Automatic Continuity of Almost n-Multiplicative Linear Functionals

Abbas Zivari-Kazempour

Sahand Communications in Mathematical Analysis

Print ISSN: 2322-5807
Online ISSN: 2423-3900
Volume: 19
Number: 4
Pages: 97-107

Sahand Commun. Math. Anal.
DOI: 10.22130/scma.2022.549567.1065

Sahand Communications in Mathematical Analysis (SCMA) Vol. 19 No. 4 (2022), 97-107
http://scma.maragheh.ac.ir
DOI: 10.22130/scma.2022.549567.1065

Automatic Continuity of Almost n-Multiplicative Linear Functionals

Abbas Zivari-Kazempour

Abstract. We generalize a theorem due to Jarosz, by proving that every almost n-multiplicative linear functional on Banach algebra A is automatically continuous. The relation between almost multiplicative and almost n-multiplicative linear functional on Banach algebra A is also investigated. Additionally, for commutative Banach algebra A, we prove that every almost Jordan homomorphism $\varphi : A \longrightarrow \mathbb{C}$ is an almost n-Jordan homomorphism.

1. Introduction

Let A and B be complex Banach algebras and $\varphi : A \longrightarrow B$ be a linear map. Then, φ is called an n-homomorphism if for all $a_1, a_2, \ldots, a_n \in A$,

$$\varphi(a_1 a_2 \cdots a_n) = \varphi(a_1)\varphi(a_2)\cdots\varphi(a_n).$$

The concept of n-homomorphism was studied for complex algebras in [6] and [11].

A linear map φ between algebras A and B is called an n-Jordan homomorphism if $\varphi(a^n) = \varphi(a)^n$, for all $a \in A$. This notion was introduced by Herstein in [10].

In the case of $n = 2$, these concepts coincide with the classical definitions of homomorphism and Jordan homomorphism, respectively.

Clearly, each homomorphism is an n-homomorphism for every $n \geq 2$, but the converse does not hold in general. For example, if $\varphi : A \longrightarrow B$ is a homomorphism, then $\psi := -\varphi$ is a 3-homomorphism which is not a homomorphism [6].

2020 *Mathematics Subject Classification.* 47B48, 46H25.

Key words and phrases. Almost n-multiplicative, Almost n-Jordan homomorphism, Automatic continuity, Semisimple.

Received: 25 February 2022, Accepted: 22 June 2022.

Also, every n-homomorphism is an n-Jordan homomorphism, but in general, the converse is false. Zelazko in [20] has given a characterization of Jordan homomorphism that we will mention.

Theorem 1.1. *Suppose that A is a Banach algebra, which need not be commutative, and suppose that B is a semi-simple commutative Banach algebra. Then each Jordan homomorphism $\varphi : A \longrightarrow B$ is a homomorphism.*

This result has been proved by the author in [22] for 3-Jordan homomorphisms with the additional hypothesis that the Banach algebra A is unital, and then it is extended for all $n \in \mathbb{N}$ in [1].

Bodaghi and İnceboz in [4], extended Theorem 1.1 for $n \in \{3, 4\}$ by considering an extra condition that $\varphi(a^2b - ba^2) = 0$ for all $a, b \in A$.

There are two basic results concerning the automatic continuity of homomorphisms between Banach algebras.

The first basic result is due to Šilov, which is expressed as follows (see also [5]).

Theorem 1.2 ([7, Theorem 2.3.3])**.** *Let A and B be two Banach algebras such that B is commutative and semi-simple. Then, every homomorphism $\varphi : A \longrightarrow B$ is automatically continuous.*

The second result is the following which is due to Johnson (see also [17]).

Theorem 1.3 ([14, Theorem 2])**.** *Let A and B be Banach algebras where B is semi-simple. Then, every surjective homomorphism $\varphi : A \longrightarrow B$ is automatically continuous.*

Theorem 1.3 was extended to n-homomorphism in [8]. Now the following question can be raised.

