Sahand Communications in Mathematical Analysis (SCMA) Vol. 10 No. 1 (2018), 1-15
http://scma.maragheh.ac.ir
DOI: 10.22130/scma.2017.27148

On generalized topological molecular lattices

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Abstract. In this paper, we introduce the concept of the generalized topological molecular lattices as a generalization of Wang's topological molecular lattices, topological spaces, fuzzy topological spaces, L-fuzzy topological spaces and soft topological spaces. Topological molecular lattices were defined by closed elements, but in this new structure we present the concept of the open elements and define a closed element by the pseudocomplement of an open element. We have two structures on a completely distributive complete lattice, topology and generalized co-topology which are not dual to each other. We study the basic concepts, in particular separation axioms and some relations among them.

1. Introduction and preliminaries

Since the set of open sets of a topological space is a frame, many important properties to topological spaces may be expressed without referring to the points. The first person who exploit possibility of applying the lattice theory to the topology was Henry Wallman. He used the lattice-theoretic ideas to construct what is now called the "Wallman compactification" of a T_1-topological space. This idea was pursued by Mckinsey, Tarski, Nöbeling, Lesier, Ehresmann, Bénabou, etc. However, the importance of attention to open sets as a lattice appeared as late as 1962 in [3, 9]. After that, many authors became interested and developed the field. The pioneering paper [6] by Isbell merits particular mention for opening several important topics. In 1983, Johnstone gave an excellent monograph "Stone spaces" which is still the standard reference book. Until then, all attempts had been about the modeling

2010 *Mathematics Subject Classification.* 06DXX, 06F30.
Key words and phrases. Topological molecular lattice, Generalized Topological molecular lattice, Generalized order homomorphism, Separation axiom.
Received: 20 December 2016, Accepted: 21 February 2017.
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of a topology but not a topological space. In a similar method, as we deal with a general topology, Wang Guo-Jun introduced the concept of the topological molecular lattices on a completely distributive complete lattice and studied some basic properties of them [10–12]. Since in the general topological lattice theory there is no concept of complement, he only introduced the concept of the topological molecular lattices by closed elements. In [1], it was introduced an other new model of point free topologies by frames.

In this paper, since a completely distributive complete lattice has pseudocomplement, we introduce the concept of the generalized topological molecular lattices by the open elements which are closed under finite meets and arbitrary joins. Also, we define a closed element by the pseudocomplement of an open element. Thus closed elements are not closed under finite joins, and hence we have two structures on a complete lattice, topology and generalized co-topology which are not dual to each other. One can easily verify that a general topology, fuzzy topology, L-fuzzy topology and soft topology all are special cases of the generalized topological molecular lattices.

A complete lattice L is said to be completely distributive if whenever $x_{ij} \in X$ for every $i \in I$ and $j \in J$, then

$$\bigvee_{i\in I}\bigwedge_{j\in J} x_{ij} = \bigwedge_{f\in J^I}\bigvee_{i\in I} x_{if(i)}.$$

A frame F is a complete lattice which satisfies the following distributive law:

$$a \wedge \left(\bigvee_{i\in I} b_i\right) = \bigvee_{i\in I}(a \wedge b_i) \quad \text{for each} \quad a, b_i \in F (i \in I).$$

Clearly, every completely distributive lattice is a frame. A pseudocomplement of an element a of a bounded lattice L is defined by $\max(a^{\perp})$, if there exists, and denoted by a^*, where $a^{\perp} = \{x \in L : x \wedge a = 0\}$. If F is a frame, then $a^* = \vee(a^{\perp})$. A distributive pseudocomplemented lattice L is said to be a Stone algebra if $a^* \vee a^{**} = 1$ for every $a \in L$ [2, 5, 8].

For two completely distributive L_1 and L_2, and a mapping $f : L_1 \to L_2$ which preserves arbitrary joins, let $\hat{f}$ denotes the right adjoint of f, then $\hat{f} : L_2 \to L_1$ is defined by $\hat{f}(y) = \bigvee\{x \in L_1 : f(x) \leq y\}$ for every $y \in L_2$. A mapping $f : L_1 \to L_2$ is said to be a generalized order homomorphism or a **gml**-map if f preserves joins and its right adjoint preserves sups and infs [6].

An element a of a lattice L is a coprime element, if $a \neq 0$ and $a \leq b \vee c$ implies that $a \leq b$ or $a \leq c$, for every $b, c \in L$. Coprime elements are also called moleculars. For a completely distributive lattice L, we denote by M the set of all coprime elements, and L is called a molecular lattice and the write it in a slightly complex form $L(M)$ to indicate the set M of molecules. It is well known that M is a join generating base for L [10].

Remark 1.1. The following simple assertions are useful throughout the paper. Let L be a complete pseudocomplemented lattice.

1) The map $*$ is decreasing and $a \leq a^{**}$ for every $a \in L$.
2) The map $**$ is identity on L^*, i.e., $a^{***} = a^*$ for all $a \in L$.
3) For every $a, b \in L$ we have

$$\begin{aligned} a \wedge b = 0 \quad &\Leftrightarrow \quad a \leq b^* \\ &\Leftrightarrow \quad b \leq a^* \\ &\Leftrightarrow \quad a^{**} \leq b^* \\ &\Leftrightarrow \quad a^{**} \wedge b = 0. \end{aligned}$$

4) If L is a frame and $S \subseteq L$, then $(\vee_{s \in S} s)^* = \wedge_{s \in S} s^*$.

