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Godunova Type Inequality for Sugeno Integral

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Abstract. In this paper, we investigate Godunova type inequality for Sugeno integrals in two cases. At the first case, we suppose that the inner integral is the Riemann integral and the remaining two integrals are of Sugeno type. At the second case, all the integrals are assumed Sugeno integrals. We present several examples to illustrate validity of our results.

1. Introduction

In 1974, M. Sugeno introduced fuzzy measures and Sugeno integrals for the first time which was an important analytical method of measuring uncertain information [21]. Sugeno integral is applied in many fields such as management decision-making, medical decision-making, control engineering and so on. Many authors such as Ralescu and Adams considered equivalent definitions for Sugeno integral [17]. Román-Flores et al. examined level-continuity of Sugeno integral and H-continuity of fuzzy measures [18, 20]. For more details of Sugeno integral, we refer readers to [1, 2, 12, 14–16].

The study of fuzzy integral is first attributed to Román-Flores et al. Many inequalities such as Markov's, Chebyshev's, Jensen's, Minkowski's, Hölder's and Hardy's inequalities have been studied by authors. Flores-Franulič and Román-Flores expressed and proved inequalities for Sugeno integral (see [10, 11] and the references therein). Recently, in [3–9], B. Daraby et al. studied some inequalities for Sugeno integral.

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E. K. Godunova in [13] established the classical Godunova inequality as follows:

$$\int_0^{\infty} \frac{1}{x}\varphi^{-1}\left(\int_0^{x} \varphi\left(f(t)\right) dt\right) dx \leq \int_0^{\infty} \frac{f(x)}{x} dx,$$

where $f, \varphi : [0, \infty) \to [0, \infty)$ are continuous functions, φ is convex function, $\varphi(0) = 0$ and $\int_0^{\infty} \frac{f(x)}{x} dx < \infty$.

In this manuscript, we prove this inequality, regardless of the classical conditions and the new conditions.

Our goal in this paper is to prove Godunova's inequality for the Sugeno integral. This paper has been organized as follows: in Section 2, a brief preliminaries is provided. In Section 3, the main results are presented with the proofs. At the last section, we give a short result.

2. Preliminaries

In this section, we provide some definitions and concepts for the next sections. The contents of this section are from [3, 9, 14, 17, 22]. Throughout this paper, let X be a non-empty set and Σ be a $\sigma-$algebra of subsets of X.

Definition 2.1. A set function $\mu : \Sigma \to [0, +\infty]$ is called a fuzzy measure if the following properties are satisfied:

(i) $\mu(\emptyset) = 0$;

(ii) $A \subseteq B \quad \Rightarrow \quad \mu(A) \leq \mu(B)$ (monotonicity);

(iii) $A_1 \subseteq A_2 \subseteq \dots \quad \Rightarrow \quad \lim_{i\to\infty} \mu(A_i) = \mu\left(\bigcup_{i=1}^{\infty} A_i\right)$ (continuity from below);

(iv) $A_1 \supseteq A_2 \supseteq \dots$ and $\mu(A_1) < \infty \quad \Rightarrow \quad \lim_{i\to\infty} \mu(A_i) = \mu\left(\bigcap_{i=1}^{\infty} A_i\right)$ (continuity from above).

When μ is a fuzzy measure, the triple (X, Σ, μ) is called a fuzzy measure space.

If f is a non-negative real-valued function on X, we will denote

$$\begin{aligned} F_\alpha &= \{x \in X \mid f(x) \geq \alpha\} \\ &= \{f \geq \alpha\}, \end{aligned}$$

the α-level of f, for $\alpha > 0$. The set $F_0 = \overline{\{x \in X \mid f(x) > 0\}} = \mathrm{supp}(f)$ is the support of f.

For a Fuzzy measure μ on X, we denote:

$$\mathfrak{F}^{\sigma}(X) = \{f : X \to [0, \infty) | \quad f \text{ is } \mu-\text{measurable}\}.$$

Definition 2.2. Let μ be a fuzzy measure on (X, Σ). If $f \in \mathfrak{F}^{\sigma}(X)$ and $A \in \Sigma$, then the Sugeno integral of f on A is defined by

$$-\!\!\!\!\!\!\int_A f d\mu = \bigvee_{\alpha \geq 0} (\alpha \wedge \mu(A \cap F_\alpha)),$$

where $\vee$ and $\wedge$ denote the operations sup and inf on $[0, \infty]$, respectively and μ is the Lebesgue measure. If $A = X$, the fuzzy integral may also be denoted by $-\!\!\!\!\!\!\int f d\mu$.

