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Invertibility of Multipliers for Continuous G-frames

Mohammad Reza Abdollahpour[1]* and Yavar Khedmati Yengejeh[2]

Abstract. In this paper, we study the concept of multipliers for the continuous g-Bessel families in Hilbert spaces. We present necessary conditions for invertibility of multipliers for the continuous g-Bessel families and sufficient conditions for invertibility of multipliers for continuous g-frames.

1. Introduction

In 1952, the concept of frames for Hilbert spaces was defined by Duffin and Schaeffer [10]. Frames are important tools in signal processing, image processing, data compression, etc. In 1993, Ali, Antoine and Gazeau developed the notion of ordinary frames to a family indexed by a measurable space which is known as continuous frames [4]. In 2006, g-frames or generalized frames were introduced by Sun [19]. Abdollahpour and Faroughi introduced and investigated continuous g-frames and Riesz-type continuous g-frames [1]. The importance of g-frames is derived from their ability to provide more choices in analyzing functions than frame expansion coefficients [19], furthermore, every fusion frame is a g-frame [9]. Also, in [13] they show how generalized translation invariant (GTI) frames can be considered as g-frames.

In the rest of the paper, (Ω, μ) is a measure space with positive measure μ, $\{\mathcal{K}_\omega : \omega \in \Omega\}$ is a family of Hilbert spaces and $GL(\mathcal{H})$ denotes the set of all invertible bounded linear operators on Hilbert space $\mathcal{H}$.

In 2007, the Bessel multiplier for Bessel sequences in Hilbert spaces was introduced by P. Balazs [6].

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* Corresponding author.

Definition 1.1. Let $\mathcal{H}$ and $\mathcal{K}$ be Hilbert spaces. Suppose that $F = \{f_i\}_{i\in I}$ and $G = \{g_i\}_{i\in I}$ are Bessel sequences for $\mathcal{H}$ and $\mathcal{K}$, respectively, and $m = \{m_i\}_{i\in I} \in l^\infty(I)$. The operator $M_{m,F,G} : \mathcal{H} \to \mathcal{K}$ defined by

$$M_{m,F,G}f = \sum_{i\in I} m_i \langle f, f_i\rangle g_i,$$

is called the Bessel multiplier for F and G.

Stoeva and Balazs investigated the invertibility of multipliers for frames in detail [18]. In [12], they generalized the concept of Bessel multipliers for p-Bessel and p-Riesz sequences in Banach spaces. In [14], fusion frame multipliers were introduced as a generalization of frame multipliers to extend the results of frame multipliers. Structures of duals of fusion frames and continuous fusion frames are discussed in [12, 14]. The concept of g-dual frames for Hilbert $C*$-modules is introduced in [11]. Also, results for g-Bessel multipliers are presented in [16]. In this paper, by generalizing results of [18], we obtain conditions for two continuous g-Bessel families to be continuous g-frames (Proposition 2.2). Also, to obtain a dual (not necessarily canonical) for each of these families (Proposition 2.3), we generalize a result of [7]. As well, we obtain necessary conditions for invertibility of multipliers for continuous g-Bessel families and sufficient conditions for invertibility of multipliers for continuous g-frames, by extending the results of [18]. In the rest of this section, we summarize some basic informations about continuous g-frames and multipliers of continuous g-Bessel families from [1, 2].

We say that $F \in \prod_{\omega\in\Omega} \mathcal{K}_\omega$ is strongly measurable if F as a mapping of Ω to $\bigoplus_{\omega\in\Omega} \mathcal{K}_\omega$ is measurable, where

$$\prod_{\omega\in\Omega} \mathcal{K}_\omega = \left\{ f : \Omega \to \bigcup_{\omega\in\Omega} \mathcal{K}_\omega : f(\omega) \in \mathcal{K}_\omega \right\}.$$

Definition 1.2. We say that $\Lambda = \{\Lambda_\omega \in B(\mathcal{H}, \mathcal{K}_\omega) : \omega \in \Omega\}$ is a continuous g-frame for $\mathcal{H}$ with respect to $\{\mathcal{K}_\omega : \omega \in \Omega\}$ if

(i) for each $f \in \mathcal{H}$, $\{\Lambda_\omega f : \omega \in \Omega\}$ is strongly measurable,

(ii) there are two constants $0 < A_\Lambda \le B_\Lambda < \infty$ such that

$$A_\Lambda \|f\|^2 \le \int_\Omega \|\Lambda_\omega f\|^2 \, d\mu(\omega) \le B_\Lambda \|f\|^2, \quad f \in \mathcal{H}. \tag{1.1}$$

We call A_Λ, B_Λ the lower and upper continuous g-frame bounds, respectively. Λ is called a tight continuous g-frame if $A_\Lambda = B_\Lambda$, and it is a Parseval continuous g-frame if $A_\Lambda = B_\Lambda = 1$. A family $\Lambda = \{\Lambda_\omega \in B(\mathcal{H}, \mathcal{K}_\omega) : \omega \in \Omega\}$ is called a continuous g-Bessel family for $\mathcal{H}$ with respect to $\{\mathcal{K}_\omega : \omega \in \Omega\}$ if the right side of the inequality (1.1) holds for all $f \in \mathcal{H}$, in this case, B_Λ is called the continuous g-Bessel constant.

Proposition 1.3 ([1])**.** *Let* $\Lambda = \{\Lambda_\omega \in B(\mathcal{H}, \mathcal{K}_\omega) : \omega \in \Omega\}$ *be a continuous g-frame. There exists a unique positive and invertible operator* $S_\Lambda : \mathcal{H} \to \mathcal{H}$ *such that*

$$\langle S_\Lambda f, g\rangle = \int_\Omega \langle \Lambda_\omega f, \Lambda_\omega g\rangle \, d\mu(\omega), \quad f, g \in \mathcal{H},$$

and $A_\Lambda I \leq S_\Lambda \leq B_\Lambda I$*.*

The operator S_Λ in the Proposition 1.3 is called the continuous g-frame operator of Λ.

We consider the space

$$\widehat{\mathcal{K}} = \left\{ F \in \prod_{\omega\in\Omega} \mathcal{K}_\omega : \text{F is strongly measurable, } \int_\Omega \|F(\omega)\|^2 \, d\mu(\omega) < \infty \right\}.$$

It is clear that $\widehat{\mathcal{K}}$ is a Hilbert space with point-wise operations and with the inner product given by

$$\langle F, G\rangle = \int_\Omega \langle F(\omega), G(\omega)\rangle \, d\mu(\omega).$$

Proposition 1.4 ([1])**.** *Let* $\Lambda = \{\Lambda_\omega \in B(\mathcal{H}, \mathcal{K}_\omega) : \omega \in \Omega\}$ *be a continuous g-Bessel family. Then, the mapping* $T_\Lambda : \widehat{\mathcal{K}} \to \mathcal{H}$ *defined by*

$$\langle T_\Lambda F, g\rangle = \int_\Omega \langle \Lambda_\omega^* F(\omega), g\rangle \, d\mu(\omega), \quad F \in \widehat{\mathcal{K}},\ g \in \mathcal{H}, \tag{1.2}$$

is linear and bounded with $\|T_\Lambda\| \leq \sqrt{B_\Lambda}$*. Also, for each* $g \in \mathcal{H}$ *and* $\omega \in \Omega$*, we have*

$$(T_\Lambda^* g)(\omega) = \Lambda_\omega g.$$

The operators T_Λ and T_Λ^* in the Proposition 1.4 are called the synthesis and analysis operators of $\Lambda = \{\Lambda_\omega \in B(\mathcal{H}, \mathcal{K}_\omega) : \omega \in \Omega\}$, respectively.

