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The spectral properties of differential operators with matrix coefficients on elliptic systems with boundary conditions

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Abstract. Let

$$(Lv)(t)=\sum_{i,j=1}^{n}(-1)^{j}d_{j}\left(s^{2\alpha}(t)b_{ij}(t)\mu(t)d_{i}v(t)\right),$$

be a non-selfadjoint differential operator on the Hilbert space $L_2(\Omega)$ with Dirichlet-type boundary conditions. In continuing of papers [10–12], let the conditions made on the operator L be sufficiently more general than [11] and [12] as defined in Section 1. In this paper, we estimate the resolvent of the operator L on the one-dimensional space $L_2(\Omega)$ using some analytic methods.

1. Introduction

The role and application of differential operators in mathematics, physics and engineering have caused that many researchers study the spectral featurs of non-selfadjoint differential operators (see [1–8] and [13]). The spectral asymptotics of non-selfadjoint elliptic differential operators have been investigated in [11, 12] in the case when, the eigenvalues of the operator are devided into two series, one of which lies outside the angle $\Phi=\{z\in\mathbb{C}:\ |\arg\ z|<\varphi,\ \ \varphi\in(0,\pi)\}$ and the other one lies on the positive semiaxis $\mathbb{R}_+=(0,\infty)$.

Let $\Omega\subset\mathbb{R}^n$ be a bounded domain with a boundary of class C^∞, let $s(t)\in C^1(\Omega)$ be a non-negative function, let $\mu(t)$ be a complex-valued function such that $\mu(t)\in C^2(\overline{\Omega})$ for all $t\in\overline{\Omega}$, and let $\leq\alpha<1$. Let us

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consider the differential operator

$$(Lv)(t) = \sum_{i,j=1}^{n} (-1)^j d_j \left(s^{2\alpha}(t) b_{ij}(t)\mu(t) d_i v(t)\right), \quad \left(v \in L_2(\Omega)\right),$$

where

$$d_j = \frac{\partial^j}{\partial t_j \partial t_{j-1} \cdots \partial t_1} = \frac{\partial}{\partial t_j}\left(\frac{\partial^{j-1}}{\partial t_{j-1} \cdots \partial t_1}\right).$$

We assume that the following conditions hold:

(i) $b_{ij}(t) \in C^n(\overline{\Omega})$, $b_{ij}(t) = \overline{b_{ji}(t)}$ (for $t \in \overline{\Omega}$ and $1 \le i, j \le n$);
(ii) $|s|^2 \le M \sum_{i,j=1}^{n} s_i \overline{s_j} b_{ij}(t)$;
(iii) $\mu(t)$ is a complex-valued function such that $\mu(t) \in C^n(\overline{\Omega})$ and lies inside the angle $\Phi_{\varphi,\psi}$, where $\Phi_{\varphi,\psi} = \{z \in \mathbb{C} : \psi < |\arg z| < \varphi, \ \varphi \in [\frac{\pi}{2} - \varepsilon, \pi) \text{ and } \psi \in (\frac{\pi}{2}, \pi) \text{ for } \varepsilon > 0\}$.

Moreover, let $\Phi_\theta = \{z \in \mathbb{C} : |\arg z| > \theta; \ 0 < \theta < \frac{\pi}{2}\}$ be a closed angle with the vertex at zero.

The weighted Sobolev space $H_s = W^{2n}_{2,\alpha}(\Omega)$ denotes the class of all functions $v(t)$ defined on Ω with Sobolev norm:

$$|v|_s = \left(\sum_{i=1}^{n} \int_\Omega s^{2\alpha}(t) |d_i v(t)|^2_{\mathbb{C}} dt + \int_\Omega |v(t)|^2_{\mathbb{C}} dt\right)^{1/2}, \quad 0 \le \alpha < 1.$$

By $\overset{\circ}{H}_s$ we denote the closure of the linear manifold $C_0^\infty(\Omega)$ in H_s with respect to the Sobolev norm. We denote by $C_0^\infty(\Omega)$ the class of all infinitely differentiable functions with compact support in Ω.