Question 1.4. *Does Theorem 1.2 generalize to n-homomorphisms?*

A linear map φ between Banach algebras A and B is called almost n-multiplicative if there exists $\varepsilon \geq 0$ such that for all $a_1, a_2, \ldots, a_n \in A$,

$$\|\varphi(a_1 a_2 \cdots a_n) - \varphi(a_1) \cdots \varphi(a_n)\| \leq \varepsilon \|a_1\| \|a_2\| \cdots \|a_n\| .$$

If $n = 2$, then φ is called simply almost multiplicative. Note that almost n-multiplicative turns out to be n-multiplicative, whenever $\varepsilon = 0$.

Jarosz [13] introduced the concept of an almost multiplicative function between Banach algebras. He investigated the automatic continuity of such maps and proved the following famous result.

Theorem 1.5 ([13, Proposition 5.5])**.** *Let φ be an almost multiplicative linear functional from Banach algebra A into $\mathbb{C}$. Then $\|\varphi\| \leq 1 + \varepsilon$, and hence φ is continuous.*

After that, Johnson obtained some results on the continuity of almost multiplicative functionals [15], and then he generalized his result to almost multiplicative maps between Banach algebras [16].

Since then, many authors have investigated almost multiplicative maps between Banach algebras, see for example [2, 18, 23].

Similarly, we have the next question which derives from Jarosz's theorem.

Question 1.6. *Does Theorem 1.5 generalize to almost n-multiplicative?*

In this paper, we give a positive answer to both Question 1.4 and Question 1.6. We also prove that every almost multiplicative linear functional on Banach algebra A is almost n-multiplicative, and the same is true for almost Jordan homomorphisms with the extra condition that A is commutative.

2. Continuity of n-Homomorphisms

We begin with the following well-known theorem.

Theorem 2.1 ([5, Proposition 3, § 16])**.** *Suppose that $\varphi : A \longrightarrow \mathbb{C}$ is a multiplicative linear functional on A. Then φ is continuous and $\|\varphi\| \leq 1$.*

A Banach algebra A is called *n-functionally continuous* if every n-multiplicative linear functional on A is continuous. If $n = 2$, then A is called functionally continuous, in the usual sense.

Theorem 2.2 ([19, Corollary 2.2])**.** *A topological algebra A is functionally continuous if and only if it is n-functionally continuous.*

Now, it follows from Theorem 2.1 and Theorem 2.2 that every n-multiplicative linear functional on A is continuous. More precisely, every n-homomorphism from a Banach algebra A into a commutative semisimple Banach algebra B is automatically continuous and so the answer to Question 1.4 is affirmative.

If A is a unital Banach algebra with unit e, then each n-multiplicative linear functional $\varphi : A \longrightarrow \mathbb{C}$ satisfies in $\varphi(a) = \varphi(e)^{n-1}\varphi(a)$, for all $a \in A$. On the other hand, one can also verify that $\psi(a) := \varphi(e)^{n-2}\varphi(a)$ is multiplicative and so continuous by Theorem 2.1. From this, we deduce that φ is continuous.

For non-unital Banach algebra A, we now outline an alternative proof for this result with direct methods. For $n = 3$, see [24, Theorem 5].

Theorem 2.3. *Let A be a Banach algebra and $\varphi : A \longrightarrow \mathbb{C}$ be an n-multiplicative linear functional. Then $\|\varphi\| \leq 1$, and hence φ is automatically continuous.*

Proof. Suppose that $\varphi : A \longrightarrow \mathbb{C}$ is an n-multiplicative. Since $\varphi \neq 0$, there exists $a \in A$ such that $\varphi(a) = 1$. For all $x \in A$, define $\psi : A \longrightarrow \mathbb{C}$ by $\psi(x) = \varphi(ax)$. Then for every $x, y \in A$,

$$\begin{aligned}\psi(xy) &= \varphi(axy) \\ &= \varphi(axy)\varphi(a)^{n-1} \\ &= \varphi(axya^{n-1}) \\ &= \varphi(ax)\varphi(ya)\varphi(a)^{n-2} \\ &= \varphi(ax)\varphi(ya).\end{aligned}$$