2. Generalized topological molecular lattices

In this section, we introduce the concept of the generalized topological molecular lattices, and investigate some basic concepts of them.

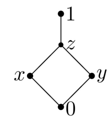
Definition 2.1. Let L be a molecular lattice. If τ is a sublattice of L, $\bigvee S \in \tau$ for each $S \subseteq \tau$ and $0, 1 \in \tau$, then (L, τ) is called a generalized topological molecular lattice space, or briefly, **gtml**$-$space. Every member of τ is said to be open and any member of $\tau^* = \{a^* : a \in \tau\}$ is said to be a closed element.

Notice that τ^* is closed under arbitrary meets, since

$$\left(\bigvee_{i \in I} a_i \right)^* = \bigwedge_{i \in I} a_i^*,$$

but it is not closed under finite joins. Thus we have two structures on a molecular lattice, topology and generalized co-topology which are not dual to each other. Furthermore, if τ^* is a sublattice of L, we say (L, τ) is a topological molecular lattice space, or briefly, **tml**$-$space.

A **gtml**$-$space need to be a **tml**$-$space. For example, consider the lattice L as follows:



It is obvious that $L^* = \{0, a, b, 1\}$ and $a \vee b = c \notin L^*$.
Thus (L, L) is a **gtml**$-$space while it is not a **tml**$-$space.

Definition 2.2. Let (L_1, τ_1) and (L_2, τ_2) be two **gtml**$-$spaces. A **gml**$-$map $f : (L_1, \tau_1) \to (L_2, \tau_2)$ is said to be:

1) continuous with respect to the topology or $i-$continuous if $b \in \tau_2$, implies $\hat{f}(b) \in \tau_1$,
2) continuous with respect to the generalized co-topology or $c-$continuous if $b \in \tau_2^*$, implies $\hat{f}(b) \in \tau_1^*$,
3) continuous if it is $i-$continuous and $c-$continuous.

Let $(L(M), \tau)$ be a **gtml**$-$space and $a \in L$. We define $a^\circ = \vee\{t \in \tau \mid t \le a\}$ and $\bar{a} = \wedge\{x \in \tau^* \mid a \le x\}$. An open element $P \in \tau$ is called a neighborhood of $a \in M$, if $a \le p$. Also a closed element $p \in \tau^*$ is called a remote-neighborhood of $a \in M$, if $a \not\le p$. The set of all remote-neighborhoods of a is denoted by $\tau^*(a)$, and the set of all neighborhoods of a is denoted by $\tau(a)$. Similarly, we define $\tau^{**}(a)$.

Definition 2.3. A **gml**$-$map $f : (L_1(M_1), \tau_1) \to (L_2(M_1), \tau_2)$ is said to be $i-$continuous at point $a \in M_1$, if $(\hat{f}(b))^\circ \in \tau_1(a)$, for every $b \in \tau_2(f(a))$.

Theorem 2.4. *Let $f : (L_1(M_1), \tau_1) \to (L_2(M_2), \tau_2)$ be a* **gml**$-$*map. Then f is $i-$continuous if and only if it is $i-$continuous at a, for every $a \in M_1$.*

Proof. Suppose that f is $i-$continuous and $a \in M_1$. Then for any $b \in \tau_2(f(a))$, $\hat{f}(b)$ is open. Clearly, $a \le (\hat{f}(b))^\circ$. Hence $(\hat{f}(b))^\circ \in \tau_1(a)$ and so f is $i-$continuous at a. Conversely, suppose that for any $a \in M_1$, f is $i-$continuous at a and $b \in \tau_2$. We may assume that $\hat{f}(b) \neq 0$ and $a \le \hat{f}(b)$ where $a \in M_1$. Then $f(a) \le b$ and so $b \in \tau_2(f(a))$. Hence $(\hat{f}(b))^\circ \in \tau_1(a)$, i.e., $a \le \hat{f}(b)$ implies that $a \le (\hat{f}(b))^\circ$, or $\hat{f}(b) \le (\hat{f}(b))^\circ$. Thus $\hat{f}(b) \in \tau_1$. $\square$

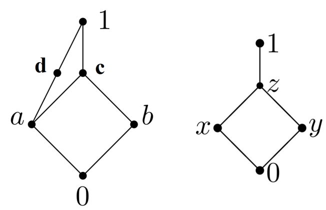
The proof of the following theorem is a result of Definition 2.2.