In this paper, we study some spectral properties of the differential operator L. To do this, by [9] we need to extend its domain to the closed domain:

$$D(L) = \left\{ v \in \overset{\circ}{H}_s \cap W^{2n}_{2,loc}(\Omega) : \sum_{i,j=1}^{n} (-1)^j d_j \left(s^{2\alpha}(t) b_{ij}(t)\mu(t) d_i v(t)\right) \in H \right\}.$$

The symbol $W^{2n}_{2,loc}(\Omega)$ denotes the space of functions $v(t) (t \in \Omega)$ satisfying the condition

$$\sum_{i=0}^{2n} \int_J |d_i v(t)|^2 dt < \infty,$$

such that J is an open subset of Ω. In other words,

$$W^{2n}_{2,loc}(\Omega) = \left\{ v : \sum_{i=0}^{2n} \int_J |d_i v(t)|^2 dx < \infty \right\}.$$

Hereinafter, the symbols $\langle ., . \rangle_{\mathbb{C}}$, $(., .)_H$, $|.|_{\mathbb{C}}$, $\|.\|_H$ and $\|L\|$ denote the inner product on the one$-$dimensional space $\mathbb{C}$, the inner product on the Hilbert space H, the norm on the space $\mathbb{C}$, the norm on the space H and the operator norm of the bounded operator L defined on the space H, respectively. The results of this paper generalize the derived results by the authors in [10–12].

2. Estimate of the resolvent of the operator L on the one-dimensional space $H = L_2(\Omega)$

Let us consider the operator

$$(Lv)(t) = \sum_{i,j=1}^{n} (-1)^j d_j \left(s^{2\alpha}(t) b_{ij}(t)\mu(t) d_i v(t)\right),$$

on the one-dimensional space $H = L_2(\Omega)$. Moreover, let us assume that the following condition on $\mu(t)$ is satisfied:

$$\mu(t) \in C^n(\overline{\Omega}), \qquad \mu(t) \in \Phi_{\varphi,\psi}, \quad \forall\, t \in \overline{\Omega}. \tag{2.1}$$

Our intention here is to estimate the resolvent of the operator L on the space $H = L_2(\Omega)$ by utilizing some analytic methods and by applying the uniformly elliptic condition.
In order to obtain the main result, we need to prove the following lemma:

Lemma 2.1. *Assume that the operator L on the space $H = L_2(\Omega)$ is as above and let the complex-valued function $\mu(t)$ satisfies the condition (2.1). In addition, assume that the function $\mu(t)(t \in \overline{\Omega})$ does not exceed $\frac{\pi}{8}$, i.e.,*

$$\left|\arg\{\mu(t_1)\mu^{-1}(t_2)\}\right| \le \frac{\pi}{8}, \quad (\forall\, t_1, t_2 \in \overline{\Omega}). \tag{2.2}$$

Then, for sufficiently large numbers in modulus $z \in \Phi_\theta$, the inverse operator $(L - zI)^{-1}$ exists and is continuous on the space $H = L_2(\Omega)$, and the following inequalities hold:

$$\left\|(L - zI)^{-1}\right\| \le N_{2,\Phi_\theta}|z|^{-1} \quad (z \in \Phi_\theta, \quad |z| > C_{\Phi_\theta}), \tag{2.3}$$

$$\left\|s^\alpha(t)\, d_i (L - zI)^{-1}\right\| \le N_{i,\Phi_\theta}|z|^{-\frac{1}{2}} \qquad (z \in \Phi_\theta, \quad |z| > C_{\Phi_\theta}). \tag{2.4}$$

Here, N_{2,Φ_θ}, N_{i,Φ_θ} and C_{Φ_θ} are sufficiently large and positive numbers depending on Φ_θ.

Proof. As in Section1, we have

$$D(L) = \left\{ v \in \overset{\circ}{H}_s \cap W^{2n}_{2,loc}(\Omega) : \sum_{i,j=1}^{n} (-1)^j d_j \left(s^{2\alpha}(t) b_{ij}(t)\mu(t) d_i v(t)\right) \in H \right\}.$$

By applying (2.2), we can find a real number $\xi \in (-\pi, \pi]$ with $\left|e^{i\xi}\right| = 1$ and a positive number c such that, for $t \in \Omega$ and $z \in \Phi_\theta$

$$c \le \mathrm{Re}\left\{e^{i\xi}\mu(t)\right\} \quad \text{and} \quad c|z| \le -\mathrm{Re}\left\{e^{i\xi}z\right\}. \tag{2.5}$$

Let $v \in D(L)$. Observe that

$$\sum_{i=1}^{n}\int_{\Omega} s^{2\alpha}(t)|d_i v(t)|_C^2 dt < \infty. \tag{2.6}$$

