpha x_k + \beta y_k) \in m(\phi, d)$. Hence, $m(\phi, d)$ is a linear space. □

Theorem 3.2. *The spaces $Z(M, d)$ for $Z = \ell_\infty, \ell_1, c, c_0$ are linear spaces.*

Proof. Let $(x_k), (y_k) \in \ell_\infty(M, d)$ and $\alpha, \beta \in C$.

Then, there exist positive numbers ρ_1 and ρ_2 such that

$$\sup_{k\geq 1} M\left(\frac{d(x_k, \theta)}{\rho_1}\right) < \infty$$

and

$$\sup_{k\geq 1} M\left(\frac{d(x_k, \theta)}{\rho_2}\right) < \infty.$$

Let, $\rho = \max(2|\alpha|\rho_1, 2|\beta|\rho_2)$.

Since, M is a non-decreasing convex function and d is the norm distance function, we have

$$\begin{aligned}M\left[\frac{d(\alpha x_k + \beta y_k, \theta)}{\rho}\right] &\leq M\left[\frac{|\alpha| d(x_k, \theta) + |\beta| d(y_k, \theta)}{\rho}\right] \\ &= M\left[\frac{|\alpha|}{\rho} d(x_k, \theta) + \frac{|\beta|}{\rho} d(y_k, \theta)\right] \\ &\leq M\left[\frac{d(x_k, \theta)}{\rho_1}\right] + M\left[\frac{d(y_k, \theta)}{\rho_2}\right]\end{aligned}$$

where $\rho_1 = \frac{|\alpha|}{\rho}$, $\rho_2 = \frac{|\beta|}{\rho}$. Then

$$\sup_{k\geq 1} M\left[\frac{d(\alpha x_k + \beta y_k, \theta)}{\rho}\right] \leq \sup_{k\geq 1} M\left[\frac{d(x_k,\theta)}{\rho_1}\right] + \sup_{k\geq 1} M\left[\frac{d(y_k,\theta)}{\rho_2}\right]$$

therefore $(\alpha x_k + \beta y_k) \in \ell_\infty(M,d)$. Hence, $\ell_\infty(M,d)$ is a linear space. □

The proof for the case $Z = \ell_1, c, c_0$ is a routine work in view of the above proof.

Theorem 3.3. *The sequence space $\ell_1(M,d)$ is a semi-normed space, by*

$$f((x_k)) = \inf\left\{\rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{d(x_k,\theta)}{\rho}\right) \leq 1\right\}.$$

Proof. Clearly, $f((x_k)) \geq 0$, for all $(x_k) \in \ell_1(M,d)$ and $f(\bar{\theta}) = 0$.

Let $x = (x_k), y = (y_k) \in \ell_1(M,d)$. Then, there exist $\rho_1 > 0$ and $\rho_2 > 0$ be such that

$$\sum_{k=1}^{\infty} M\left(\frac{d(x_k,\theta)}{\rho_1}\right) \leq 1$$

and

$$\sum_{k=1}^{\infty} M\left(\frac{d(y_k,\theta)}{\rho_2}\right) \leq 1.$$

Let, $\rho = \rho_1 + \rho_2$. Since d is the norm distance function so we have,

$$\begin{aligned}
&\sum_{k=1}^{\infty} M\left(\frac{d(x_k + y_k,\theta)}{\rho}\right) \\
&\quad \leq \sum_{k=1}^{\infty} M\left(\frac{d(x_k,\theta) + d(y_k,\theta)}{\rho_1+\rho_2}\right) \\
&\quad \leq \sum_{k=1}^{\infty} \left\{\frac{\rho_1}{\rho_1+\rho_2} M\left(\frac{d(x_k,\theta)}{\rho_1}\right) + \frac{\rho_2}{\rho_1+\rho_2} M\left(\frac{d(y_k,\theta)}{\rho_2}\right)\right\}. \\
&\quad \leq \left(\frac{\rho_1}{\rho_1+\rho_2}\right) \sum_{k=1}^{\infty} M\left(\frac{d(x_k,\theta)}{\rho_1}\right) + \left(\frac{\rho_2}{\rho_1+\rho_2}\right) \sum_{k=1}^{\infty} M\left(\frac{d(y_k,\theta)}{\rho_2}\right) \\
&\quad \leq 1.
\end{aligned}$$