Definition 1.5. Let $\Lambda = \{\Lambda_\omega \in B(\mathcal{H}, \mathcal{K}_\omega) : \omega \in \Omega\}$ and $\Theta = \{\Theta_\omega \in B(\mathcal{H}, \mathcal{K}_\omega) : \omega \in \Omega\}$ be two continuous g-Bessel families such that

$$\langle f, g\rangle = \int_\Omega \langle \Theta_\omega f, \Lambda_\omega g\rangle \, d\mu(\omega), \quad f, g \in \mathcal{H},$$

then, Θ is called a dual of Λ.

Let $\Lambda = \{\Lambda_\omega \in B(\mathcal{H}, \mathcal{K}_\omega) : \omega \in \Omega\}$ be a continuous g-frame. Then, $\widetilde{\Lambda} = \Lambda S_\Lambda^{-1} = \left\{\Lambda_\omega S_\Lambda^{-1} \in B(\mathcal{H}, \mathcal{K}_\omega) : \omega \in \Omega\right\}$ is a continuous g-frame and $\widetilde{\Lambda}$ is a dual of Λ. We call $\widetilde{\Lambda}$ the canonical dual of Λ.

Two continuous g-Bessel families $\Lambda = \{\Lambda_\omega \in B(\mathcal{H}, \mathcal{K}_\omega) : \omega \in \Omega\}$ and $\Theta = \{\Theta_\omega \in B(\mathcal{H}, \mathcal{K}_\omega) : \omega \in \Omega\}$ are called weakly equal, if for all $f \in \mathcal{H}$,

$$\Lambda_\omega f = \Theta_\omega f, \quad \text{a.e. } \omega \in \Omega.$$

Definition 1.6 ([3])**.** Let $\Lambda = \{\Lambda_\omega \in B(\mathcal{H}, \mathcal{K}_\omega) : \omega \in \Omega\}$ and $\Theta = \{\Theta_\omega \in B(\mathcal{H}, \mathcal{K}_\omega) : \omega \in \Omega\}$ be two continuous g-Bessel families. The family Θ is called a generalized dual of Λ (or a g-dual of Λ), whenever the well-defined operator $S_{\Lambda\Theta} : \mathcal{H} \to \mathcal{H}$,

$$\langle S_{\Lambda\Theta} f, g\rangle = \int_\Omega \langle \Theta_\omega f, \Lambda_\omega g\rangle \, d\mu(\omega), \quad f, g \in \mathcal{H},$$

is invertible.

In the case that, the continuous g-Bessel family Θ is a g-dual of the continuous g-Bessel family Λ, then, Θ is a dual of a continuous g-Bessel family $\Lambda S_{\Theta\Lambda}^{-1} = \left\{\Lambda_\omega S_{\Theta\Lambda}^{-1} \in B(\mathcal{H}, \mathcal{K}_\omega) : \omega \in \Omega\right\}$, i.e.

$$\langle f, g\rangle = \int_\Omega \left\langle \Theta_\omega f, \Lambda_\omega S_{\Theta\Lambda}^{-1} g\right\rangle d\mu(\omega), \quad f, g \in \mathcal{H}. \tag{1.3}$$

As continuous frames are generalized by continuous g-frames, the above definition is the generalization of reproducing pair of weakly measurable functions [5].

Proposition 1.7 ([2])**.** *Let $\Lambda = \{\Lambda_\omega \in B(\mathcal{H}, \mathcal{K}_\omega) : \omega \in \Omega\}$ and $\Theta = \{\Theta_\omega \in B(\mathcal{H}, \mathcal{K}_\omega) : \omega \in \Omega\}$ be continuous g-Bessel families and $m \in L^\infty(\Omega, \mu)$. The operator $M_{m,\Lambda,\Theta} : \mathcal{H} \to \mathcal{H}$ defined by*

$$\langle M_{m,\Lambda,\Theta} f, g\rangle = \int_\Omega m(\omega) \langle \Theta_\omega f, \Lambda_\omega g\rangle \, d\mu(\omega), \quad f, g \in \mathcal{H},$$

is a bounded operator with bound $\|m\|_\infty \sqrt{B_\Lambda B_\Theta}$.

The operator $M_{m,\Lambda,\Theta}$ in the Proposition 1.7 is called the continuous g-Bessel multiplier for Λ and Θ with respect to m. Note that $M_{1,\Lambda,\Theta} = S_{\Lambda\Theta}$.

Proposition 1.8 ([2])**.** *Let $\Lambda = \{\Lambda_\omega \in B(\mathcal{H}, \mathcal{K}_\omega) : \omega \in \Omega\}$ and $\Theta = \{\Theta_\omega \in B(\mathcal{H}, \mathcal{K}_\omega) : \omega \in \Omega\}$ be continuous g-Bessel families and $m \in L^\infty(\Omega, \mu)$. Then*

$$M^*_{m,\Lambda,\Theta} = M_{\overline{m},\Theta,\Lambda}.$$

2. Invertibility of Multipliers for Continuous g-Bessel Families

In this section, we are going to get some results relevant to invertibility of continuous g-Bessel multipliers by generalizing results of [18].

For every $\Lambda_\omega \in B(\mathcal{H}, \mathcal{K}_\omega), \omega \in \Omega$ and $m \in L^\infty(\Omega, \mu)$, we have

$$\begin{aligned}\|(m(\omega)\Lambda_\omega) f\| &= \|m(\omega)\Lambda_\omega f\| \\ &= |m(\omega)| \, \|\Lambda_\omega f\| \\ &\le \|m\|_\infty \|\Lambda_\omega\| \, \|f\|, \quad f \in \mathcal{H},\end{aligned}$$

so, $m(\omega)\Lambda_\omega \in B(\mathcal{H}, \mathcal{K}_\omega)$.

Proposition 2.1. *Let* $\Lambda = \{\Lambda_\omega \in B(\mathcal{H}, \mathcal{K}_\omega) : \omega \in \Omega\}$ *be a continuous g-Bessel family and* $m \in L^\infty(\Omega, \mu)$*. Then*

(i) $m\Lambda = \{m(\omega)\Lambda_\omega \in B(\mathcal{H}, \mathcal{K}_\omega) : \omega \in \Omega\}$ *is a continuous g-Bessel family with the continuous g-Bessel constant* $B_\Lambda \|m\|_\infty^2$*.*

(ii) $M_{m,\Lambda,\Theta} = M_{1,\overline{m}\Lambda,\Theta} = M_{1,\Lambda,m\Theta}$*, where* $\overline{m}\Lambda = \{\overline{m(\omega)}\Lambda_\omega \in B(\mathcal{H}, \mathcal{K}_\omega) : \omega \in \Omega\}$ *and* $m\Theta = \{m(\omega)\Theta_\omega \in B(\mathcal{H}, \mathcal{K}_\omega) : \omega \in \Omega\}$*.*

Proof. (i) For every $f \in \mathcal{H}$, we have

$$\begin{aligned}\int_\Omega \|m(\omega)\Lambda_\omega f\|^2 \, d\mu(\omega) &= \int_\Omega |m(\omega)|^2 \|\Lambda_\omega f\|^2 \, d\mu(\omega) \\ &\leq \|m\|_\infty^2 \int_\Omega \|\Lambda_\omega f\|^2 \, d\mu(\omega) \\ &\leq B_\Lambda \|m\|_\infty^2 \|f\|^2 .\end{aligned}$$