The following proposition gives the elementary properties of the Sugeno integral.

Proposition 2.3. *Let (X, Σ, μ) be a fuzzy measure space, $A, B \in \sum$ and $f, g \in \mathfrak{F}^{\sigma}(X)$. We have*

(i) $-\!\!\!\!\!\!\int_A f d\mu \leq \mu(A)$;
(ii) $-\!\!\!\!\!\!\int_A k d\mu = k \wedge \mu(A)$, *for any constant* $k \in [0, \infty)$;
(iii) $-\!\!\!\!\!\!\int_A f d\mu < \alpha \Leftrightarrow$ *there exists* $\gamma < \alpha$ *such that* $(A \cap \{f \geq \gamma\}) < \alpha$;
(iv) $-\!\!\!\!\!\!\int_A f d\mu > \alpha \Leftrightarrow$ *there exists* $\gamma > \alpha$ *such that* $(A \cap \{f \geq \gamma\}) > \alpha$.

Remark 2.4. Consider the distribution function F associated to f on A, that is to say,

$$F(\alpha) = \mu(A \cap \{f \geq \alpha\}).$$

Then

$$F(\alpha) = \alpha \quad \Rightarrow \quad -\!\!\!\!\!\!\int_A f d\mu = \alpha.$$

Thus, from a numerical (or computational) point of view, the Sugeno integral can be calculated by solving the equation $F(\alpha) = \alpha$ (if the solution exists).

Theorem 2.5 ([19])**.** *Let (X, Σ, μ) be a fuzzy measure space and let $f \in \mathfrak{F}^{\mu}(X)$ be such that $-\!\!\!\!\!\!\int f d\mu = p$. If $\Phi : [0, \infty) \to [0, \infty)$ is a strictly increasing function such that $\Phi(x) \leq x$, for every $x \in [0, p]$, then:*

$$\Phi\left(-\!\!\!\!\!\!\int f d\mu\right) \leq -\!\!\!\!\!\!\int \Phi(f) d\mu. \tag{2.1}$$

Notation 2.6. We use $\mathrm{SINT}_0\ f(x)dx$ for the Sugeno integral on $[0, \infty)$ and $\mathrm{SINT}_1\ f(x)dx$ for the Sugeno integral on $[1, \infty)$ with respect to standard Lebesgue measure.

3. Main Results

In this section, we prove Godunova type inequality in two cases for Sugeno integral. In first case, we give the proof of Theorem 3.1 and illustrate the validity of theorem by an example. In the continue, we show that the increasing condition of $f(x)$ and $\varphi(x)$ is necessary condition in

Theorem 3.1. Note that, in Theorem 3.1, we use the correct symbols to represent fuzzy integrals other than is used in the [23]. Also, note that in the second integral, we consider x from interval $(0,1]$.

Theorem 3.1 (Godunova type inequality for Sugeno integral: first case)**.** *Let $f,\varphi : [0,\infty) \to [0,\infty)$ be increasing measurable functions and $\mathrm{SINT}_0\ \dfrac{f(x)}{x}dx < \infty$. Then the inequality*

$$\mathrm{SINT}_0\ \frac{1}{x}\varphi^{-1}\left(\frac{1}{x}\int_0^x \varphi(f(t))dt\right)dx \leq \mathrm{SINT}_0\ \frac{f(x)}{x}dx,$$

holds.