Since, $\rho > 0$, $\rho_1 > 0$ and $\rho_2 > 0$, so we have,

$$\begin{aligned}
&\inf\left\{\rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{d(x_k + y_k,\theta)}{\rho}\right) \leq 1\right\} \\
&\quad \leq \left\{\inf\left\{\rho_1 > 0 : \sum_{k=1}^{\infty} M\left(\frac{d(x_k,\theta)}{\rho_1}\right) \leq 1\right\}\right.
\end{aligned}$$

$$+\inf\left\{\rho_2>0:\sum_{k=1}^{\infty}M\left(\frac{d(y_k,\theta)}{\rho_2}\right)\le 1\right\}\Bigg\}.$$

Then $f(x+y)\le f(x)+f(y)$.

Let $\lambda\in C$. Without loss of generality, suppose $\lambda\neq 0$, then

$$\begin{aligned} f(\lambda x) &= f(\lambda(x_k))\\ &=\inf\left\{\rho>0:\sum_{k=1}^{\infty}M\left(\frac{d(\lambda x_k,\theta)}{\rho}\right)\le 1\right\}\\ &=\inf\left\{|\lambda|r>0:\sum_{k=1}^{\infty}M\left(\frac{d(x_k,\theta)}{r}\right)\le 1\right\},\quad \text{where } r=\frac{\rho}{|\lambda|}\\ &=|\lambda|\inf\left\{r>0:\sum_{k=1}^{\infty}M\left(\frac{d(x_k,\theta)}{r}\right)\le 1\right\}\\ &=|\lambda|f(x). \end{aligned}$$

□

The proof of the following result is a consequence of the above theorem.

Proposition 3.4. *The spaces $Z(M,d)$ for $Z=\ell_\infty,\ell_1$ are semi normed spaces by*

$$p((x_k))=\inf\left\{\rho>0:\sup_k M\left(\frac{d(x_k,\theta)}{\rho}\right)\le 1\right\}.$$

The proof of the following theorem is a consequence of Theorem 3.1 and Theorem 3.2.

Theorem 3.5. *The space $m(M,\phi,d)$ is a linear space.*

The proof of the following theorem is a consequence of Theorem 3.3.

Theorem 3.6. *The space $m(M,\phi,d)$ is a semi normed space by*

$$g((x_k))=\left\{\rho>0:\sup_{s\ge 1,\sigma\in P_s}\frac{1}{\phi_s}\sum_{k\in\sigma}M\left(\frac{d(x_k,\theta)}{\rho}\right)\le 1\right\}$$

Theorem 3.7. *The space $m(M,\rho,d)$ is solid and symmetric.*

Proof. Let $(x_k)\in m(M,\rho,d)$.

Then,

$$\sup_{s\ge 1,\sigma\in P_s}\frac{1}{\phi_s}\sum_{k\in\sigma}M\left(\frac{d(x_k,\theta)}{\rho}\right)<\infty,\quad \text{for some}\rho>0.$$

Let, (α_k) be a sequence of scalars with $|\alpha_k|\le 1$, for all $k\in N$.

Now,

$$\begin{aligned}\sup_{s\geq 1,\sigma\in P_s}\frac{1}{\phi_s}\sum_{k\in\sigma}M\left(\frac{d(\alpha_k x_k,\theta)}{\rho}\right) &\leq \sup_{s\geq 1,\sigma\in P_s}\frac{1}{\phi_s}\sum_{k\in\sigma}|\alpha_k|M\left(\frac{d(x_k,\theta)}{\rho}\right)\\ &\quad (\text{since } M(\lambda x)\leq \lambda M(x))\\ &\leq \sup_{s\geq 1,\sigma\in P_s}\frac{1}{\phi_s}\sum_{k\in\sigma}M\left(\frac{d(x_k,\theta)}{\rho}\right)\\ &< \infty.\end{aligned}$$

Therefore, $(\alpha_k x_k) \in m(M,\phi,d)$. Hence, the space $m(M,\phi,d)$ is solid.

The symmetricity of the space follows from the definition of the space $m(M,\phi,d)$ and symmetric sequence space. □

The following result follows from the Theorem 3.7 and the Lemma 2.4.

Corollary 3.8. *The space $m(M,\phi,d)$ is monotone.*

The proof of the following result is a routine work.