As

$$\begin{aligned}\varphi(ya) &= \varphi(a)^{n-1}\varphi(ya) \\ &= \varphi(a)^{n-2}\varphi(ay)\varphi(a) \\ &= \varphi(ay),\end{aligned}$$

we get

$$\begin{aligned}\psi(xy) &= \varphi(ax)\varphi(ay) \\ &= \psi(x)\psi(y),\end{aligned}$$

hence ψ is a multiplicative linear functional on A. Thus, ψ is continuous and $\|\psi\| \leq 1$. On the other hand, for all $x \in A$, we have

$$\begin{aligned}\psi(x) &= \varphi(ax) \\ &= \varphi(a)^{n-1}\varphi(ax) \\ &= \varphi(a^2)\varphi(a)^{n-2}\varphi(x) \\ &= \varphi(a^2)\varphi(x),\end{aligned} \tag{2.1}$$

which proves that $\varphi(a^2) \neq 0$. Let $w = \varphi(a^2)$. Since ψ is multiplicative, by (2.1) for all $x_1, x_2, \ldots, x_n \in A$, we get

$$\begin{aligned}w\varphi(x_1x_2\cdots x_n) &= \psi(x_1x_2\cdots x_n) \\ &= \psi(x_1)\psi(x_2)\cdots\psi(x_n) \\ &= w^n\varphi(x_1)\varphi(x_2)\cdots\varphi(x_n).\end{aligned}$$

Consequently, $|w| = 1$, so we conclude that $\|\varphi\| \leq 1$. □

We get the following result in a similar mannar to [24, Corollary 1].

Corollary 2.4. *Suppose that A is a Banach algebra and B is a semi-simple commutative Banach algebra. Then each n-homomorphism $\varphi : A \longrightarrow B$ is continuous.*

3. Continuity of Almost n-Multiplicative

Our main theorem in this section is to generalize Theorem 1.5 for almost n-multiplicative linear functionals. First, we prove it for the unital Banach algebra A.

Proposition 3.1. *Let A be a unital Banach algebra and $\varphi : A \longrightarrow \mathbb{C}$ be an almost n-multiplicative linear functional. Then φ is automatically continuous.*

Proof. For all $a \in A$, define $\psi : A \longrightarrow \mathbb{C}$ by $\psi(a) = \varphi(e)^{n-2}\varphi(a)$, where e is the unit of A. Then

$$\begin{aligned}|\psi(ab) - \psi(a)\psi(b)| &= \left|\varphi(e)^{n-2}\varphi(ab) - \varphi(e)^{n-2}\varphi(a)\varphi(e)^{n-2}\varphi(b)\right| \\ &\le \varepsilon \left|\varphi(e)^{n-2}\right| \left|\varphi(ae^{n-2}b) - \varphi(a)\varphi(e)^{n-2}\varphi(b)\right| \\ &\le \varepsilon \left|\varphi(e)^{n-2}\right| \|a\| \, \|e\|^{n-2} \, \|b\| \\ &\le \varepsilon' \, \|a\| \, \|b\| ,\end{aligned}$$

where $\varepsilon' = \varepsilon \left|\varphi(e)^{n-2}\right| \|e\|^{n-2}$. Therefore ψ is almost multiplicative and it is continuous by Theorem 1.5. Now the continuity of ψ implies that of φ. □

Lemma 3.2. *Let A be a Banach algebra and $\varphi : A \longrightarrow \mathbb{C}$ be an almost n-multiplicative linear functional. Then for all $a_1, a_2, \ldots, a_n, t \in A$, we have*

$$\begin{aligned}&|\varphi(t)|^{n-1} \cdot |\varphi(a_1 a_2 \cdots a_n) - \varphi(a_1)\varphi(a_2)\cdots\varphi(a_n)| \\ &\quad \le \varepsilon\big(2\,\|a_1\| \cdots \|a_{n-1}\| + |\varphi(a_1)\varphi(a_2)\cdots\varphi(a_{n-1})|\,\big)\, \|a_n\| \, \|t\|^{n-1} .\end{aligned}$$

Proof. Clearly, this is Lemma 3.1 of [12]. □

The next result is a generalization of Theorem 1.5. The case $n = 3$ is [24, Theorem 7].