(ii) By (i), $\overline{m}\Lambda$ and $m\Theta$ are continuous g-Bessel families. For every $f, g \in \mathcal{H}$, we have

$$\begin{aligned}\langle M_{m,\Lambda,\Theta} f, g\rangle &= \int_\Omega m(\omega) \langle \Theta_\omega f, \Lambda_\omega g\rangle \, d\mu(\omega) \\ &= \int_\Omega \left\langle \Theta_\omega f, \overline{m(\omega)}\Lambda_\omega g \right\rangle d\mu(\omega) \\ &= \int_\Omega \langle m(\omega)\Theta_\omega f, \Lambda_\omega g\rangle \, d\mu(\omega).\end{aligned}$$

Therefore, $M_{m,\Lambda,\Theta} = M_{1,\overline{m}\Lambda,\Theta} = M_{1,\Lambda,m\Theta}$. □

By generalizing a result of [18], the following proposition gives necessary conditions for invertibility of multipliers for continuous g-Bessel families.

Proposition 2.2. *Let* $\Lambda = \{\Lambda_\omega \in B(\mathcal{H}, \mathcal{K}_\omega) : \omega \in \Omega\}$ *and* $\Theta = \{\Theta_\omega \in B(\mathcal{H}, \mathcal{K}_\omega) : \omega \in \Omega\}$ *be continuous g-Bessel families and* $0 \neq m \in L^\infty(\Omega, \mu)$*. If* $M_{m,\Lambda,\Theta} \in GL(\mathcal{H})$*, then*

(i) $m\Theta = \{m(\omega)\Theta_\omega \in B(\mathcal{H}, \mathcal{K}_\omega) : \omega \in \Omega\}$ *and* $\overline{m}\Lambda = \{\overline{m(\omega)}\Lambda_\omega \in B(\mathcal{H}, \mathcal{K}_\omega) : \omega \in \Omega\}$ *are continuous g-frames with lower continuous g-frame bounds* $\left(B_\Lambda \left\|M_{m,\Lambda,\Theta}^{-1}\right\|^2\right)^{-1}$ *and* $\left(B_\Theta \left\|M_{m,\Lambda,\Theta}^{-1}\right\|^2\right)^{-1}$*, respectively.*

(ii) Λ *and* Θ *are continuous g-frames with lower continuous g-frame bounds* $\left(B_\Lambda \|m\|_\infty^2 \left\|M_{m,\Lambda,\Theta}^{-1}\right\|^2\right)^{-1}$ *and* $\left(B_\Theta \|m\|_\infty^2 \left\|M_{m,\Lambda,\Theta}^{-1}\right\|^2\right)^{-1}$*, respectively.*

Proof. (i) Since $M_{m,\Lambda,\Theta} \in GL(\mathcal{H})$, by Proposition 2.1 (ii), the operators $M_{1,\Lambda,m\Theta}$ and $M_{1,\overline{m}\Lambda,\Theta}$ are invertible. Let $f \in \mathcal{H}$ and $f \neq 0$, then, from Proposition 1.8, we have

$$\begin{aligned}
\|f\|^2 &= |\langle f, f\rangle| \\
&= \left|\left\langle f, M_{1,\Lambda,m\Theta}^{-1} M_{1,\Lambda,m\Theta} f\right\rangle\right| \\
&= \left|\left\langle M_{1,m\Theta,\Lambda} M_{1,m\Theta,\Lambda}^{-1} f, f\right\rangle\right| \\
&= \left|\int_\Omega \left\langle \Lambda_\omega M_{1,m\Theta,\Lambda}^{-1} f, m(\omega)\Theta_\omega f\right\rangle d\mu(\omega)\right| \\
&= \left|\left\langle T_\Lambda^* M_{1,m\Theta,\Lambda}^{-1} f, T_{m\Theta}^* f\right\rangle\right| \\
&\leq \left\|T_\Lambda^* M_{1,m\Theta,\Lambda}^{-1} f\right\| \|T_{m\Theta}^* f\| \\
&\leq \sqrt{B_\Lambda} \left\|M_{1,m\Theta,\Lambda}^{-1}\right\| \|f\| \|T_{m\Theta}^* f\|,
\end{aligned}$$

therefore, we get

$$\begin{aligned}
\frac{1}{B_\Lambda \left\|M_{m,\Lambda,\Theta}^{-1}\right\|^2} \|f\|^2 &= \frac{1}{B_\Lambda \left\|M_{1,m\Theta,\Lambda}^{-1}\right\|^2} \|f\|^2 \\
&\leq \|T_{m\Theta}^* f\|^2 \\
&= \int_\Omega \|m(\omega)\Theta_\omega f\|^2 d\mu(\omega).
\end{aligned} \tag{2.1}$$

Similarly, we have

$$\begin{aligned}
\frac{1}{B_\Theta \left\|M_{m,\Lambda,\Theta}^{-1}\right\|^2} \|f\|^2 &\leq \|T_{\overline{m}\Lambda}^* f\|^2 \\
&= \int_\Omega \left\|\overline{m(\omega)}\Lambda_\omega f\right\|^2 d\mu(\omega).
\end{aligned} \tag{2.2}$$

It is clear that the inequlities (2.1) and (2.2) also hold for $f = 0$. So, by Proposition 2.1 (i), $m\Theta$ and $\overline{m}\Lambda$ are continuous g-frames.

(ii) For every $f \in \mathcal{H}$ by inequality (2.1), we have

$$\begin{aligned}
\frac{1}{B_\Lambda \left\|M_{m,\Lambda,\Theta}^{-1}\right\|^2} \|f\|^2 &\leq \int_\Omega \|m(\omega)\Theta_\omega f\|^2 d\mu(\omega) \\
&\leq \|m\|_\infty^2 \int_\Omega \|\Theta_\omega f\|^2 d\mu(\omega),
\end{aligned}$$

therefore,

$$\frac{1}{B_\Lambda \|m\|_\infty^2 \left\|M_{m,\Lambda,\Theta}^{-1}\right\|^2} \|f\|^2 \leq \int_\Omega \|\Theta_\omega f\|^2 \, d\mu(\omega).$$

Similarly, by inequality (2.2), we have

$$\frac{1}{B_\Theta \|m\|_\infty^2 \left\|M_{m,\Lambda,\Theta}^{-1}\right\|^2} \|f\|^2 \leq \int_\Omega \|\Lambda_\omega f\|^2 \, d\mu(\omega).$$

Thus Λ and Θ are continuous g-frames. □

Note that Proposition 2.2 (ii), generalizes Proposition 3.2 of [1]. In the following proposition, by generalizing a conclusion from [7], we get a dual for continuous g-Bessel families Λ and Θ when $M_{m,\Lambda,\Theta} \in GL(\mathcal{H})$.