Proposition 3.9. *The spaces $Z(M,d)$, where $Z=\ell_\infty,\ell_1$ are solid and as such are monotone.*

Theorem 3.10. *Let (X,d) be a complete seminormed space, then the spaces $Z(M,d)$ for $Z=\ell_\infty$ are complete seminormed spaces under the seminorm p defined as*

$$p((x_k)) = \inf\left\{\rho>0:\sup_k M\left(\frac{d(x_k,\theta)}{\rho}\right)\leq 1\right\}.$$

Proof. Let $\bar{x}=(x_k^i)\in\ell_\infty(M,d)$, for each $i\in N$ be a Cauchy sequence.

Let, $r>0$ and $x_0>0$ be fixed such that $M(rx_0)\geq 1$. Then for each $\frac{\varepsilon}{rx_0}>0$, there exists a positive integer n_0 such that

$$f\left(x_k^i - x_k^j\right) < \frac{\varepsilon}{rx_0}, \quad \text{for all } i,j\geq n_0$$

then

$$\inf\left\{\rho>0:\sup_{k\geq 1} M\left(\frac{d(x_k^i - x_k^j,\theta)}{\rho}\right)\leq 1\right\} < \frac{\varepsilon}{rx_0}. \tag{3.1}$$

Now, for $i,j\geq n_0$ we have,

$$\begin{aligned}\sup_{k\geq 1} M\left(\frac{d(x_k^i - x_k^j,\theta)}{f(x_k^i - x_k^j)}\right) &\leq 1\\ &\leq M(rx_0)\end{aligned}$$

then

$$M\left(\frac{d(x_k^i, x_k^j)}{f(x_k^i - x_k^j)}\right) \le M(rx_0) \quad \Rightarrow \quad d(x_k^i, x_k^j) < rx_0.\frac{\varepsilon}{rx_0}$$
$$\Rightarrow \quad d(x_k^i, x_k^j) < \varepsilon, \quad \text{for all } i, j \ge n_0.$$

Therefore, (x_k^i) is a Cauchy sequence in X, for all $k \in N$.

Since, (X, d) is complete , so there exists $(x_k) \in X$, for each $k \in N$ such that $d(x_k^i, x_k) < \varepsilon$.

Using continuity of M and d we have,

$$\sup_{k\ge1} M\left(\frac{d(x_k^i, \lim_{j\to\infty} x_k^j)}{\rho}\right) \le 1.$$

then

$$\sup_{k\ge1} M\left(\frac{d(x_k^i, x_k)}{\rho}\right) \le 1, \quad \text{for some } \rho > 0. \tag{3.2}$$

Now, taking infimum of such ρ's by Eq (3.1) we get,

$$\inf\left\{\rho > 0 : \sup_{k\ge1} M\left(\frac{d(x_k^i, x_k)}{\rho}\right) \le 1\right\} < \frac{\varepsilon}{rx_0}.$$

Then $f(x_k^i, x_k) < \varepsilon$, since, $r > 0$ and $x_0 > 0$. Then $x_k^i \to x_k$ as $i \to \infty$.

Now, according to (3.2) we have

$$\sup_{k\ge1} M\left(\frac{d(x_k - x_k^i, \theta)}{\rho}\right) \le 1 < \infty.$$

Therefore, $(x_k - x_k^i) \in \ell_\infty(M, d)$.

Since, $\ell_\infty(M, d)$ is a linear space and $(x_k^i), (x_k - x_k^i) \in \ell_\infty(M, d)$. then $(x_k^i + x_k - x_k^i) \in \ell_\infty(M, d)$.

Therefore, $(x_k) \in \ell_\infty(M, d)$. Hence, $\ell_\infty(M, d)$ is complete. □

Theorem 3.11. *Let M_1 and M_2 be two Orlicz functions. Then we have*

(i) *$Z(M_2, d) \subseteq Z(M_1 \circ M_2, d)$, for $Z = \ell_\infty, \ell_1, c$ and c_0.*

(ii) *$Z(M_1, d) \cap Z(M_2, d) \subseteq Z(M_1 + M_2, d)$, for for $Z = \ell_\infty, \ell_1, c$ and c_0.*

(iii) *$Z(M_1, d_1) \cap Z(M_1, d_2) \subseteq Z(M_1, d_1 + d_2)$, for for $Z = \ell_\infty, \ell_1, c$ and c_0, where d_1, d_2 are metrics on X.*