Proof. Let $\alpha = \mathrm{SINT}_0\ \dfrac{f(x)}{x}dx$. If $\mathrm{SINT}_0\ \dfrac{1}{x}\varphi^{-1}\left(\frac{1}{x}\int_0^x \varphi(f(t))dt)dx\right) >$ α, then there exists $\gamma > \alpha$ such that

(3.1)

$$\mu\left\{\frac{1}{x}\varphi^{-1}\left(\frac{1}{x}\int_0^x \varphi(f(t))dt\right) > \gamma\right\} > \alpha \quad \text{(from proposition 2.3 (4))}.$$

Now, if

$$x \in \left\{\frac{1}{x}\varphi^{-1}\left(\frac{1}{x}\int_0^x \varphi(f(t))dt\right) > \gamma\right\},$$

then we can write

$$\varphi^{-1}\left(\frac{1}{x}\int_0^x \varphi(f(t))dt\right) > x\gamma.$$

By multiplying φ on both sides of the above inequality, we get

$$\frac{1}{x}\int_0^x \varphi(f(t))dt > \varphi(x\gamma).$$

Using the classical integral properties, we have

$$\int_0^x \varphi(f(t))dt \geq x\varphi(x\gamma).$$

Because f,φ are increasing functions, we conclude that $x\varphi(f(x)) >$ $x\varphi(x\gamma)$ and thus $\varphi(f(x)) > \varphi(x\gamma)$. So $f(x) > x\gamma$, it follows that $\dfrac{f(x)}{x} > \gamma$. Therefore

$$\left\{\frac{f(x)}{x} > \gamma\right\} \supseteq \left\{\frac{1}{x}\varphi^{-1}\left(\frac{1}{x}\int_0^x \varphi(f(t))dt\right) > \gamma\right\}.$$

Now, from monotonicity of μ, we have

$$\mu\left\{\frac{f(x)}{x} > \gamma\right\} \geq \mu\left\{\frac{1}{x}\varphi^{-1}\left(\frac{1}{x}\int_0^x \varphi(f(t))dt\right) > \gamma\right\}.$$

Thereby, from (3.1), we obtain that

$$\mu\left\{\frac{f(x)}{x}\geq\gamma\right\}>\alpha.$$

Which is a contradiction with our initial hypothesis. The proof is now complete. □

Now, by an example, we illustrate the validity of the above theorem.

Example 3.2. Let $f(x)=\sqrt{x}$ and $\varphi(x)=\dfrac{x}{2}$ be defined from $[0,2]$ into $[0,2]$. A straightforward calculation shows that

$$\begin{aligned}
\fint_0^2 \frac{f(x)}{x}dx &= \fint_0^2 \frac{\sqrt{x}}{x}dx = \fint_0^2 \frac{1}{\sqrt{x}}dx \\
&= \sup_{\alpha\in[0,2]}\left(\alpha\wedge\mu\left([0,2]\cap\left\{x:\ \frac{1}{\sqrt{x}}\geq\alpha\right\}\right)\right) \\
&= \sup_{\alpha\in[0,2]}\left(\alpha\wedge\mu\left(\left[0,\frac{1}{\alpha^2}\right]\right)\right) \\
&= \sup_{\alpha\in[0,2]}\left(\alpha\wedge\left(\frac{1}{\alpha^2}\right)\right) \\
&= 1.
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
&\int_0^x \varphi(f(t))dt = \int_0^x \frac{\sqrt{t}}{2}dt = \frac{x\sqrt{x}}{3}, \\
&\frac{1}{x}\int_0^x \varphi(f(t))dt = \frac{\sqrt{x}}{3}, \\
&\varphi^{-1}\left(\frac{1}{x}\int_0^x \varphi(f(t))dt\right) = 2\left(\frac{\sqrt{x}}{3}\right) = \frac{2\sqrt{x}}{3}, \\
&\frac{1}{x}\varphi^{-1}\left(\frac{1}{x}\int_0^x \varphi(f(t))dt\right) = \frac{2}{3\sqrt{x}}.
\end{aligned}$$

For calculating the integral $\fint_0^2 \dfrac{2}{3\sqrt{x}}dx$, we have

$$\begin{aligned}
\fint_0^2 \frac{2}{3\sqrt{x}}dx &= \sup_{\alpha\in[0,2]}\left(\alpha\wedge\mu\left([0,2]\cap\left\{x:\ \frac{2}{3\sqrt{x}}\geq\alpha\right\}\right)\right) \\
&= \sup_{\alpha\in[0,2]}\left(\alpha\wedge\mu\left(\left[0,\frac{4}{9\alpha^2}\right]\right)\right) \\
&= \sup_{\alpha\in[0,2]}\left(\alpha\wedge\left(\frac{4}{9\alpha^2}\right)\right)
\end{aligned}$$

$$= 0.7631.$$

Hence, we have $0.7631 \leq 1$.