Proposition 2.3. *Let* $\Lambda = \{\Lambda_\omega \in B(\mathcal{H}, \mathcal{K}_\omega) : \omega \in \Omega\}$ *and* $\Theta = \{\Theta_\omega \in B(\mathcal{H}, \mathcal{K}_\omega) : \omega \in \Omega\}$ *be continuous g-Bessel families and* $0 \neq m \in L^\infty(\Omega, \mu)$. *If* $M_{m,\Lambda,\Theta} \in GL(\mathcal{H})$, *then,* Θ *and*

$$\overline{m}\Lambda M_{\overline{m},\Theta,\Lambda}^{-1} = \left\{\overline{m(\omega)}\Lambda_\omega M_{\overline{m},\Theta,\Lambda}^{-1} \in B(\mathcal{H}, \mathcal{K}_\omega) : \omega \in \Omega\right\},$$

are dual. Also, Λ *and* $m\Theta M_{m,\Lambda,\Theta}^{-1} = \{m(\omega)\Theta_\omega M_{m,\Lambda,\Theta}^{-1} \in B(\mathcal{H}, \mathcal{K}_\omega) : \omega \in \Omega\}$ *are dual.*

Proof. By Proposition 2.2 (ii), Λ and Θ are continuous g-frames. Since $M_{m,\Lambda,\Theta} \in GL(\mathcal{H})$ and $M_{\overline{m},\Theta,\Lambda} = M_{m,\Lambda,\Theta}^* \in GL(\mathcal{H})$ and then, by Proposition 2.2 (i) and [1, Proposition 3.3], we conclude

$$\overline{m}\Lambda M_{\overline{m},\Theta,\Lambda}^{-1} = \left\{\overline{m(\omega)}\Lambda_\omega M_{\overline{m},\Theta,\Lambda}^{-1} \in B(\mathcal{H}, \mathcal{K}_\omega) : \omega \in \Omega\right\},$$

is a continuous g-frame. For every $f, g \in \mathcal{H}$, we have

$$\begin{aligned}\int_\Omega \left\langle \Theta_\omega f, \overline{m(\omega)}\Lambda_\omega M_{\overline{m},\Theta,\Lambda}^{-1} g\right\rangle d\mu(\omega) &= \int_\Omega m(\omega) \left\langle \Theta_\omega f, \Lambda_\omega M_{\overline{m},\Theta,\Lambda}^{-1} g\right\rangle d\mu(\omega) \\ &= \left\langle M_{m,\Lambda,\Theta} f, M_{\overline{m},\Theta,\Lambda}^{-1} g\right\rangle \\ &= \langle f, g\rangle .\end{aligned}$$

Also,

$$\begin{aligned}\int_\Omega \left\langle m(\omega)\Theta_\omega M_{m,\Lambda,\Theta}^{-1} f, \Lambda_\omega g\right\rangle d\mu(\omega) &= \int_\Omega m(\omega) \left\langle \Theta_\omega M_{m,\Lambda,\Theta}^{-1} f, \Lambda_\omega g\right\rangle d\mu(\omega) \\ &= \left\langle M_{m,\Lambda,\Theta} M_{m,\Lambda,\Theta}^{-1} g\right\rangle \\ &= \langle f, g\rangle .\end{aligned}$$

□

The following result is the generalization of [8, Theorem 1.1.] and [3, Proposition 8.] with similar proof.

Proposition 2.4. *Let* $\Lambda = \{\Lambda_\omega \in B(\mathcal{H},\mathcal{K}_\omega) : \omega \in \Omega\}$ *and* $\Theta = \{\Theta_\omega \in B(\mathcal{H},\mathcal{K}_\omega) : \omega \in \Omega\}$ *be continuous g-Bessel families and* $0 \neq m \in L^\infty(\Omega,\mu)$*. If* $M_{m,\Lambda,\Theta} \in GL(\mathcal{H})$*, then,*

(i) *There is a dual* $\widehat{\Theta}$ *of* Θ*, such that for every dual* Λ^d *of* Λ *we have* $M^{-1}_{m,\Lambda,\Theta} = M_{\frac{1}{\overline{m}},\Lambda^d,\widehat{\Theta}}$*.*

(ii) *There is a dual* $\widehat{\Lambda}$ *of* Λ*, such that for every dual* Θ^d *of* Θ *we have* $M^{-1}_{m,\Lambda,\Theta} = M_{\frac{1}{\overline{m}},\widehat{\Lambda},\Theta^d}$*.*

Proof. (i) By Proposition 2.3, $\widehat{\Theta} = \overline{m}\Lambda M^{-1}_{\overline{m},\Theta,\Lambda}$ and Θ are dual. Similar to proof of [3, Proposition 8.] and by Propositions 2.1 for every dual Λ^d of Λ we have

$$\begin{aligned} M^{-1}_{m,\Lambda,\Theta} &= M^{-1}_{1,\overline{m}\Lambda,\Theta} \\ &= S^{-1}_{(\overline{m}\Lambda)\Theta} \\ &= T_{\Lambda^d} T^*_{\Lambda S^{-1}_{(\overline{m}\Lambda)\Theta}} \\ &= T_{\Lambda^d} T^*_{\Lambda M^{-1}_{m,\Lambda,\Theta}} \\ &= T_{\Lambda^d} T^*_{\frac{1}{\overline{m}}\widehat{\Theta}} \\ &= M_{\frac{1}{\overline{m}},\Lambda^d,\widehat{\Theta}}. \end{aligned}$$

(ii) The proof is similar to the proof of (i). □

By generalizing a result of [18], the following results give sufficient conditions for invertibility of multipliers for continuous g-frames. The [18, Proposition 2.2.] gives the criterion for the invertibility of operators and we apply this proposition in the proof of the following results.

Theorem 2.5. *Let* $\Lambda = \{\Lambda_\omega \in B(\mathcal{H},\mathcal{K}_\omega) : \omega \in \Omega\}$ *be a continuous g-frame and* $\Theta = \{\Theta_\omega \in B(\mathcal{H},\mathcal{K}_\omega) : \omega \in \Omega\}$ *be a family of operators such that for each* $f \in \mathcal{H}$, $\{\Theta_\omega f\}_{\omega\in\Omega}$ *is strongly measurable and there exists* $\nu \in \left[0, \frac{A^2_\Lambda}{B_\Lambda}\right)$ *such that*

$$\int_\Omega \|(\Lambda_\omega - \Theta_\omega)f\|^2 \, d\mu(\omega) \leq \nu \|f\|^2, \quad f \in \mathcal{H}. \tag{2.3}$$

Suppose $m \in L^\infty(\Omega,\mu)$ *such that for some positive constants* δ *we have* $m(\omega) \geq \delta > 0$ *a.e. and* $\frac{\|m\|_\infty}{\delta}\sqrt{\nu} < \frac{A_\Lambda}{\sqrt{B_\Lambda}}$*. Then,* $M_{m,\Lambda,\Theta} \in GL(\mathcal{H})$ *and*

$$\frac{1}{\|m\|_\infty B_\Lambda + \|m\|_\infty \sqrt{\nu B_\Lambda}} \|f\| \leq \left\| M^{-1}_{m,\Lambda,\Theta} f \right\| \leq \frac{1}{\delta A_\Lambda - \|m\|_\infty \sqrt{\nu B_\Lambda}} \|f\|,$$

for every $f \in \mathcal{H}$, *and*

$$M_{m,\Lambda,\Theta}^{-1} = \sum_{k=0}^{\infty} \left[S_{\sqrt{m}\Lambda}^{-1}\left(S_{\sqrt{m}\Lambda} - M_{m,\Lambda,\Theta}\right)\right]^k S_{\sqrt{m}\Lambda}^{-1}.$$