(iv) *If d_1 is stronger than d_2, then $Z(M_1, d_1) \subseteq Z(M_1, d_2)$, for $Z = \ell_\infty, \ell_1, c$ and c_0.*

Proof. (i) We prove it for the case $Z = c$, the other cases can be proved following similar technique. Let, $\varepsilon > 0$ be given and

$(x_k) \in c(M_2, d)$. Therefore,

$$\lim_{k\to\infty} M_2\left(\frac{d(x_k, L)}{\rho}\right) = 0, \quad \text{for some } \rho > 0, L \in X.$$

$$\Rightarrow \quad M_1\left[\lim_{k\to\infty} M_2\left(\frac{d(x_k, L)}{\rho}\right)\right] = M_1(0).$$

$$\Rightarrow \quad \lim_{k\to\infty} (M_1 \circ M_2\left(\frac{d(x_k, L)}{\rho}\right) = 0, \quad \text{since } M_1(0) = 0.$$

Therefore, $(x_k) \in c(M_1 \circ M_2, d)$. Thus, $c(M_2, d) \subseteq c(M_1 \circ M_2, d)$ Hence, $Z(M_2, d) \subseteq Z(M_1 \circ M_2, d)$, for $Z = \ell_\infty, \ell_1, c$ and c_0.

(ii) We prove the result for $Z = c$. The other cases can be established following similar technique.

Let, $(x_k) \in c(M_1) \cap c(M_2, d)$ then $(x_k) \in c(M_1)$ and $(x_k) \in c(M_2, d)$

Let, $\varepsilon > 0$ be given. Then,

$$\lim_{k\to\infty} M_1\left(\frac{d(x_k, L)}{\rho_1}\right) = 0, \quad \text{for some} \rho_1 > 0, L \in X.$$

and

$$\lim_{k\to\infty} M_2\left(\frac{d(x_k, L)}{\rho_2}\right) = 0, \quad \text{for some } \rho_2 > 0, L \in X.$$

Then

$$M_1\left(\frac{d(x_k, L)}{\rho_1}\right) < \frac{\varepsilon}{2}, \quad \text{for some } \rho_1 > 0, L \in X.$$

and

$$M_2\left(\frac{d(x_k, L)}{\rho_2}\right) < \frac{\varepsilon}{2}, \quad \text{for some } \rho_2 > 0, L \in X.$$

Let, $\rho = \max\{\rho_1, \rho_2\}$. Then, for $k \in N$ and $\rho > 0$ we have,

$$\begin{aligned}(M_1 + M_2)\left(\frac{d(x_k, L)}{\rho}\right) &\leq M_1\left(\frac{d(x_k, L)}{\rho}\right) + M_2\left(\frac{d(x_k, L)}{\rho}\right)\\ &< M_1\left(\frac{d(x_k, L)}{\rho}\right) + M_2\left(\frac{d(x_k, L)}{\rho_2}\right).\end{aligned}$$

Then we have

$$(M_1 + M_2)\left(\frac{d(x_k, L)}{\rho}\right) < \varepsilon, \quad \text{for some } \rho > 0, L \in X.$$

This implies

$$\lim_{k\to\infty} (M_1 + M_2)\left(\frac{d(x_k, L)}{\rho}\right) < \varepsilon, \quad \text{for some } \rho > 0, L \in X.$$

Hence we have, $(x_k) \in c(M_1+M_2,d)$. Thus, $c(M_1,d) \cap c(M_2,d) \subseteq c(M_1+M_2,d)$ Hence, $Z(M_1,d) \cap Z(M_2,d) \subseteq Z(M_1+M_2,d)$, for for $Z = \ell_\infty, \ell_1, c$ and c_0.

(iii) The result can be proved by using standard technique.

(iv) We prove the result for $Z = \ell_\infty$.