Theorem 2.5. *Let $f : L_1 \to L_2$ be a* **gml**$-$*map. Then:*

1) *f is $i-$continuous if and only if $\hat{f}(a^\circ) \le (\hat{f}(a))^\circ$, for every $a \in L_2$.*
2) *f is $c-$continuous if and only if $\overline{\hat{f}(a)} \le \hat{f}(\bar{a})$, for every $a \in L_2$.*

3) *f is $c-$continuous if and only if $f(\bar{a}) \leq \overline{f(a)}$, for every $a \in L_1$.*

Example 2.6. An $i-$continuous map need not to be a $c-$continuous map. For instance, consider L_1 and L_2 respectively, as follows:



Let $\tau_1 = \tau_2 = \{0, b, 1\}$ and define the map $f : L_1 \to L_2$ by:

$$f(t) = \begin{cases} 1, & t \neq d, \\ t, & \text{oth.} \end{cases}$$

Clearly, f is a **gml**$-$ map, $\hat{f}(t) = t$, for each $t \in L_2$, $\tau_1^* = \{0, d, 1\}$ and $\tau_2^* = \{0, a, 1\}$. Then f is $i-$continuous but it is not $c-$continuous, because $a \in \tau_2^*$ and $\hat{f}(a) = a \notin \tau_1^*$.

Conversely, a $c-$continuous map need not to be an $i-$continuous map. If we put $\tau_1 = \{0, a, 1\}$ and $\tau_2 = \{0, a, c, 1\}$, then $\tau_1^* = \tau_2^* = \{0, b, 1\}$ and f is $c-$continuous but it is not $i-$continuous, because $c \in \tau_2$ and $\hat{f}(c) = c \notin \tau_1$.

Recall that, a Boolean algebra is a distributive lattice with complements, thus in a Boolean algebra L, we have $a^{**} = a$ for all $a \in L$. Let (L, τ) be a gtml-space. Since $a^{***} = a^*$ for all $a \in L$, the map $* : \tau^* \to \tau^{**}$ is bijective. Also, since τ and τ^* are not dual to each other, the map $* : \tau \to \tau^*$ is not bijective. Clearly, in a Boolean algebra, $\tau = \tau^{**}$. If the map $* : \tau \to \tau^*$ is bijective, then L is called topologically injective.

Theorem 2.7 ([7])**.** *Let $f : L_1 \to L_2$ be a* **gml**$-$*map. Then the following statements hold.*

1) $\hat{f}(a^*) \leq (\hat{f}(a))^*$.
2) *If L_2 is a Boolean algebra, then $\hat{f}(a^*) = (\hat{f}(a))^*$.*

Theorem 2.8. *Let $f : L_1 \to L_2$ be a* **gml**$-$*map and L_2 a Boolean algebra. Then f is $i-$continuous if and only if it is $c-$continuous.*

Proof. By Theorem 2.7, the result follows. □

Definition 2.9. Let $f : (L_1, \tau_1) \to (L_2, \tau_2)$ be a **gml**$-$map. Then:

1) f is an open map if $f(a) \in \tau_2$, for every $a \in \tau_1$.
2) f is a closed map if $f(a) \in \tau_2^*$, for every $a \in \tau_1^*$.

The following theorems are immediate consequences of the definitions of open and closed maps.

Theorem 2.10. *Let $f : L_1 \to L_2$ be an $i-$continuous* **gml**$-$*map. Then the following statements are equivalent.*

1) *f is an open map,*
2) *$f(a^\circ) \leq (f(a))^\circ$, for every $a \in L_1$,*
3) *$(\hat{f}(b))^\circ = \hat{f}(b^\circ)$, for every $b \in L_2$.*

Theorem 2.11. *Let $f : L_1 \to L_2$ be a $c-$continuous* **gml**$-$*map. Then the following statements are equivalent.*

1) *f is a closed map,*
2) *$\overline{f(a)} = f(\bar{a})$, for every $a \in L_1$.*

Recall that an element b of L is called crisp if for any $a \in M, a \not\leq b$ implies that $a \wedge b = 0$, or equivalently, $a \wedge b \neq 0$ implies that $a \leq b$.

Definition 2.12. Let $(L(M), \tau)$ be a **gtml**$-$space. Then L is said to be:

1) $i-$crisp if p is crisp, for all $p \in \tau$,
2) $c-$crisp if p is crisp, for all $p \in \tau^*$,
3) crisp if it is $i-$crisp and $c-$crisp.

Definition 2.13 ([10])**.** Let $L(M)$ be a molecular lattice, D be a directed set and $S : D \to M$ be a mapping. Then S is called a molecular net in L and is denoted by $S = \{s(n), n \in D\}$. S is said to be in $a \in L$, if for any $n \in D$, $s(n) \leq a$.

Definition 2.14. Let $(L(M), \tau)$ be a **gtml**$-$space, $S = \{s(n), n \in D\}$ a molecular net and $a \in M$. Then a is said to be:

1) an $i-$limit point of S (or S $i-$converges to a), if for any $p \in \tau(a)$, $s(n) \leq p$ is eventually true,
2) a $c-$limit point of S (or S $c-$converges to a), if for any $p \in \tau^*(a)$, $s(n) \not\leq p$ is eventually true,
3) an $**-$limit point of S (or S $**-$converges to a), if for any $p \in \tau^{**}(a)$, $s(n) \leq p$ is eventually true.

The union of all $c-$limit points of S is denoted by $c-\lim S$. Similarly, we have $i - \lim S$ and $** - \lim S$.

The proof of the following theorem is a result of Definition 2.14.