Also,

$$\left\| M_{m,\Lambda,\Theta}^{-1} - \sum_{k=0}^{n} \left[S_{\sqrt{m}\Lambda}^{-1}\left(S_{\sqrt{m}\Lambda} - M_{m,\Lambda,\Theta}\right)\right]^k S_{\sqrt{m}\Lambda}^{-1} \right\| \leq \left(\frac{\|m\|_\infty \sqrt{\nu B_\Lambda}}{\delta A_\Lambda}\right)^{n+1} \frac{1}{\delta A_\Lambda - \|m\|_\infty \sqrt{\nu B_\Lambda}}, \quad n \in \mathbb{N}.$$

Proof. If $\nu = 0$, then, by ineqaulity (2.3), Λ and Θ are weakly equal so, for every $f, g \in \mathcal{H}$, we have

$$\begin{aligned}\langle M_{m,\Lambda,\Theta} f, g\rangle &= \int_\Omega m(\omega) \langle \Theta_\omega f, \Lambda_\omega g\rangle \, d\mu(\omega) \\ &= \int_\Omega m(\omega) \langle \Lambda_\omega f, \Lambda_\omega g\rangle \, d\mu(\omega) \\ &= \langle M_{m,\Lambda,\Lambda} f, g\rangle .\end{aligned}$$

Therefore, by [2, Proposition 3.3.], $M_{m,\Lambda,\Theta} = M_{m,\Lambda,\Lambda} = S_{\sqrt{m}\Lambda}$ is an invertible operator with lower and upper bounds δA_Λ and $\|m\|_\infty B_\Lambda$, respectively, where $\sqrt{m}\Lambda = \{\sqrt{m(\omega)}\Lambda_\omega \in B(\mathcal{H}, \mathcal{K}_\omega) : \omega \in \Omega\}$. Therefore, for every $f \in \mathcal{H}$, we have

$$\begin{aligned}\frac{1}{\|m\|_\infty B_\Lambda} \|f\| &\leq \left\| M_{m,\Lambda,\Lambda}^{-1} f \right\| \\ &= \left\| S_{\sqrt{m}\Lambda}^{-1} f \right\| \\ &\leq \frac{1}{\delta A_\Lambda} \|f\| .\end{aligned} \tag{2.4}$$

For $\nu > 0$, by inequality (2.3), the family $\Lambda - \Theta = \{\Lambda_\omega - \Theta_\omega \in B(\mathcal{H}, \mathcal{K}_\omega) : \omega \in \Omega\}$ is a continuous g-Bessel family so, Θ is a continuous g-Bessel family. Thus by Proposition 1.7, $M_{m,\Lambda,\Theta}$ is a well-defined bounded operator. By (2.3), for any $f, g \in \mathcal{H}$, we have

$$\begin{aligned}&\left|\left\langle M_{m,\Lambda,\Theta} f - S_{\sqrt{m}\Lambda} f, g \right\rangle\right| \\ &\quad = \left| \int_\Omega m(\omega) \langle \Theta_\omega f, \Lambda_\omega g\rangle \, d\mu(\omega) - \int_\Omega m(\omega) \langle \Lambda_\omega f, \Lambda_\omega g\rangle \, d\mu(\omega) \right| \\ &\quad = \left| \int_\Omega m(\omega) \langle (\Theta_\omega - \Lambda_\omega) f, \Lambda_\omega g\rangle \, d\mu(\omega) \right| \\ &\quad \leq \int_\Omega |m(\omega)| \, |\langle (\Theta_\omega - \Lambda_\omega) f, \Lambda_\omega g\rangle| \, d\mu(\omega)\end{aligned}$$

$$\begin{aligned}
&\leq \|m\|_\infty \int_\Omega \|(\Theta_\omega - \Lambda_\omega) f\| \, \|\Lambda_\omega g\| \, d\mu(\omega) \\
&\leq \|m\|_\infty \left(\int_\Omega \|(\Theta_\omega - \Lambda_\omega) f\|^2 \, d\mu(\omega) \right)^{\frac{1}{2}} \left(\int_\Omega \|\Lambda_\omega g\|^2 \, d\mu(\omega) \right)^{\frac{1}{2}} \\
&\leq \|m\|_\infty \sqrt{\nu B_\Lambda} \, \|f\| \, \|g\| \, .
\end{aligned}$$

Therefore, we have

$$\left\| M_{m,\Lambda,\Theta} f - S_{\sqrt{m}\Lambda} f \right\| \leq \|m\|_\infty \sqrt{\nu B_\Lambda} \, \|f\| \, . \tag{2.5}$$

Since $\|m\|_\infty \sqrt{\nu B_\Lambda} < \delta A_\Lambda \leq \frac{1}{\left\| S^{-1}_{\sqrt{m}\Lambda} \right\|}$, by [18, Proposition 2.2.], $M_{m,\Lambda,\Theta} \in GL(\mathcal{H})$ and

$$M^{-1}_{m,\Lambda,\Theta} = \sum_{k=0}^{\infty} \left[S^{-1}_{\sqrt{m}\Lambda} (S_{\sqrt{m}\Lambda} - M_{m,\Lambda,\Theta}) \right]^k S^{-1}_{\sqrt{m}\Lambda}.$$

Also, by inequality (2.4) for every $f \in \mathcal{H}$, we have

$$\begin{aligned}
\frac{1}{\|m\|_\infty \sqrt{\nu B_\Lambda} + \|m\|_\infty B_\Lambda} \|f\| &\leq \frac{1}{\|m\|_\infty \sqrt{B_\Lambda \nu} + \left\| S_{\sqrt{m}\Lambda} \right\|} \|f\| \\
&\leq \left\| M^{-1}_{m,\Lambda,\Theta} f \right\| \\
&\leq \frac{1}{\frac{1}{\left\| S^{-1}_{\sqrt{m}\Lambda} \right\|} - \|m\|_\infty \sqrt{B_\Lambda \nu}} \|f\| \\
&\leq \frac{1}{\delta A_\Lambda - \|m\|_\infty \sqrt{\nu B_\Lambda}} \|f\| \, .
\end{aligned}$$

Since $\frac{\|m\|_\infty}{\delta} \sqrt{\nu} < \frac{A_\Lambda}{\sqrt{B_\Lambda}}$ and $\frac{\|m\|_\infty \sqrt{\nu B_\Lambda}}{\delta A_\Lambda} < 1$. By inequalities (2.4) and (2.5) for $n \in \mathbb{N}$, we have

$$\begin{aligned}
&\left\| M^{-1}_{m,\Lambda,\Theta} - \sum_{k=0}^{n} \left[S^{-1}_{\sqrt{m}\Lambda} (S_{\sqrt{m}\Lambda} - M_{m,\Lambda,\Theta}) \right]^k S^{-1}_{\sqrt{m}\Lambda} \right\| \\
&\quad = \left\| \sum_{k=n+1}^{\infty} \left[S^{-1}_{\sqrt{m}\Lambda} (S_{\sqrt{m}\Lambda} - M_{m,\Lambda,\Theta}) \right]^k S^{-1}_{\sqrt{m}\Lambda} \right\| \\
&\quad \leq \left\| S^{-1}_{\sqrt{m}\Lambda} \right\| \sum_{k=n+1}^{\infty} \left\| S^{-1}_{\sqrt{m}\Lambda} \right\|^k \left\| S_{\sqrt{m}\Lambda} - M_{m,\Lambda,\Theta} \right\|^k \\
&\quad \leq \frac{1}{\delta A_\Lambda} \sum_{k=n+1}^{\infty} \left(\frac{\|m\|_\infty \sqrt{\nu B_\Lambda}}{\delta A_\Lambda} \right)^k
\end{aligned}$$

$$= \left(\frac{\|m\|_\infty \sqrt{\nu B_\Lambda}}{\delta A_\Lambda} \right)^{n+1} \frac{1}{\delta A_\Lambda - \|m\|_\infty \sqrt{\nu B_\Lambda}}. \qquad \square$$

Note that by considering $\Theta = \Lambda$, in Theorem 2.5, we get the Proposition 3.3 of [2].