The other cases can be established by following similar technique. Since, d_1 is stronger than d_2, Then $d_2 < d_1$, by (i) Let $(x_k) \in \ell_\infty(M_1, d_1)$

$$\sup_{k\geq 1} M_1\left(\frac{d(x_k,L)}{\rho}\right) < \infty, \quad \text{for some } \rho > 0.$$

From (i) we have,

$$d_2\left(\frac{d(x_k,L)}{\rho}\right) < d_1\left(\frac{d(x_k,L)}{\rho}\right), \quad \text{for some } \rho > 0.$$

Then

$$\begin{aligned}\sup_{k\geq 1} M_1\left(d_2\left(\frac{d(x_k,L)}{\rho}\right)\right) &< \sup_{k\geq 1} M_1\left(d_1\left(\frac{d(x_k,L)}{\rho}\right)\right)\\ &< \infty,\end{aligned}$$

for some $\rho > 0$. Then $(x_k) \in \ell_\infty(M_1, d_2)$.

Therefore, $\ell_\infty(M_1,d_1) \subseteq \ell_\infty(M_1,d_2)$. Hence, if d_1 is stronger than d_2, then $Z(M_1,d_1) \subseteq Z(M_1,d_2)$, for $Z = \ell_\infty, \ell_1, c$ and c_0. □

Theorem 3.12. *$m(M,\phi,d) \subseteq m(M,\psi,d)$ if and only if*

$$\sup_{k\geq 1} \frac{\phi_s}{\psi_s} < \infty.$$

Proof. Let, $\sup_{k\geq 1} \frac{\phi_s}{\psi_s} < \infty$ and $(x_k) \in m(M,\phi,d)$. Then,

$$\sup_{s\geq 1, \sigma\in P_s} \frac{1}{\phi_s} \sum_{k\in\sigma} M\left(\frac{d(x_k,\theta)}{\rho}\right) < \infty, \quad \text{for some } \rho > 0.$$

This implies,

$$\begin{aligned}\sup_{s\geq 1, \sigma\in P_s} \frac{1}{\psi_s} \sum_{k\in\sigma} M\left(\frac{d(x_k,\theta)}{\rho}\right) &\leq \left\{\sup_{s\geq 1} \frac{\phi_s}{\psi_s}\right\}\left\{\sup_{s\geq 1, \sigma\in P_s} \frac{1}{\phi_s} \sum_{k\in\sigma} M\left(\frac{d(x_k,\theta)}{\rho}\right)\right\}\\ &< \infty.\end{aligned}$$

Therefore, $(x_k) \in m(M,\psi,d)$. Hence, $m(M,\phi,d) \subseteq m(M,\psi,d)$. Conversely, let $m(M,\phi,d) \subseteq m(M,\psi,d)$. Suppose,

$$\sup_{k\geq 1} \frac{\phi_s}{\psi_s} = \infty.$$

Then, there exists a sequence of natural numbers s_i such that

$$\lim_{i\to\infty} \frac{\phi_{s_i}}{\psi_{s_i}} = \infty.$$

Now, $(x_k) \in m(M, \phi, d)$

Then there exists $\rho > 0$ such that

$$\sup_{s\geq 1, \sigma\in P_s} \frac{1}{\phi_s} \sum_{k\in\sigma} M\left(\frac{d(x_k, \theta)}{\rho}\right) < \infty, \quad \text{for some } \rho > 0\ .$$

Now, we have,

$$\sup_{s\geq 1, \sigma\in P_s} \frac{1}{\psi_s} \sum_{k\in\sigma} M\left(\frac{d(x_k, \theta)}{\rho}\right) \geq \left\{\sup_{i\geq 1} \frac{\phi_{s_i}}{\psi_{s_i}}\right\} \left\{\sup_{i\geq 1, \sigma\in P_{s_i}} \frac{1}{\phi_{s_i}} \sum_{k\in\sigma} M\left(\frac{d(x_k, \theta)}{\rho}\right)\right\} < \infty.$$

Therefore, $(x_k) \notin m(M, \psi, d)$, which is a contradiction to our assumption that $m(M, \phi, d) \subseteq m(M, \psi, d)$. This contradiction implies that

$$\sup_{k\geq 1} \frac{\phi_s}{\psi_s} < \infty.$$

The proof of the following result is a routine work. □

Theorem 3.13. *Let M_1 and M_2 be two Orlicz functions. Then the following hold:*

(i) $m(M_2, \phi, d) \subseteq m(M_1 \circ M_2, \phi, d)$.
(ii) $m(M_1, \phi, d) \cap m(M_2, \phi, d) \subseteq m(M_1 + M_2, \phi, d)$.
(iii) $m(M, \phi, d_1) \cap m(M, \phi, d_2) \subseteq m(M, \phi, d_1 + d_2)$.
(iv) *If d_1 is stronger than d_2, then $m(M, \phi, d_1) \subseteq m(M, \phi, d_2)$.*

Taking $M_2(x) = x$, for $x \in [0, \infty)$ and $M_1 = M$ in Theorem 3.13 (i) we get the following result.