Theorem 3.3. *Every almost n-multiplicative linear functional from a Banach algebra A into $\mathbb{C}$ is automatically continuous.*

Proof. Let $\varphi : A \longrightarrow \mathbb{C}$ be an almost n-homomorphism. Then, there exists $\varepsilon > 0$ such that

$$|\varphi(a_1 a_2 \cdots a_n) - \varphi(a_1)\varphi(a_2)\cdots\varphi(a_n)| \le \varepsilon \, \|a_1\| \, \|a_2\| \cdots \|a_n\| , \tag{3.1}$$

for all $a_1, a_2, \ldots, a_n \in A$. Set $\xi = \frac{1+\sqrt{1+4\varepsilon}}{2}$. If for all $a \in A$,

$$|\varphi(a)| \le \xi \, \|a\| , \tag{3.2}$$

then $\|\varphi\| \leq 1+\varepsilon$, and hence φ is continuous. If (3.2) does not hold, then by applying Lemma 3.2 and a method similar to [24, Theorem 7], we conclude that φ is n-multiplicative. Now, the continuity of φ follows from Theorem 2.3. □

Corollary 3.4. *Suppose that A and B are Banach algebras, where B is commutative and semisimple. Then each almost n-homomorphism $\varphi : A \longrightarrow B$ is continuous.*

Every multiplicative linear functional is n-multiplicative. Next, we prove the same result for almost multiplicative.

Theorem 3.5. *Let A be a Banach algebra and $\varphi : A \longrightarrow \mathbb{C}$ be an almost multiplicative. Then φ is almost n-multiplicative, for all $n \geq 2$.*

Proof. Let φ be an almost multiplicative. Hence there exists $\varepsilon > 0$ such that

$$|\varphi(ab) - \varphi(a)\varphi(b)| \leq \varepsilon \|a\| \|b\|, \quad a, b \in A. \tag{3.3}$$

Then by Theorem 1.5, φ is continuous and $\|\varphi\| \leq 1+\varepsilon$. Therefore, for all $a \in A$,

$$|\varphi(a)| \leq (1+\varepsilon)\|a\|. \tag{3.4}$$

By (3.3) and (3.4), for all $a, b, c \in A$, we have

$$\begin{aligned} |\varphi(abc) - \varphi(a)\varphi(b)\varphi(c)| &\leq |\varphi(abc) - \varphi(ab)\varphi(c)| \\ &\quad + |\varphi(ab)\varphi(c) - \varphi(a)\varphi(b)\varphi(c)| \\ &\leq \varepsilon \|ab\| \|c\| + |\varphi(ab) - \varphi(a)\varphi(b)| |\varphi(c)| \\ &\leq \varepsilon \|a\| \|b\| \|c\| + \varepsilon(1+\varepsilon) \|a\| \|b\| \|c\| \\ &\leq \varepsilon' \|a\| \|b\| \|c\|, \end{aligned}$$

where $\varepsilon' = \varepsilon(2+\varepsilon)$. Thus, φ is almost 3-multiplicative. Now, assume that φ is an almost n-multiplicative for some fixed $n \in \mathbb{N}$. Then there exists $\varepsilon_1 > 0$ such that

$$|\varphi(a_1a_2\cdots a_n) - \varphi(a_1)\varphi(a_2)\cdots\varphi(a_n)| \leq \varepsilon_1 \|a_1\| \|a_2\| \cdots \|a_n\|, \tag{3.5}$$

for all $a_1, a_2, \ldots, a_n \in A$. Hence by (3.3), (3.4) and (3.5), we get

$$\begin{aligned} &|\varphi(a_1a_2\cdots a_{n+1}) - \varphi(a_1)\varphi(a_2)\cdots\varphi(a_{n+1})| \\ &\quad \leq |\varphi(a_1a_2\cdots a_{n+1}) - \varphi(a_1a_2)\varphi(a_3)\cdots\varphi(a_{n+1})| \\ &\qquad + |\varphi(a_1a_2)\varphi(a_3)\cdots\varphi(a_{n+1}) - \varphi(a_1)\varphi(a_2)\cdots\varphi(a_{n+1})| \\ &\quad \leq \varepsilon_1 \|a_1a_2\| \|a_3\| \cdots \|a_{n+1}\| \\ &\qquad + \left|\varphi(a_1a_2) - \varphi(a_1)\varphi(a_2)\right| \left(|\varphi(a_3)\right| \cdots |\varphi(a_{n+1})|\,) \\ &\quad \leq \varepsilon_1 \|a_1\| \|a_2\| \|a_3\| \cdots \|a_{n+1}\| \end{aligned}$$

$$+\,\varepsilon \|a_1\| \|a_2\| \left((1+\varepsilon)^{n-1} \|a_3\| \cdots \|a_{n+1}\|\right)$$
$$\leq \varepsilon'' \|a_1\| \|a_2\| \|a_3\| \cdots \|a_{n+1}\| .$$