Proposition 2.6. *Let $\Lambda = \{\Lambda_\omega \in B(\mathcal{H}, \mathcal{K}_\omega) : \omega \in \Omega\}$ be a continuous g-frame. Let $m \in L^\infty(\Omega, \mu)$ such that $\|m - 1\|_\infty \leq \lambda < \frac{A_\Lambda}{B_\Lambda}$ for some λ. Then, $M_{m,\Lambda,\Lambda} \in GL(\mathcal{H})$ and*

$$\frac{1}{(\lambda + 1)B_\Lambda} \|f\| \leq \left\| M^{-1}_{m,\Lambda,\Lambda} f \right\| \leq \frac{1}{A_\Lambda - \lambda B_\Lambda} \|f\|, \quad f \in \mathcal{H},$$

and

$$M^{-1}_{m,\Lambda,\Lambda} = \sum_{k=0}^{\infty} \left[S^{-1}_\Lambda (S_\Lambda - M_{m,\Lambda,\Lambda}) \right]^k S^{-1}_\Lambda.$$

Also,

$$\left\| M^{-1}_{m,\Lambda,\Lambda} - \sum_{k=0}^{n} \left[S^{-1}_\Lambda (S_\Lambda - M_{m,\Lambda,\Lambda}) \right]^k S^{-1}_\Lambda \right\| \leq \left(\frac{\lambda B_\Lambda}{A_\Lambda} \right)^{n+1} \frac{1}{A_\Lambda - \lambda B_\Lambda}, \quad n \in \mathbb{N}.$$

Proof. For every $f, g \in \mathcal{H}$, we have

$$\begin{aligned}
&|\langle M_{1,\Lambda,m\Lambda} f - S_\Lambda f, g \rangle| \\
&\quad = \left| \int_\Omega \langle (m(\omega) - 1)\Lambda_\omega f, \Lambda_\omega g \rangle \, d\mu(\omega) \right| \\
&\quad \leq \int_\Omega |m(\omega) - 1| \, |\langle \Lambda_\omega f, \Lambda_\omega g \rangle| \, d\mu(\omega) \\
&\quad \leq \|m - 1\|_\infty \int_\Omega \|\Lambda_\omega f\| \, \|\Lambda_\omega g\| \, d\mu(\omega) \\
&\quad \leq \|m - 1\|_\infty \left(\int_\Omega \|\Lambda_\omega f\|^2 \, d\mu(\omega) \right)^{\frac{1}{2}} \left(\int_\Omega \|\Lambda_\omega g\|^2 \, d\mu(\omega) \right)^{\frac{1}{2}} \\
&\quad \leq \lambda B_\Lambda \|f\| \, \|g\| .
\end{aligned}$$

Therefore, we have

$$\|M_{1,\Lambda,m\Lambda} f - S_\Lambda f\| \leq \lambda B_\Lambda \|f\| .$$

Since $0 \leq \lambda B_\Lambda < A_\Lambda \leq \frac{1}{\|S^{-1}_\Lambda\|}$, similar to the proof of the Theorem 2.5, by [18, Proposition 2.2.] and Proposition 2.1 (ii), the proof is completed. $\square$

Theorem 2.7. *Let $\Lambda = \{\Lambda_\omega \in B(\mathcal{H}, \mathcal{K}_\omega) : \omega \in \Omega\}$ be a continuous g-frame and $\Theta = \{\Theta_\omega \in B(\mathcal{H}, \mathcal{K}_\omega) : \omega \in \Omega\}$ be a family of operators such that for each $f \in \mathcal{H}$, $\{\Theta_\omega f\}_{\omega \in \Omega}$ is strongly measurable. Suppose there exists $\nu \in [0, \frac{A^2_\Lambda}{B_\Lambda})$ such that the inequality (2.3) is satisfied. Let*

$m \in L^{\infty}(\Omega,\mu)$ that $\|m-1\|_{\infty} \leq \lambda < \frac{A_\Lambda - \sqrt{\nu B_\Lambda}}{B_\Lambda + \sqrt{\nu B_\Lambda}}$ for some λ. Then, $M_{m,\Lambda,\Theta} \in GL(\mathcal{H})$ and for every $f \in \mathcal{H}$,

$$\frac{1}{(\lambda+1)(B_\Lambda + \sqrt{\nu B_\Lambda})} \|f\| \leq \left\| M_{m,\Lambda,\Theta}^{-1} f \right\| \leq \frac{1}{A_\Lambda - \lambda B_\Lambda - (\lambda+1)\sqrt{\nu B_\Lambda}} \|f\|,$$

and

$$M_{m,\Lambda,\Theta}^{-1} = \sum_{k=0}^{\infty} \left[M_{m,\Lambda,\Lambda}^{-1} (M_{m,\Lambda,\Lambda} - M_{m,\Lambda,\Theta}) \right]^k M_{m,\Lambda,\Lambda}^{-1}.$$

Also,

$$\left\| M_{m,\Lambda,\Theta}^{-1} - \sum_{k=0}^{n} \left[M_{m,\Lambda,\Lambda}^{-1} (M_{m,\Lambda,\Lambda} - M_{m,\Lambda,\Theta}) \right]^k M_{m,\Lambda,\Lambda}^{-1} \right\| \leq \left(\frac{(\lambda+1)\sqrt{\nu B_\Lambda}}{A_\Lambda - \lambda B_\Lambda} \right)^{n+1} \frac{1}{A_\Lambda - \lambda B_\Lambda - (\lambda+1)\sqrt{\nu B_\Lambda}}, \quad n \in \mathbb{N}.$$

Proof. If $\nu = 0$, by the inequality (2.3), Λ and Θ are weakly equal. Also, for $\nu = 0$ we have $\|m-1\|_{\infty} \leq \lambda < \frac{A_\Lambda}{B_\Lambda}$. Then, by Proposition 2.6, for $\nu = 0$ the proof is completed. For $\nu \neq 0$ by inequality (2.3), the family $\Lambda - \Theta$ is a continuous g-Bessel family so, Θ is a continuous g-Bessel family. Similar to the proof of Theorem 2.5, for every $f, g \in \mathcal{H}$, we have

$$\begin{aligned} &|\langle M_{m,\Lambda,\Theta} f - M_{m,\Lambda,\Lambda} f, g \rangle| \\ &= \left| \int_{\Omega} m(\omega) \langle (\Theta_\omega - \Lambda_\omega) f, \Lambda_\omega g \rangle \, d\mu(\omega) \right| \\ &\leq \|m\|_{\infty} \left(\int_{\Omega} \|(\Theta_\omega - \Lambda_\omega) f\|^2 \, d\mu(\omega) \right)^{\frac{1}{2}} \left(\int_{\Omega} \|\Lambda_\omega g\|^2 \, d\mu(\omega) \right)^{\frac{1}{2}} \\ &\leq \|m\|_{\infty} \sqrt{\nu B_\Lambda} \|f\| \|g\|. \end{aligned}$$