Corollary 3.14. *Let M be an Orlicz function, then $m(\phi, d) \subseteq m(M, \phi, d)$.*

From Theorem 3.12 and Corollary 3.14 we have,

Corollary 3.15. *Let M be an Orlicz function, then $m(\phi, d) \subseteq m(M, \psi, d)$ if and only if*

$$\sup_{s\geq 1} \frac{\phi_s}{\psi_s} < \infty.$$

Theorem 3.16. $\ell_1(M, d) \subseteq m(M, \phi, d) \subseteq \ell_\infty(M, d)$.

Proof. Let $(x_k) \in \ell_1(M, d)$. Then,

$$\sum_{k=1}^{\infty} M\left(\frac{d(x_k, \theta)}{\rho}\right) \leq \infty, \quad \text{for some } \rho > 0.$$

Since, (ϕ_n) is monotonic increasing, so we have,

$$\begin{aligned}\sup_{s\geq 1,\sigma\in\phi_s}\frac{1}{\psi_s}\sum_{k\in\sigma}M\left(\frac{d(x_k,\theta)}{\rho}\right) &\leq \sup_{s\geq 1,\sigma\in\phi_s}\frac{1}{\psi_1}\sum_{k\in\sigma}M\left(\frac{d(x_k,\theta)}{\rho}\right)\\ &\leq \frac{1}{\psi_1}\sum_{k=1}^{\infty}M\left(\frac{d(x_k,\theta)}{\rho}\right)\\ &\leq \infty.\end{aligned}$$

Hence,

$$\sup_{s\geq 1,\sigma\in P_s}\frac{1}{\phi_s}\sum_{k\in\sigma}M\left(\frac{d(x_k,\theta)}{\rho}\right) < \infty.$$

Thus, $(x_k) \in m(M,\phi,d)$. Therefore, $\ell_1(M,d) \subseteq m(M,\phi,d)$. Next, let $(y_k) \in m(M,\phi,d)$. Then we have,

$$\begin{aligned}&\sup_{s\geq 1,\sigma\in P_s}\frac{1}{\phi_s}\sum_{k\in\sigma}M\left(\frac{d(y_k,\theta)}{\rho}\right) < \infty,\ \text{for some } \rho > 0.\\ \Rightarrow\quad &\sup_{k\in N}\frac{1}{\phi_1}M\left(\frac{d(y_k,\theta)}{\rho}\right) < \infty,\ \text{for some } \rho > 0,\\ &\qquad\text{(since, on taking cardinality of } \sigma \text{ to be 1).}\\ \Rightarrow\quad &(y_k) \in \ell_\infty(M,d).\end{aligned}$$

Hence, $m(M,\phi,d) \subseteq \ell_\infty(M,d)$. Thus, $\ell_1(M,d) \subseteq m(M,\phi,d) \subseteq \ell_\infty(M,d)$. □

The proof of the following result is a consequence of Theorem 3.10.

Theorem 3.17. *Let (X,d) be complete metric space, then $m(M,\phi,d)$ is also complete.*

4. Conclusion

We have introduced the sequence space $m(\phi,d)$ and $m(M,\phi,d)$ in a metric space on generalising the sequence space $m(\phi)$, defined by the Orlicz function. Different algebraic and topological properties such as solidity, completeness etc. have been investigated. We have obtained some inclusion results involving the space $m(M,\phi,d)$. The results can be further investigated from other properties as well as applied for investigation in other areas of research.

Acknowledgment. The authors thank the unanimous reviewer for his comments, those improved the presentation of the article.

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[1] Department of Mathematics, Jagannath Barooah College, Jorhat 785001, Assam, India.

Email address: rinkudey_math@rediffmail.com and rinkudey1984@rediffmail.com

[2] Department of Mathematics, Tripura University, Suryamaninagar, Agartala799022, Tripura, India.

Email address: tripathybc@yahoo.com and tripathybc@rediffmail.com