Theorem 2.15. *If $(L(M), \tau)$ is an $i-$crisp* **gtml**$-$*space and S is a molecular net, then a is a $c-$limit point of S if and only if it is an $i-$limit point of S.*

Theorem 2.16. *Let $f : (L_1(M_1), \tau_1) \to (L_2(M_2), \tau_2)$ be a* **gml**$-$*map. Then f is $i-$continuous at $a \in M_1$ if and only if for each molecular net*

S in L_1 which $i-$converges to a we have $f(S)$ is $i-$converges to $f(a)$, where $f(S) = \{f(s(n)), n \in D\}$ is a molecular net in L_2.

Proof. Suppose that f is $i-$continuous at a and S $i-$converges to a. Let $b \in \tau_2(f(a))$. By hypothesis $(\hat{f}(b))^\circ \in \tau_1(a)$ and so $a \leq \hat{f}(b)$. Hence $s(n) \leq \hat{f}(b)$ and consequently $f(s(n)) \leq b$, which shows that $f(S)$ $i-$converges to $f(a)$. Conversely, suppose that f is not $i-$continuous at a. Then there exists $b \in \tau_2(f(a))$ such that $(\hat{f}(b))^\circ \notin \tau_1(a)$. Hence $a \nleq (\hat{f}(b))^\circ$ and so for any $p \in \tau(a)$, $p \nleq \hat{f}(b)$. Therefore, there exists $s(p) \in M_1$ such that $s(p) \nleq \hat{f}(b)$ and $s(p) \leq p$. Now the molecular net $\{s(p), p \in \tau(a)\}$ is $i-$converges at a but $f(S)$ does not $i-$converges to $f(a)$. □

Definition 2.17. Let $(L(M), \tau)$ be a **gtml**$-$space, $b \in L$ and $a \in M$. Then a is said to be:

1) an $i-$adherence point of b, if $p \in \tau(a)$, implies $b \wedge p \neq 0$,
2) a $c-$adherence point of b, if $p \in \tau^*(a)$, implies $b \nleq p$,
3) an $**$-adherence point of b, if $p \in \tau^{**}(a)$, implies $b \wedge p \neq 0$.

Clearly, the element 0 has no $c-$adherence points and $i-$adherence points.

Theorem 2.18. *Let $(L(M), \tau)$ be a* **gtml**$-$*space and $a \in M$. Then the following statements hold.*

1) *If L is $i-$crisp, then a is a $c-$adherent point of b if and only if a is an $i-$adherent point of b.*
2) *If L is $c-$crisp and a is $**-$adherent point of b, then a is a $c-$adherent point of b.*
3) *If L is a Stone algebra and a is $**-$adherent point of b, then a is a $c-$adherent point of b.*

Proof.
1) Clearly, $c-$adherent point implies $i-$adherent point. Conversely, let a be an $i-$adherent point of b and $p \in \tau^*(a)$. Then there exists $c \in \tau$ such that $p = c^*$. Since $a \nleq c^*$, it follows that $a \wedge c \neq 0$ and so, by hypothesis $a \leq c$. Hence $b \wedge c \neq 0$ and consequently $b \nleq c^* = p$, which shows that a is a $c-$adherent point of b.
2) Let $p \in \tau^*(a)$. Then $a \wedge p = 0$ and consequently $a \leq p^*$. Now by hypothesis, $b \wedge p^* \neq 0$. Hence $b \nleq p^{**} = p$, which shows that a is a $c-$adherent point of b.
3) Let $p \in \tau^*(a)$. Then we show that $a \leq p^*$. If $a \nleq p^*$, then $a \nleq (p \vee p^*) = 1$, which is a contradiction, and so by assumption we have $b \wedge p^* \neq 0$. Thus $b \nleq p^{**} = p$, which shows that a is a $c-$adherent point of b.

□

Theorem 2.19. *If $(L(M),\tau)$ is an $i-$crisp* **gtml**$-$*space, then a is an $i-$adherent point of b if and only if $a \leq \bar{b}$.*

Proof. By Definition 2.17, a is an $i-$adherent point of b if and only if $a \leq p$ implies $p \wedge b \neq 0$, or equivalently, $p \wedge b = 0$ implies $a \nleq p$ for every $p \in \tau$ and so, by hypothesis, $a \wedge p = 0$ for every $p \in \tau$. Hence $b \leq p^*$ implies $a \leq p^*$ for every $p^* \in \tau^*$. This means that $a \leq \bar{b}$. □

Definition 2.20. Let $(L(M),\tau)$ be a **gtml**$-$space and $\{b_i\}_{i\in I}$ be a subset of L. Then $\{b_i\}_{i\in I}$ is said to be:

1) $i-$locally finite if every point $a \in M$ has a neighborhood $p \in \tau(a)$ such that $b_i \leq p$ holds for at most a finite number of i,
2) $c-$locally finite if every point $a \in M$ has a remote-neighborhood $p \in \tau^*(a)$ such that $b_i \nleq p$ holds for at most a finite number of i,
3) $**-$locally finite if every point $a \in M$ has a neighborhood $p \in \tau^{**}(a)$ such that $b_i \leq p$ holds for at most a finite number of i.