Note that, in Theorem 3.1, the increasing condition of $f(x)$ and $\varphi(x)$ is necessary. This condition is confirmed by the following example.

Example 3.3. (i) Let $f, \varphi : \left[\frac{1}{4}, 2\right] \to \left[\frac{1}{4}, 2\right]$ be defined as $f(x) = \dfrac{1}{x}$ and $\varphi(x) = \dfrac{1}{2x}$. We have

$$\int_0^x \varphi(f(t))dt = \int_0^x \frac{t}{2}dt = \frac{x^2}{4},$$
$$\frac{1}{x}\int_0^x \varphi(f(t))dt = \frac{x}{4},$$
$$\varphi^{-1}\left(\frac{1}{x}\int_0^x \varphi(f(t))dt\right) = \frac{2}{x},$$
$$\frac{1}{x}\varphi^{-1}\left(\frac{1}{x}\int_0^x \varphi(f(t))dt\right) = \frac{2}{x^2}.$$

Now, we calculate the integrals $\fint_{\frac{1}{4}}^{2} \dfrac{2}{x^2}dx$ and $\fint_{\frac{1}{4}}^{2} \dfrac{f(x)}{x}dx$. So, we have

$$\begin{aligned}
\fint_{\frac{1}{4}}^{2} \frac{2}{x^2}dx &= \sup_{\alpha\in\left[\frac{1}{4},2\right]} \left(\alpha \wedge \left(\left[\frac{1}{4}, 2\right] \cap \left\{x : \frac{2}{x^2} \geq \alpha\right\}\right)\right)\\
&= \sup_{\alpha\in\left[\frac{1}{4},2\right]} \left(\alpha \wedge \mu\left(\left[\frac{1}{4}, \sqrt{\frac{2}{\alpha}}\right]\right)\right)\\
&= \sup_{\alpha\in\left[\frac{1}{4},2\right]} \left(\alpha \wedge \left(\sqrt{\frac{2}{\alpha}} - \frac{1}{4}\right)\right)\\
&= 0.8297,
\end{aligned}$$

and

$$\begin{aligned}
\fint_{\frac{1}{4}}^{2} \frac{f(x)}{x}dx &= \fint_{\frac{1}{4}}^{2} \frac{1}{x^2}dx\\
&= \sup_{\alpha\in[0,1]} \left(\alpha \wedge \mu(\left[\frac{1}{4}, 2\right] \cap \left\{x : \frac{1}{x^2} \geq \alpha\right\}\right)\\
&= \sup_{\alpha\in\left[\frac{1}{4},2\right]} \left(\alpha \wedge \mu\left(\left[\frac{1}{4}, \sqrt{\frac{1}{\alpha}}\right]\right)\right)
\end{aligned}$$

$$
\begin{aligned}
&= \sup_{\alpha\in\left[\frac{1}{4},2\right]} \left(\alpha \wedge \left(\sqrt{\frac{1}{\alpha}} - 1\right)\right) \\
&= 0.6074.
\end{aligned}
$$

Therefore, $0.8297 \leq 0.6074$. This is a contradiction with Theorem 3.1.

(ii) Let $f(x) = \dfrac{1}{x}, \varphi(x) = \sqrt{x}$ be defined from $[0,2]$ to $[0,2]$. We have

$$
\begin{aligned}
&\int_0^x \varphi(f(t))dt = \int_0^x \frac{2}{\sqrt{\frac{1}{t}}}dt = 2\sqrt{x}, \\
&\frac{1}{x}\int_0^x \varphi(f(t))dt = \frac{2}{\sqrt{x}}, \\
&\varphi^{-1}\left(\frac{1}{x}\int_0^x \varphi(f(t))dt\right) = \frac{4}{x}, \\
&\frac{1}{x}\varphi^{-1}\left(\frac{1}{x}\int_0^x \varphi(f(t))dt\right) = \frac{4}{x^2},
\end{aligned}
$$

and

$$
\begin{aligned}
\fint_0^2 \frac{1}{x}\varphi^{-1}\left(\frac{1}{x}\int_0^x \varphi(f(t))dt\right)dx &= \fint_0^2 \frac{4}{x^2}dx \\
&= 1.587.
\end{aligned}
$$

Now we calculate $\fint_0^2 \dfrac{f(x)}{x}dx$.

$$
\begin{aligned}
\fint_0^2 \frac{f(x)}{x}dx &= \fint_0^2 \frac{1}{x^2}dx \\
&= 1.
\end{aligned}
$$

It follows that, the inequality is not valid in Theorem 3.1.