Consequently, φ is almost $(n+1)$-multiplicative for $\varepsilon'' = \varepsilon_1 + \varepsilon(1+\varepsilon)^{n-1}$. This finishes the proof. □

The converse of Theorem 3.5 fails, in general. This is illustrated by the following example.

Example 3.6. Let X be the normed algebra of all polynomials defined on $[0,1]$, and let $T : X \longrightarrow \mathbb{C}$ be a linear unbounded functional on X. Let

$$A = \left\{ \begin{bmatrix} 0 & f \\ 0 & 0 \end{bmatrix} : \quad f \in X \right\} \quad \text{and} \quad B = \left\{ \begin{bmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{bmatrix} : \quad a, b, c \in \mathbb{C} \right\},$$

and define $\varphi : A \longrightarrow B$ by

$$\varphi\left(\begin{bmatrix} 0 & f \\ 0 & 0 \end{bmatrix} \right) = \begin{bmatrix} 0 & z & z \\ 0 & 0 & z \\ 0 & 0 & 0 \end{bmatrix},$$

where $z = T(f)$. Then, φ is n-homomorphism for every $n \geq 3$, and hence it is almost n-homomorphism for all $\varepsilon \geq 0$. But, it is easy to check that φ is not almost homomorphism.

4. Almost n-Jordan Homomorphisms

Let A and B be Banach algebras and $\varphi : A \longrightarrow B$ be a linear map. Then φ is called almost mixed n-Jordan homomorphism if there exists $\varepsilon > 0$ such that

$$\|\varphi(a^n b) - \varphi(a)^n \varphi(b)\| \leq \varepsilon \|a\|^n \|b\| , \quad a, b \in A.$$

Moreover, φ is said to be almost n-Jordan homomorphism if there exists $\varepsilon > 0$ such that

$$\|\varphi(a^n) - \varphi(a)^n\| \leq \varepsilon \|a\|^n , \quad a \in A.$$

The following theorem gives a relation between almost mixed n-Jordan homomorphisms and almost n-homomorphisms.

Proposition 4.1. *Let A be an unital Banach algebra with unit e, and let $\varphi : A \longrightarrow \mathbb{C}$ be almost n-multiplicative such that $\varphi(e) = 1$. Then φ is almost multiplicative.*

Proof. This follows from Proposition 3.1. □

Theorem 4.2. *Let A and B be two commutative algebras and φ be an almost mixed n-Jordan homomorphism from A into B. Then for all $a_1, a_2, \ldots, a_n \in A$,*

$$\|\varphi(a_1 a_2 \cdots a_n) - \varphi(a_1)\varphi(a_2)\varphi(a_3 \cdots a_n)\| \leq 3\varepsilon \|a_1\| \|a_2\| \cdots \|a_n\| .$$

Proof. Let φ be an almost mixed n-Jordan homomorphism. Then there exists $\varepsilon > 0$ such that

$$\|\varphi(a^n b) - \varphi(a)^n \varphi(b)\| \leq \varepsilon \|a\|^n \|b\| , \tag{4.1}$$

for every $a, b \in A$. Since A and B are commutative, we get

$$\begin{aligned} &\varphi(xya_3 \cdots a_n) - \varphi(x)\varphi(y)\varphi(a_3 \cdots a_n) \\ &\quad = \frac{1}{2}\big[\varphi((x+y)^2 a_3 \cdots a_n) - \varphi(x+y)^2 \varphi(a_3 \cdots a_n) \\ &\qquad + \varphi(x)^2 \varphi(a_3 \cdots a_n) - \varphi(x^2 a_3 \cdots a_n) + \varphi(y)^2 \varphi(a_3 \cdots a_n) \\ &\qquad - \varphi(y^2 a_3 \cdots a_n)\big]. \end{aligned}$$