Theorem 2.21. *Let $(L(M),\tau)$ be an $i-$crisp* **gtml**$-$*space. Then the following statements are equivalent.*

1) *$\{b_i\}_{i\in I}$ is a $c-$locally finite family,*
2) *$\{b_i\}_{i\in I}$ is an $i-$locally finite family,*
3) *$\{b_i\}_{i\in I}$ is $**-$locally finite family.*

Proof. $1 \Rightarrow 2)$ Let $a \in M$ and $p \in \tau^*(a)$. Then there exists $c \in \tau$ such that $p = c^*$. Since $a \nleq c^*$, it follows that $a \wedge c \neq 0$ and so, by hypothesis, $a \leq c$. On the other hand since $\{b_i\}_{i\in I}$ is a $c-$locally finite, it follows that $b_i \nleq p = c^*$ and so $b_i \wedge c \neq 0$ holds for at most a finite number of i. Hence $b_i \leq c$ holds for at most a finite number of i, which shows that $\{b_i\}_{i\in I}$ is $i-$locally finite.

$2 \Rightarrow 3)$ It is clear.

$3 \Rightarrow 1)$ Assume that $a \in M$, $p \in \tau^{**}(a)$. Then there exists $c \in \tau$ such that $p = c^{**}$. Hence $a \nleq c^*$. By hypothesis $b_i \leq p = c^{**}$ and consequently $b_i \nleq c^{***} = c^*$ holds for at most a finite number of i, which shows that $\{b_i\}_{i\in I}$ is $c-$locally finite.

□

Theorem 2.22. *If $\{b_i\}_{i\in I}$ is an $i-$locally finite family and $(L(M),\tau)$ an $i-$crisp* **gtml**$-$*space, then $\{\overline{b_i}\}_{i\in I}$ is an $i-$locally finite family.*

Proof. Let $a \in M$, $p \in \tau(a)$. By hypothesis $b_i \leq p$ holds for at most a finite number of i. Hence $b_i \nleq p^*$ and so $\overline{b_i} \nleq \overline{p^*} = p^*$ holds for at most a finite number of i. Consequently, $\overline{b_i} \wedge p \neq 0$ and so $\overline{b_i} \leq p$ holds for at most a finite number of i, which shows that $\{\overline{b_i}\}_{i\in I}$ is $i-$locally finite. □

Definition 2.23. Let (L, τ) be a **gtml**$-$space. Then a subset $\{b_i\}_{i\in I}$ of L is said to be:

1) an $i-$cover of an element $a \in L$ if $a \leq \vee_{i\in I} b_i$,
2) a $c-$cover of an element $a \in L$ if $(\wedge_{i\in I} b_i) \wedge a = 0$,
3) $**-$cover of an element $a \in L$ if $a \leq (\vee_{i\in I} b_i)^{**}$.

Notice that in the above definition if $a = 1$, $\{b_i\}_{i\in I}$ is called a $c-$cover of L. Similarly we have $i-$cover and $**-$cover of L.

Theorem 2.24. *Let (L, τ) be a* **gtml**$-$*space and $\{b_i\}_{i\in I}$ a subset of L.*

1) *If L is a Boolean algebra, then $\{b_i^*\}_{i\in I}$ is a $c-$cover of an element $a \in L$ if and only if $\{b_i\}_{i\in I}$ is an $i-$cover of an element $a \in L$.*
2) *Let L be a $c-$crisp. If $\{b_i\}_{i\in I}$ is an $i-$cover of an element $a \in L$, then $\{b_i^*\}_{i\in I}$ is a $c-$cover of an element $a \in L$.*
3) *$\{b_i^*\}_{i\in I}$ is a $c-$cover of an element $a \in L$ if and only if $\{b_i\}_{i\in I}$ is $**-$cover of an element $a \in L$.*
4) *If $\{b_i\}_{i\in I}$ is an $i-$cover of an element $a \in L$, then $\{b_i\}_{i\in I}$ is $**-$cover of an element $a \in L$.*

Proof. 1) Since we have $a \wedge (\wedge b_i^*) = 0 \quad \Leftrightarrow \quad a \leq (\wedge b_i^*)^* = (\vee b_i)$, the result follows.

2) By hypothesis, we have $a \leq \vee_{i\in I} b_i$, thus $a \wedge (\vee_{i\in I} b_i) \neq 0$ and so $a \nleq (\vee_{i\in I} b_i)^* = \wedge b_i^*$. Consequently, $a \wedge (\wedge b_i^*) = 0$, which shows that $\{b_i^*\}_{i\in I}$ is a $c-$cover of an element $a \in L$.

The parts 3 and 4 are evident.

□

Recall that a **gtml**$-$space (L, τ) has the finite c-cover property if every $c-$cover of L consisting of the closed elements has a finite $c-$subcover. Also, we say that (L, τ) has the finite $i-$cover property if every $i-$cover of L consisting of the open elements has a finite $i-$subcover. Similarly, we define the finite $**-$cover property. We say $a \in L$ is compact if $S \subseteq \tau$ and $a \leq \vee S$, imply that there exists a finite subset D of S such that $a \leq \vee D$. If 1 is a compact element in (L, τ), then we say (L, τ) is compact. Similarly, we define $**-$compact space.

Theorem 2.25. *Let (L, τ) be a* **gtml**$-$*space. Then the following statements hold.*

1) *If L has the finite c-cover property, then it is $**-$compact space.*
2) *If the map $*$ is topologically injective and L has the finite $c-$cover property, then it is compact.*

Proof.