In the following, we prove Godunova type inequality for Sugeno integral in the second case.

Theorem 3.4 (Godunova type inequality for Sugeno integral: second case)**.** *Let $f, \varphi : [1,\infty) \to [1,\infty)$ be measurable functions and φ be a increasing function. Then the inequality*

$$
\mathrm{SINT}_1\ \frac{1}{x}\varphi^{-1}\left(\frac{1}{x}\fint_0^x \varphi(f(t))dt\right)dx \leq \mathrm{SINT}_1\ \frac{f(x)}{x}dx,
$$

holds.

Proof. From Proposition 2.3(1), it's clear that

$$\fint_0^x \varphi(f(t))dt \le x.$$

By straightforward calculus, we get

$$\begin{aligned}
&\frac{1}{x}\fint_0^x \varphi(f(t))dt \le 1\\
&\varphi^{-1}\left(\frac{1}{x}\fint_0^x \varphi(f(t))dt\right) \le \varphi^{-1}(1)\\
&\frac{1}{x}\left(\varphi^{-1}\left(\frac{1}{x}\fint_0^x \varphi(f(t))dt\right)\right) \le \frac{\varphi^{-1}(1)}{x}.
\end{aligned}$$

By fuzzy integration of both sides of above inequality from 1 to ∞, we obtain that

$$\mathrm{SINT}_1\ \frac{1}{x}\left(\varphi^{-1}\left(\frac{1}{x}\fint_0^x \varphi(f(t))dt\right)\right)dx \le \mathrm{SINT}_1\ \frac{\varphi^{-1}(1)}{x}dx. \tag{3.2}$$

Calculating the $\mathrm{SINT}_1\ \dfrac{\varphi^{-1}(1)}{x}dx$, we get

$$\mathrm{SINT}_1\ \frac{\varphi^{-1}(1)}{x}dx = \sqrt{\varphi^{-1}(1)}.$$

Thereby, from (3.2), we have

$$\mathrm{SINT}_1\ \frac{1}{x}\left(\varphi^{-1}\left(\frac{1}{x}\fint_0^x \varphi(f(t))dt\right)\right)dx \le \sqrt{\varphi^{-1}(1)}.$$

Now, the proof can be divided in two parts.

1. Suppose that $\mathrm{SINT}_1\ \dfrac{f(x)}{x}dx > \sqrt{\varphi^{-1}(1)}$. In this case, it is not difficult to see that

$$\begin{aligned}
\mathrm{SINT}_1\ \frac{1}{x}\left(\varphi^{-1}\left(\frac{1}{x}\fint_0^x \varphi(f(t))dt\right)\right)dx &\le \sqrt{\varphi^{-1}(1)}\\
&< \mathrm{SINT}_1\ \frac{f(x)}{x}dx.
\end{aligned}$$

2. Suppose that $\mathrm{SINT}_1\ \dfrac{f(x)}{x}dx \le \sqrt{\varphi^{-1}(1)}$. In this case, we have

$$\begin{aligned}
\sqrt{\varphi^{-1}(1)} &\ge \mathrm{SINT}_1\ \frac{f(x)}{x}dx\\
&\ge \fint_0^x \frac{f(t)}{t}dt\\
&\ge \fint_0^x f(t)dt
\end{aligned}$$

$$= \fint_0^x \varphi^{-1}\left(\varphi(f(t))\right) dt.$$

From Inequality (2.1), we have

$$\sqrt{\varphi^{-1}(1)} \geq \varphi^{-1} \fint_0^x \varphi(f(t))dt.$$

It's easy to see that

$$\begin{aligned}
&\varphi\sqrt{\varphi^{-1}(1)} \geq \varphi\varphi^{-1} \fint_0^x \varphi(f(t))dt \\
&\frac{1}{x}\varphi\sqrt{\varphi^{-1}(1)} \geq \frac{1}{x} \fint_0^x \varphi(f(t))dt \\
&\varphi^{-1}\left(\frac{1}{x}\varphi\sqrt{\varphi^{-1}(1)}\right) \geq \varphi^{-1}\left(\frac{1}{x}\fint_0^x \varphi(f(t))dt\right) \\
&\frac{1}{x}\varphi^{-1}\left(\frac{1}{x}\varphi\sqrt{\varphi^{-1}(1)}\right) \geq \frac{1}{x}\varphi^{-1}\left(\frac{1}{x}\fint_0^x \varphi(f(t))dt\right).
\end{aligned}$$