For all $x, y, a_3, \ldots, a_n \in A$ with $\|x\| = \|y\| = 1$, it follows from (4.1) and the above equality that

$$\begin{aligned} \|\varphi(xya_3 \cdots a_n) &- \varphi(x)\varphi(y)\varphi(a_3 \cdots a_n)\| \\ &\leq \frac{1}{2}\left\|\varphi((x+y)^2 a_3 \cdots a_n) - \varphi(x+y)^2 \varphi(a_3 \cdots a_n)\right\| \\ &\quad + \frac{1}{2}\big(\left\|\varphi(x)^2 \varphi(a_3 \cdots a_n) - \varphi(x^2 a_3 \cdots a_n)\right\| \\ &\quad + \left\|\varphi(y)^2 \varphi(a_3 \cdots a_n) - \varphi(y^2 a_3 \cdots a_n)\right\| \big) \\ &\leq \frac{1}{2}\varepsilon\big(\|x+y\|^2 + \|x\|^2 + \|y\|^2 \big) \|a_3 \cdots a_n\| \\ &\leq 3\varepsilon \|a_3\| \cdots \|a_n\| . \end{aligned} \tag{4.2}$$

Now, let $a_1, a_2, \ldots, a_n \in A$ be arbitrary. By setting $x = \frac{a_1}{\|a_1\|}$ and $y = \frac{a_2}{\|a_2\|}$ in (4.2), we get the result. □

As a consequence of Theorem 4.2, we get the following result.

Corollary 4.3. *Let A and B be two commutative algebras and φ from A into B be an almost mixed 3-Jordan homomorphism. Then φ is almost 3-homomorphism.*

The following result follows from Corollary 4.3 and Theorem 3.3.

Corollary 4.4. *Every almost mixed 3-Jordan homomorphism from commutative Banach algebra A into $\mathbb{C}$ is continuous.*

Corollary 4.5. *Suppose that A is a unital commutative Banach algebra such that $\varphi(e) = 1$. Then each almost mixed n-Jordan homomorphism $\varphi : A \longrightarrow \mathbb{C}$ is continuous.*

Combining Theorem 2.5 of [21] and Theorem 3.5, we get the following result.

Proposition 4.6. *Let A be a commutative Banach algebra. Then every almost Jordan homomorphism $\varphi : A \longrightarrow \mathbb{C}$ is an almost n-Jordan homomorphism.*

The converse of the previous proposition is not true. For example, let A, B and φ be as in Example 3.6. Then φ is almost n-Jordan homomorphism for all $n \geq 3$, but it is not almost Jordan homomorphism.

Recall that every continuous linear map between Banach algebras A and B is an almost n-Jordan homomorphism. In other words, let φ be a continuous linear map from A into B. Then there exists $\delta > 0$ such that $\|\varphi(a)\| \leq \delta \|a\|$, for all $a \in A$. Hence

$$\begin{aligned}\|\varphi(a^n) - \varphi(a)^n\| &\leq \|\varphi(a^n)\| + \|\varphi(a)^n\| \\ &\leq (\delta + \delta^n) \|a\|^n ,\end{aligned}$$

so φ is an almost n-Jordan homomorphism.

By [3, Theorem 2.4] or [9, Theorem 2.1], every n-Jordan homomorphism between two commutative Banach algebras is an n-homomorphism. Now, the following question can be raised.

Question 4.7. *Let $\varphi : A \longrightarrow B$ be an almost n-Jordan homomorphism between commutative Banach algebras.*

(i) *Is φ almost n-homomorphism?*
(ii) *If $B = \mathbb{C}$, then is φ automatically continuous?*

If the answer of (1) is positive, then the answer of (2) is affirmative by Theorem 3.3. For $n = 2, 3$, both parts (1) and (2) are valid. Indeed, if A is a commutative Banach algebra, then by [21, Theorem 2.5], each almost Jordan homomorphism $\varphi : A \longrightarrow \mathbb{C}$ is almost homomorphism and hence φ is continuous by Theorem 1.5. The case $n = 3$, is [24, Theorem 11].

Acknowledgment. The author gratefully acknowledges the helpful comments of the anonymous referees.