Thus by $\|m-1\|_{\infty} \leq \lambda$, we have

$$\|M_{m,\Lambda,\Theta} f - M_{m,\Lambda,\Lambda} f\| \leq \|m\|_{\infty} \sqrt{\nu B_\Lambda} \|f\| \leq (\lambda+1)\sqrt{\nu B_\Lambda} \|f\|.$$

By $\lambda < \frac{A_\Lambda - \sqrt{\nu B_\Lambda}}{B_\Lambda + \sqrt{\nu B_\Lambda}}$, we have $(\lambda+1)\sqrt{\nu B_\Lambda} < A_\Lambda - \lambda B_\Lambda$ and since

$$\|m-1\|_{\infty} \leq \lambda < \frac{A_\Lambda - \sqrt{\nu B_\Lambda}}{B_\Lambda + \sqrt{\nu B_\Lambda}} < \frac{A_\Lambda}{B_\Lambda},$$

by Proposition 2.6, we have $(\lambda+1)\sqrt{\nu B_\Lambda} < A_\Lambda - \lambda B_\Lambda \leq \frac{1}{\|M_{m,\Lambda,\Lambda}^{-1}\|}$ and $\|M_{m,\Lambda,\Lambda}\| \leq (\lambda+1) B_\Lambda$. Therefore, by [18, Proposition 2.2.], $M_{m,\Lambda,\Theta} \in GL(\mathcal{H})$ and for every $f \in \mathcal{H}$, we have

$$\frac{1}{(\lambda+1)(B_\Lambda + \sqrt{\nu B_\Lambda})} \|f\| = \frac{1}{(\lambda+1)\sqrt{\nu B_\Lambda} + (\lambda+1) B_\Lambda} \|f\|$$

$$\begin{aligned}
&\leq \frac{1}{(\lambda+1)\sqrt{\nu B_\Lambda}+\|M_{m,\Lambda,\Lambda}\|}\,\|f\| \\
&\leq \left\|M^{-1}_{m,\Lambda,\Theta}f\right\| \\
&\leq \frac{1}{\frac{1}{\left\|M^{-1}_{m,\Lambda,\Lambda}\right\|}-(\lambda+1)\sqrt{\nu B_\Lambda}}\,\|f\| \\
&\leq \frac{1}{A_\Lambda-\lambda B_\Lambda-(\lambda+1)\sqrt{\nu B_\Lambda}}\,\|f\|,
\end{aligned}$$

and

$$M^{-1}_{m,\Lambda,\Theta}=\sum_{k=0}^{\infty}\left[M^{-1}_{m,\Lambda,\Lambda}(M_{m,\Lambda,\Lambda}-M_{m,\Lambda,\Theta})\right]^k M^{-1}_{m,\Lambda,\Lambda}.$$

Also, for $n\in\mathbb{N}$, we have

$$\begin{aligned}
&\left\|M^{-1}_{m,\Lambda,\Theta}-\sum_{k=0}^{n}\left[M^{-1}_{m,\Lambda,\Lambda}(M_{m,\Lambda,\Lambda}-M_{m,\Lambda,\Theta})\right]^k M^{-1}_{m,\Lambda,\Lambda}\right\| \\
&\quad=\left\|\sum_{k=n+1}^{\infty}\left[M^{-1}_{m,\Lambda,\Lambda}(M_{m,\Lambda,\Lambda}-M_{m,\Lambda,\Theta})\right]^k M^{-1}_{m,\Lambda,\Lambda}\right\| \\
&\quad\leq \left\|M^{-1}_{m,\Lambda,\Lambda}\right\|\sum_{k=n+1}^{\infty}\left\|M^{-1}_{m,\Lambda,\Lambda}\right\|^k\|M_{m,\Lambda,\Lambda}-M_{m,\Lambda,\Theta}\|^k \\
&\quad\leq \frac{1}{A_\Lambda-\lambda B_\Lambda}\sum_{k=n+1}^{\infty}\left(\frac{(\lambda+1)\sqrt{\nu B_\Lambda}}{A_\Lambda-\lambda B_\Lambda}\right)^k \\
&\quad=\left(\frac{(\lambda+1)\sqrt{\nu B_\Lambda}}{A_\Lambda-\lambda B_\Lambda}\right)^{n+1}\frac{1}{A_\Lambda-\lambda B_\Lambda-(\lambda+1)\sqrt{\nu B_\Lambda}}. \qquad \square
\end{aligned}$$

Proposition 2.8. *Let $\Lambda=\{\Lambda_\omega\in B(\mathcal{H},\mathcal{K}_\omega):\omega\in\Omega\}$ be a continuous g-frame and $S\in GL(\mathcal{H})$. Also, suppose $m\in L^\infty(\Omega,\mu)$ satisfies one of the following conditions:*

(i) *for some positive constants δ, $m(\omega)\geq\delta>0$ a.e.*
(ii) $\|m-1\|_\infty\leq\lambda<\frac{A_\Lambda}{B_\Lambda}$ *for some λ.*

Then, the operators $M_{m,\Lambda,\Lambda S}$ and $M_{m,\Lambda S,\Lambda}$ are invertible and

$$M^{-1}_{m,\Lambda,\Lambda S}=S^{-1}M^{-1}_{m,\Lambda,\Lambda},\qquad M^{-1}_{m,\Lambda S,\Lambda}=M^{-1}_{m,\Lambda,\Lambda}(S^{-1})^*,$$

where $\Lambda S=\{\Lambda_\omega S\in B(\mathcal{H},K_\omega):\omega\in\Omega\}$.

Proof. By [1, Proposition 3.3], ΛS is a continuous g-frame. For every $f,g\in\mathcal{H}$, we have

$$\langle M_{m,\Lambda,\Lambda S}f,g\rangle=\int_\Omega m(\omega)\,\langle\Lambda_\omega Sf,\Lambda_\omega g\rangle\,d\mu(\omega)$$

$$= \langle M_{m,\Lambda,\Lambda} S f, g\rangle ,$$

$$\begin{aligned}\langle M_{m,\Lambda S,\Lambda} f, g\rangle &= \int_\Omega m(\omega) \langle \Lambda_\omega f, \Lambda_\omega S g\rangle \, d\mu(\omega) \\ &= \langle M_{m,\Lambda,\Lambda} f, Sg\rangle \\ &= \langle S^* M_{m,\Lambda,\Lambda} f, g\rangle .\end{aligned}$$

Therefore, $M_{m,\Lambda,\Lambda S} = M_{m,\Lambda,\Lambda} S$ and $M_{m,\Lambda S,\Lambda} = S^* M_{m,\Lambda,\Lambda}$. If (i) is satisfied, then, by [2, Proposition 3.3], $M_{m,\Lambda,\Lambda} \in GL(\mathcal{H})$, and if (ii) is satisfied, then, by Proposition 2.6, $M_{m,\Lambda,\Lambda} \in GL(\mathcal{H})$, so, the proof is completed. □

Corollary 2.9. *Let $\Lambda = \{\Lambda_\omega \in B(\mathcal{H}, \mathcal{K}_\omega) : \omega \in \Omega\}$ be a continuous g-frame. Also suppose $m \in L^\infty(\Omega, \mu)$ satisfies one of the following conditions:*

(i) *for some positive constants δ, $m(\omega) \geq \delta > 0$ a.e.*
(ii) $\|m-1\|_\infty \leq \lambda < \frac{A_\Lambda}{B_\Lambda}$ *for some λ.*