It is not difficult to check that

$$\begin{aligned}
\frac{1}{x}\varphi^{-1}\left(\frac{1}{x}\varphi\sqrt{\varphi^{-1}(1)}\right) &\leq \frac{1}{x}\varphi^{-1}\left(\varphi\sqrt{\varphi^{-1}(1)}\right) \\
&= \frac{1}{x}\sqrt{\varphi^{-1}(1)}.
\end{aligned}$$

Thus, we have

$$\frac{1}{x}\sqrt{\varphi^{-1}(1)} \geq \frac{1}{x}\varphi^{-1}\left(\frac{1}{x}\fint_0^x \varphi(f(t))dt\right).$$

Now, by fuzzy integration of both sides of the above inequality from 0 to ∞, we get

$$\mathrm{SINT}_1\ \frac{1}{x}\sqrt{\varphi^{-1}(1)} \geq \mathrm{SINT}_1\ \frac{1}{x}\varphi^{-1}\left(\frac{1}{x}\fint_0^x \varphi(f(t))dt\right) dx.$$

Calculating the left of the above relation, we have

$$\mathrm{SINT}_1\ \frac{1}{x}\sqrt{\varphi^{-1}(1)}dx = \sqrt{\varphi^{-1}(1)}.$$

Hence, the proof is now complete. □

Now, with the following example, we illustrate the validity of the Theorem 3.4.

Example 3.5. (i) Suppose that function f is defined from $[1, 5]$ to $[1, 5]$ as $f(x) = \dfrac{x}{2}$ and $\varphi(x) = 2x$, then

$$\fint_1^5 \frac{f(x)}{x}dx = \fint_1^5 \frac{1}{2}dx$$

$$= \frac{1}{2},$$

and

$$\begin{aligned}\fint_1^5 \frac{1}{x}\varphi^{-1}\left(\frac{1}{x}\fint_0^x \varphi(f(t))dt\right)dx &= \fint_1^5 \frac{1}{4x}dx \\ &= 0.2071.\end{aligned}$$

(ii). Let $f(x) = \dfrac{1}{x}, \varphi(x) = \sqrt{x}$ be defined from $[1,2]$ to $[1,2]$. A simple calculations show that

$$\begin{aligned}\fint_1^2 \frac{f(x)}{x}dx &= \fint_1^2 \frac{1}{x^2}dx \\ &= 0.4656,\end{aligned}$$

and

$$\begin{aligned}\fint_1^2 \frac{1}{x}\varphi^{-1}\left(\frac{1}{x}\fint_0^x \varphi(f(t))dt\right)dx &= \fint_1^2 \frac{1}{x^3}dx \\ &= 0.3803.\end{aligned}$$

Therefore, the theorem is valid.

4. Conclusion

At this paper, we proved the Godunova type inequality in two cases for Sugeno integral. In first case, we proved:

$$\mathrm{SINT}_0\ \frac{1}{x}\varphi^{-1}\left(\frac{1}{x}\int_0^x \varphi(f(t))dt\right)dx \leq \mathrm{SINT}_0\ \frac{f(x)}{x}dx,$$

where $f, \varphi : [0,\infty) \to [0,\infty)$ are increasing functions and at the second case, we proved:

$$\mathrm{SINT}_1\ \frac{1}{x}\varphi^{-1}\left(\frac{1}{x}\fint_0^x \varphi(f(t))dt\right)dx \leq \mathrm{SINT}_1\ \frac{f(x)}{x}dx,$$

where $f, \varphi : [1,\infty) \to [1,\infty)$ are measurable functions and φ is an increasing function. In the future works, one can discuss about these inequalities for pseudo and Choquet integrals.

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