References

1. G. An, *Characterization of n-Jordan homomorphism*, Linear Multilinear Algebra, 66(4) (2018), pp. 671-680.
2. E. Ansari-Piri and N. Eghbali, *A note on multiplicative and almost multiplicative linear maps*, Honam Math. J., 27(4) (2005), pp. 641-647.

3. A. Bodaghi and H. İnceboz, *n-Jordan homomorphisms on commutative algebras*, Acta Math. Univ. Comen., 87(1) (2018), pp. 141-146.
4. A. Bodaghi and H. İnceboz, *Extension of Zelazko's theorem to n-Jordan homomorphisms*, Adv. Pure Appl. Math., 10(2) (2019), pp. 165-170.
5. F.F. Bonsall and J. Duncan, *Complete normed algebra*, Springer-Verlag, York, 1973.
6. J. Bračič and M.S. Moslehian, *On automatic continuity of* 3-*homomorphisms on Banach algebras*, Bull. Malays. Math. Sci. Soc., 30(2) (2007), pp. 195-200.
7. H.G. Dales, *Banach Algebras and Automatic Continuity*, London Mathematical Society, Monograph 24, Oxford, 2000.
8. M. Eshaghi Gordji, A. Jabbari and E. Karapinar, *Automatic continuity of surjective n-homomorphisms on Banach algebras*, Bull. Iran. Math. Soc., 41(5) (2015), pp. 1207-1211.
9. E. Gselmann, *On approximate n-Jordan homomorphisms*, Ann. Math. Sil., 28 (2014), pp. 47-58.
10. I.N. Herstein, *Jordan homomorphisms*, Trans. Amer. Math. Soc., 81(1) (1956), pp. 331-341.
11. Sh. Hejazian, M. Mirzavaziri and M.S. Moslehian, *n-homomorphisms*, Bull. Iran. Math. Soc., 31(1) (2005), pp. 13-23.
12. T.G. Honary, M. Omidi and A.H. Sanatpour, *Almost n-multiplicative maps between Fréchet algebras*, Int. J. Nonlinear Anal. Appl., 8(1) (2017), pp. 187-195.
13. K. Jarosz, *Perturbation of Banach algebras*, Lecture Notes in Mathematics, Springer-verlag, 1985.
14. B.E. Johnson, *The uniqueness of the (complete) norm topology*, Bull. Amer. Math. Soc., 73 (1967) pp. 537-539.
15. B.E. Johnson, *Approximately multiplicative functionals*, J. Lond. Math. Soc., 34(2) (1986), pp. 489-510.
16. B.E. Johnson, *Approximately multiplicative maps between Banach algebras*, J. London Math. Soc., 37(2) (1988), pp. 294-316.
17. T.J. Ransford, *A short proof of Johnson's uniqueness of norm theorem*, Bull. Lond. Math. Soc., 21(5) (1989), 487-488.
18. P. Semrl, *Almost multiplicative functions and almost linear multiplicative functionals*, Aequationes Math., 63(1) (2002), pp. 180-192.
19. H. Shayanpour, T.G. Honary and M.S. Hashemi, *Certain properties of n-characters and n-homomorphisms on topological algebras*, Bull. Malays. Math. Sci. Soc., 38(2) (2015), pp. 985-999.

20. W. Zelazko, *A characterization of multiplicative linear functionals in complex Banach algebras*, Studia Math., 30 (1968), pp. 83-85.
21. A. Zivari-Kazempour, *A note on δ-Jordan homomorphisms on Banach algebras*, Gol. J. Math. Anal., 2(2) (2014), pp. 70-72.
22. A. Zivari-Kazempour, *A characterization of* 3*-Jordan homomorphism on Banach algebras*, Bull. Aust. Math. Soc., 93(2) (2016), pp. 301-306.
23. A. Zivari-Kazempour, *Automatic continuiuy of n-Jordan homomorphisms on Banach algebras*, Commun. Korean Math. Soc., 33(1) (2018), pp. 165-170.
24. A. Zivari-Kazempour, *Automatic continuiuy of almost* 3*-homomorphisms and almost* 3*-Jordan homomorphisms on Banach algebras*, Adv. Oper. Theory, 5(4) (2020), pp. 1340-1349.

Department of Mathematics, Ayatollah Borujerdi University, Borujerd, Iran.

Email address: zivari@abru.ac.ir, zivari6526@gmail.com