1) The proof is clear.
2) Suppose that $\vee_{i\in I} b_i = 1$. Then $\wedge_{i\in I} b_i^* = (\vee_{i\in I} b_i)^* = 1^* = 0$ and so, by hypothesis, $\wedge_{i=1}^{n} b_i^* = 0$, thus $(\vee_{i=1}^{n} b_i)^* = 0$ and consequently $\vee_{i=1}^{n} b_i = 1$.

□

3. SEPARATION AXIOMS

In this section, we introduce some kinds of separation axioms in a gtml-space and investigate their properties. Moreover, we discuss the relations among them.

Definition 3.1. Let $(L(M), \tau)$ be a **gtml**−space. Then L is said to be:

1) $c - T_{-1}$, if $a, b \in M$ and $a < b$, imply that there exists $t \in \tau$ such that $b \nleq t^*$ and $a \leq t^*$,
2) $i - T_{-1}$, if $a, b \in M$ and $a < b$, imply that there exists $t \in \tau$ such that $b \leq t$ and $a \wedge t = 0$,
3) $**-T_{-1}$, if $a, b \in M$ and $a < b$, imply that there exists $t \in \tau$ such that $b \leq t^{**}$ and $a \wedge t^{**} = 0$.

Clearly, we have the implications: $i - T_{-1} \Rightarrow ** - T_{-1} \Rightarrow c - T_{-1}$.

Definition 3.2 ([10])**.** Let $L(M)$ be a molecular lattice, $0 \neq a \in L$ and $m \in M$. Then m is called a component of a if (i) $m \leq a$, and (ii) $m' \in M$, $m' \geq m$ and $m' \leq a$ imply that $m' = m$.

Theorem 3.3. *Let $(L(M), \tau)$ be a* **gtml**−*space.*

1) *If L is $c - T_{-1}$ and $i-$crisp, then it is $i - T_{-1}$.*
2) *If L is $i-$crisp, then it is $i - T_{-1}$ if and only if for all $x \in M$, x is a component of $\bar{x}$.*

Proof.

1) Let $a, b \in M$ and $a < b$. Then there exists $t \in \tau$ such that $b \nleq t^*$ and $a \leq t^*$. Therefore $b \wedge t \neq 0$ and $a \wedge t = 0$. Now by hypothesis, we have that $b \leq t$ and $a \wedge t = 0$, which shows that L is $i - T_{-1}$.
2) By the part 1 and Theorem 6.16 [10], the result follows.

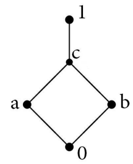
□

Definition 3.4. Let $(L(M), \tau)$ be a **gtml**−space. Then L is said to be:

1) $c - T_0$, if $a, b \in M$ and $a \neq b$, imply that there exists $p \in \tau$ such that $a \nleq p^*$ and $b \leq p^*$ or there exists $q \in \tau$ such that $b \nleq q^*$ and $a \leq q^*$,
2) $i - T_0$, if $a, b \in M$ and $a \neq b$, imply that there exists $t \in \tau$ such that $a \leq t$ and $b \wedge t = 0$ or there exists $t' \in \tau$ such that $b \leq t'$ and $a \wedge t' = 0$,

3) $**-T_0$, if $a, b \in M$ and $a \neq b$, imply that there exists $t \in \tau$ such that $a \leq t^{**}$ and $b \wedge t^{**} = 0$ or there exists $q \in \tau$ such that $b \leq q^{**}$ and $a \wedge q^{**} = 0$.

Clearly, we have the implications: $i - T_0 \Rightarrow **-T_0 \Rightarrow c - T_0$. In general, we have that $i-T_{-1}$ does not imply $i-T_0$. For example, consider the lattice L as follows:



Let $\tau = \{0, 1\}$ and so $\tau^* = \{0, 1\}$. Clearly, (L, τ) is $i - T_{-1}$ but it is not $i - T_0$, because $x \leq 1$, $y \leq 1$ and $y \wedge 1 = y$, $x \wedge 1 = x$.

Theorem 3.5. *Let $(L(M), \tau)$ be a* **gtml**$-$*space.*

1) *If L is $c - T_0$ and $i-$crisp, then it is $i - T_0$.*
2) *If L is $i-$crisp, then it is $i - T_0$ if and only if for all $a, b \in M$, $a \neq b$, we have $a \nleq \bar{b}$ or $b \nleq \bar{a}$.*

Proof. 1) Let $a, b \in M$ and $a \neq b$. Then there exists $t \in \tau$ such that $a \nleq t^*$ and $b \leq t^*$. Therefore $a \wedge t \neq 0$ and $b \wedge t = 0$. Now by hypothesis, we have that $a \leq t$ and $b \wedge t = 0$. On the other hand, we suppose that there exists $q \in \tau$ such that $a \leq q^*$ and $b \nleq q^*$. Similarly, we have that $a \wedge q = 0$ and $b \leq q$, which shows that L is $**-T_0$.

2) The proof is easy and hence is omitted.