Then, the operators $M_{m,\Lambda,\widetilde{\Lambda}}$ and $M_{m,\widetilde{\Lambda},\Lambda}$ are invertible and

$$M^{-1}_{m,\Lambda,\widetilde{\Lambda}} = S_\Lambda M^{-1}_{m,\Lambda,\Lambda}, \qquad M^{-1}_{m,\widetilde{\Lambda},\Lambda} = M^{-1}_{m,\Lambda,\Lambda} S_\Lambda .$$

Proof. By Proposition 2.8, for $S = S_\Lambda^{-1}$ the proof is completed. □

Theorem 2.10. *Let $\Lambda = \{\Lambda_\omega \in B(\mathcal{H}, \mathcal{K}_\omega) : \omega \in \Omega\}$ and $\Theta = \{\Theta_\omega \in B(\mathcal{H}, \mathcal{K}_\omega) : \omega \in \Omega\}$ be dual continuous g-frames. Let $m \in L^\infty(\Omega, \mu)$ such that $\|m-1\|_\infty \leq \lambda < \frac{1}{\sqrt{B_\Lambda B_\Theta}}$ for some λ. Then, $M_{m,\Lambda,\Theta} \in GL(\mathcal{H})$ and*

$$\frac{1}{1+\lambda\sqrt{B_\Lambda B_\Theta}} \|f\| \leq \left\| M^{-1}_{m,\Lambda,\Theta} f \right\| \leq \frac{1}{1-\lambda\sqrt{B_\Lambda B_\Theta}} \|f\| , \quad f \in \mathcal{H}, \tag{2.6}$$

and

$$M^{-1}_{m,\Lambda,\Theta} = \sum_{k=0}^{\infty} \left(M_{(1-m),\Lambda,\Theta} \right)^k . \tag{2.7}$$

Also,

$$\left\| M^{-1}_{m,\Lambda,\Theta} - \sum_{k=0}^{n} \left(M_{(1-m),\Lambda,\Theta} \right)^k \right\| \leq \frac{(\lambda\sqrt{B_\Lambda B_\Theta})^{n+1}}{1-\lambda\sqrt{B_\Lambda B_\Theta}}, \quad n \in \mathbb{N}.$$

Proof. For every $f, g \in \mathcal{H}$, we have

$$\begin{aligned}&|\langle M_{m,\Lambda,\Theta} f - f, g\rangle| \\ &\quad = |\langle M_{m,\Lambda,\Theta} f, g\rangle - \langle f, g\rangle| \\ &\quad = \left| \int_\Omega (m(\omega) - 1) \langle \Theta_\omega f, \Lambda_\omega g\rangle \, d\mu(\omega) \right|\end{aligned}$$

$$\leq ||m(\omega)-1||_\infty \int_\Omega ||\Theta_\omega f|| ||\Lambda_\omega g|| d\mu(\omega)$$
$$\leq ||m(\omega)-1||_\infty \left(\int_\Omega ||\Theta_\omega f||^2 d\mu(\omega)\right)^{\frac{1}{2}} \left(\int_\Omega ||\Lambda_\omega g||^2 d\mu(\omega)\right)^{\frac{1}{2}}$$
$$\leq \lambda\sqrt{B_\Lambda B_\Theta}\, \|f\| \, \|g\| .$$

Therefore,

$$\|M_{m,\Lambda,\Theta}f - f\| \leq \lambda\sqrt{B_\Lambda B_\Theta}\, \|f\| .$$

Since $\lambda\sqrt{B_\Lambda B_\Theta} < 1 = \frac{1}{\|I^{-1}\|}$ and $I - M_{m,\Lambda,\Theta} = M_{(1-m),\Lambda,\Theta}$, by [18, Proposition 2.2.], inequality (2.6) and equality (2.7) are satisfied. Also, for $n \in \mathbb{N}$ we have

$$\begin{aligned}\left\| M^{-1}_{m,\Lambda,\Theta} - \sum_{k=0}^{n} \left(M_{(1-m),\Lambda,\Theta}\right)^k \right\| &= \left\| \sum_{k=n+1}^{\infty} \left(M_{(1-m),\Lambda,\Theta}\right)^k \right\| \\ &\leq \sum_{k=n+1}^{\infty} \left\|M_{(1-m),\Lambda,\Theta}\right\|^k \\ &\leq \sum_{k=n+1}^{\infty} (\lambda\sqrt{B_\Lambda B_\Theta})^k \\ &= \frac{(\lambda\sqrt{B_\Lambda B_\Theta})^{n+1}}{1-\lambda\sqrt{B_\Lambda B_\Theta}}.\end{aligned}$$

□

Proposition 2.11. *Let $\Lambda = \{\Lambda_\omega \in B(\mathcal{H}, \mathcal{K}_\omega) : \omega \in \Omega\}$ be a continuous g-frame and $\Theta = \{\Theta_\omega \in B(\mathcal{H}, \mathcal{K}_\omega) : \omega \in \Omega\}$ be a family of operators such that for each $f \in \mathcal{H}$, $\{\Theta_\omega f\}_{\omega\in\Omega}$ is strongly measurable that inequality* (2.3) *is satisfied for some $\nu > 0$. If $\nu < A_\Lambda$, then, Θ is a continuous g-frame.*

Proof. By ineqaulity (2.3), the family $\Lambda - \Theta = \{\Lambda_\omega - \Theta_\omega \in B(\mathcal{H}, \mathcal{K}_\omega) : \omega \in \Omega\}$ is a continuous g-Bessel family so, Θ is a continuous g-Bessel family. For every $f, g \in \mathcal{H}$, we have

$$\begin{aligned}\left|\left\langle M_{1,\widetilde{\Lambda},\Theta}f - f, g\right\rangle\right| &= \left|\left\langle M_{1,\widetilde{\Lambda},\Theta}f - M_{1,\widetilde{\Lambda},\Lambda}f, g\right\rangle\right| \\ &= \left|\int_\Omega \left\langle (\Theta_\omega - \Lambda_\omega)f, \widetilde{\Lambda}_\omega g\right\rangle d\mu(\omega)\right| \\ &\leq \left(\int_\Omega ||(\Theta_\omega - \Lambda_\omega)f||^2 d\mu(\omega)\right)^{\frac{1}{2}} \left(\int_\Omega ||\widetilde{\Lambda}_\omega g||^2 d\mu(\omega)\right)^{\frac{1}{2}} \\ &\leq \sqrt{\nu\frac{1}{A_\Lambda}}\, \|f\| \, \|g\| .\end{aligned}$$

Thus

$$\left\| I - M_{1,\widetilde{\Lambda},\Theta} \right\| \leq \sqrt{\nu \frac{1}{A_\Lambda}} < 1.$$

It shows that $M_{1,\widetilde{\Lambda},\Theta} \in GL(\mathcal{H})$ therefore, according to Proposition 2.2 (ii), Θ is a continuous g-frame. □

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[1] Department of Mathematics, Faculty of Sciences, University of Mohaghegh Ardabili, 56199-11367,Ardabil, Iran.

Email address: mrabdollahpour@yahoo.com; m.abdollah@uma.ac.ir

[2] Department of Mathematics, Faculty of Sciences, University of Mohaghegh Ardabili, 56199-11367,Ardabil, Iran.

Email address: khedmati.y@uma.ac.ir; khedmatiy.y@gmail.com