□

Definition 3.6. Let $(L(M), \tau)$ be a **gtml**$-$space. Then L is said to be:

1) $i - T_1$, if $a, b \in M$ and $a \nleq b$, imply that there exists $t \in \tau$ such that $a \leq t$ and $b \wedge t = 0$,
2) $c - T_1$, if $a, b \in M$ and $a \nleq b$, imply that there exists $t \in \tau$ such that $a \nleq t^*$ and $b \leq t^*$,
3) $**-T_1$, if $a, b \in M$ and $a \nleq b$, imply that there exists $t \in \tau$ such that $a \leq t^{**}$ and $b \wedge t^{**} = 0$.

Clearly, we have the implications: $i - T_1 \Rightarrow **-T_1 \Rightarrow c - T_1$.

Theorem 3.7. *Let $(L(M), \tau)$ be a* **gtml**$-$*space. Then the following statements hold.*

1) *If L is $c - T_1$ and $i-$crisp, then it is $i - T_1$.*
2) *L is $c - T_1$ if and only if for every $x \in M$, x is closed.*

3) *If L is $i-$crisp, then L is $i - T_1$ if and only if for every $x \in M$, x is closed.*

Proof. 1) The proof is similar to Theorem 3.5.

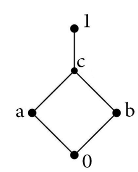
2) Suppose that L is $c-T_1$ and $b \in M$. for any $a \in M$, if $a \nleq b$, then there exists $t \in \tau$ such that $a \nleq t^*$ and $b \leq t^*$. By Definition 2.17, this shows that a is not a $c-$adherent point of b. In other words, b contains all its $c-$adherent points and hence is closed. Conversely, let $a, b \in M$ and $a \nleq b$. Then $b \in \tau^*(a)$ and $b \leq b$, hence L is $c - T_1$.

3) It follows easily from 1 and 2.

□

In general, we have that $c - T_0$ does not imply $c - T_1$. For example, consider the lattice $L = L_2$ given in Example 2.6. Let $\tau = \{0, a, c, 1\}$ and so $\tau^* = \{0, b, 1\}$. Clearly, (L, τ) is $c-T_0$ but it is not $c-T_1$, because by Theorem 3.7, $a \in M$ but a is not closed.

Also, $i - T_0$ does not imply $i - T_1$. For example, consider the lattice L as follows:



Let $\tau = \{0, x, 1\}$ and so $\tau^* = \{0, y, 1\}$. Clearly, (L, τ) is $i - T_0$ but it is not $i - T_1$ because by Theorem 3.7, $x \in M$ but x is not closed.

Definition 3.8. Let $(L(M), \tau)$ be a **gtml**$-$space. Then L is said to be:

1) $i-T_2$, if $a, b \in M$ and $a \wedge b = 0$, imply that there exist $t_1, t_2 \in \tau$ such that $a \leq t_1$, $b \leq t_2$ and $t_1 \wedge t_2 = 0$,
2) $c-T_2$, if $a, b \in M$ and $a \wedge b = 0$, imply that there exist $t_1, t_2 \in \tau$ such that $a \nleq t_1^*$, $b \nleq t_2^*$ and $t_1^* \vee t_2^* = 1$,
3) $**-T_2$, if $a, b \in M$ and $a \wedge b = 0$, imply that there exist $t_1, t_2 \in \tau$ such that $a \leq t_1^{**}$, $b \leq t_2^{**}$ and $t_1^{**} \wedge t_2^{**} = 0$.

Theorem 3.9. *Let $(L(M), \tau)$ be a* **gtml**$-$*space. Then the following statements hold.*

1) *If L is a tml-space and $i-$crisp, then L is $c - T_2$ if and only if it is $i - T_2$.*
2) *If L is $i - T_2$, then it is $** - T_2$.*

Proof. 1) The proof follows from this fact that $(t_1 \wedge t_2)^* = t_1^* \vee t_2^*$.

2) Since for every $a, b \in L$, we have $a \wedge b = 0$ if and only if $a^{**} \wedge b^{**} = 0$, the result holds.

$\square$

Definition 3.10. Let $(L(M), \tau)$ be a **gtml**$-$space. Then L is said to be:

1) $i-$regular, if $a, b \in M$ and $b \in \tau^*(a)$, imply that there exist $t_1, t_2 \in \tau$ such that $a \leq t_1$, $b \leq t_2$ and $t_1 \wedge t_2 = 0$,
2) $c-$regular, if $a, b \in M$ and $b \in \tau^*(a)$, imply that there exist $t_1, t_2 \in \tau$ such that $a \not\leq t_1^*$, $b \not\leq t_2^*$ and $t_1^* \vee t_2^* = 1$,
3) $**-$regular, if $a, b \in M$ and $b \in \tau^*(a)$, imply that there exist $t_1, t_2 \in \tau$ such that $a \leq t_1^{**}$, $b \leq t_2^{**}$ and $t_1^{**} \wedge t_2^{**} = 0$.

A $c - T_0$ regular gtml-space is said to be $c - T_3$. Similarly, we define $i - T_3$ and $** - T_3$.

Theorem 3.11. *Let $(L(M), \tau)$ be a* **gtml**$-$*space. Then the following statements hold.*

1) *If L is a tml$-$space and $i-$crisp, then L is $c - T_3$ if and only if it is $i - T_3$.*
2) *If L is $i - T_3$, then it is $** - T_3$.*

Proof. The proof is similar to Theorem 3.9 $\square$

In general, $i - T_2$ does not imply $i - T_3$. For example, consider the lattice $L = L_1$ given in Example 2.6. Let $\tau = \{0, b, d, 1\}$ and so $\tau^* = \{0, b, d, 1\}$. Clearly, (L, τ) is $i - T_2$ but it is not $i - T_3$, because for $a \neq d$, we have $a \leq d$, $d \leq 1$ but $a \wedge d = a \neq 0$ and $a \wedge 1 = a \neq 0$. Similarly, we can show that (L, τ) is $c - T_2$ but it is not $c - T_1$.

By the above definitions, we can directly obtain the following results.

Corollary 3.12. *For a* **gtml**$-$*space L, the following implications hold.*

1) $c - T_1 \Rightarrow c - T_0 \Rightarrow c - T_{-1}$.
2) $i - T_1 \Rightarrow i - T_0 \Rightarrow i - T_{-1}$.
3) $i - T_2 + i - T_{-1} \Rightarrow i - T_1$.
4) *If L is also $i-$crisp, then $c - T_2 + c - T_{-1} \Rightarrow c - T_1$.*
5) $c - T_3 \Rightarrow c - T_2$.
6) $i - T_3 \Rightarrow i - T_2$.

In general, $i - T_2$ dose not imply $i - T_1$. For instance, let $L = [0, 1]$ and $\tau = \{0, 1\}$. Clearly, $(L(M), \tau)$ is not $i - T_{-1}$ and hence it is not $i - T_1$. But there are no disjoint points, so L is $i - T_2$, where $a, b \in M$ are called disjoint, if $a \wedge b = 0$.

Theorem 3.13 ([4]). *Let $L(M)$ be a* **gtml**$-$*space. Then it is $c - T_2$ if and only if for each molecular net S, $c - \lim S$ contains no disjoint points.*

Theorem 3.14. *Let $L(M)$ be a* **gtml***$-$space. Then it is $i-T_2$ if and only if for each molecular net S, $i-\lim S$ contains no disjoint points.*

Proof. Let $S=\{s(n),n\in D\}$ be a molecular net such that $i-\lim S$ contains two disjoint points a and b. Suppose that $p\in\tau(a)$ and $q\in\tau(b)$. Since a is an $i-$limit point of S, there exists $n_1\in D$ such that $s(n)\leq p$ whenever $n\geq n_1$; similarly, there exists $n_2\in D$ such that $s(n)\leq q$ whenever $n\geq n_2$. Since D is directed, there exists $n_3\in D$ such that $s(n_3)\leq p$ and $s(n_3)\leq q$. Hence $s(n_3)\leq p\wedge q$. Therefore, $p\wedge q\neq 0$ and consequently L is not $i-T_2$. Conversely, if L is not $i-T_2$, then there exist $a,b\in M$ with $a\wedge b=0$ and for any $p,q\in\tau$ such that $a\leq p$ and $b\leq q$, we have $p\wedge q\neq 0$. Hence we can choose a molecular $s((p,q))$ such that $s((p,q))\leq p\wedge q$. Define $S=\{s((p,q)),(p,q)\in\tau(a)\times\tau(b)\}$. Then S is a molecular net which converges to both a and b, and hence $i-\lim S$ contains at least two disjoint points. □

By the previous statements, we have the following result.

Corollary 3.15. *Let L be an $i-$crisp* **tml***$-$space and $j\in\{-1,0,1,2,3\}$. Then L is $i-T_j$ if and only if it is $c-T_j$.*

Definition 3.16. Let $f:(L_1(M),\tau_1)\to(L_2(M),\tau_2)$ be a bijective gml-map. Then f is said to be a homeomorphism if f and $\hat{f}$ are continuous.

Notice that if f is a homeomorphism, then $\hat{f}=f^{-1}$ is also a homeomorphism, where f^{-1} is the inverse of f.

Theorem 3.17. *Let $f:(L_1(M),\tau_1)\to(L_2(M),\tau_2)$ be a homeomorphism gml-map and $j\in\{-1,0,1,2,3\}$.*

1) *If L_1 is $i-T_j$, then so is L_2.*
2) *If L_1 is $c-T_j$, then so is L_2.*

Proof. We only show the case of $i-T_2$ and the others are similar. Let L_1 be an $i-T_2$ and $x,y\in M_2$ with $x\wedge y=0$. Since f is bijective, then there exist $a,b\in M_1$ such that $f(a)=x$ and $f(b)=y$, and so, $a\wedge b=0$. Then there exist $p,q\in\tau_1$ such that $a\leq p$, $b\leq q$ and $p\wedge q=0$. Hence $x\leq f(p)$, $y\leq f(q)$ and $f(p\wedge q)=f(p)\wedge f(q)=0$, which shows that L_2 is $i-T_2$. □

4. Conclusion

In this paper, we have introduced the concept of the generalized topological molecular lattices and investigated some basic properties of them. We have presented some kinds of separation axioms and investigated some relations among them. In particular, we have showed that a **gtml**$-$space is $i-T_2$ and $c-T_2$ if and only if for each molecular net S, $i-\lim S$ and $c-\lim S$ have no disjoint points.

Acknowledgment. We are very grateful to the referee(s) for the careful reading of the paper and for the useful comments.

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