Making use of the uniformly elliptic condition, it follows that

$$c' \sum_{i=1}^{n} |d_i v(t)|_{\mathbb{C}^\ell}^2 \le \sum_{i,j=1}^{n} b_{ij}(t) d_i v(t)\overline{d_j v(t)}, \tag{2.7}$$

where c' is a positive constant. By multiplying both sides of the first inequality in (2.5) by (2.6), by utilizing (2.7) and by applying the famous theorem of the m-sectorial operators [9], one arrives to the inequality

$$\begin{aligned}\sum_{i=1}^{n}\int_{\Omega} s^{2\alpha}(t)|d_i v(t)|_{\mathbb{C}^\ell}^2 dt &\le N_1 \mathrm{Re}\left\{e^{i\xi}\sum_{i,j=1}^{n} < s^\alpha b_{ij}\mu d_i v, s^\alpha d_j v >_C\right\} \\ &= N_1 \mathrm{Re}\left\{e^{i\xi}(Lv, v)_H\right\}.\end{aligned}$$

Letting

$$J(v) = \sum_{i=1}^{n}\int_{\Omega} s^{2\alpha}(t)|d_i v(t)|_C^2 dt, \tag{2.8}$$

we get

$$\begin{aligned} J(v) &\le N_1 \mathrm{Re}\left\{e^{i\xi}(Lv, v)_H\right\} \\ &\le N_1\|Lv\|_H\|v\|_H. \end{aligned} \tag{2.9}$$

By applying (2.5), one obtains

$$\begin{aligned} J(v) + |z|\|v\|_H^2 &\le N_2 \mathrm{Re}\left\{e^{i\xi}((L - zI)v, v)_H\right\} \\ &\le N_2|e^{i\xi}|\|v\|_H\|(L - zI)v\|_H \\ &= N_2\|v\|_H\|(L - zI)v\|_H. \end{aligned} \tag{2.10}$$

Thus,

$$J(v) \le N_2\|v\|_H\|Lv\|_H. \tag{2.11}$$

By omitting the positive term $J(v)$ from the inequality (2.10), we derive

$$|z|\|v\|_H \le N_2\|(L \quad - z\ I)v\|_H. \tag{2.12}$$

Setting $v = (L - zI)^{-1}f,\ f \in H$ in (2.12), then the inequality (2.12) leads to

$$|z|\left\|(L - zI)^{-1}f\right\|_H \leq N_2\left\|(L \quad - zI)(L - zI)^{-1}f\right\|_H.$$

From $(L - zI)(L - zI)^{-1}f = I(f) = f$, it follows that

$$|z|\left\|(L \quad - zI)^{-1}f\right\|_H \leq N_2\|f\|_H,$$

i.e.,

$$\left\|(L \quad - zI)^{-1}\right\| \leq N_{2,\Phi_\theta}|z|^{-1},$$

for every $z \in \Phi_\theta$. This proves (2.3).

In the next step, we prove (2.4). Insertting $v = (L - zI)^{-1}f, f \in H$ in (2.8), it follows that

$$J\left((L - zI)^{-1}f\right) = \sum_{i=1}^{n}\int_\Omega s^{2\alpha}(t)\left|\frac{\partial}{\partial t_i}(L - zI)^{-1}f(t)\right|_C^2 dt.$$

Since

$$\left\|s^\alpha(t)\frac{\partial}{\partial t_i}(L - zI)^{-1}f\right\|_H^2 = \int_\Omega s^{2\alpha}(t)\left|\frac{\partial}{\partial t_i}(L - zI)^{-1}f(t)\right|_C^2 dt,$$

we get

$$\left\|s^\alpha(t)\frac{\partial}{\partial t_i}(L - zI)^{-1}f\right\|_H^2 \leq J\left((L - zI)^{-1}f\right). \tag{2.13}$$

By (2.9), we have

$$J(v) \leq N_1\|v\|_H\|Lv\|_H. \tag{2.14}$$

On the other hand, by (2.3), we have

$$\left\|L(L - zI)^{-1}\right\| \leq N. \tag{2.15}$$

From (2.14)-(2.15) and $\left\|s^\alpha(t)\frac{\partial}{\partial t_i}(L - zI)^{-1}f\right\|_H^2 \leq J\left((L - zI)^{-1}f\right)$, one has

$$\begin{aligned}\left\|s^\alpha(t)\frac{\partial}{\partial t_i}(L - zI)^{-1}f\right\|_H^2 &\leq J\left((L - zI)^{-1}f\right)\\ &\leq N_1\left\|(L - zI)^{-1}f\right\|_H\left\|L(L - zI)^{-1}f\right\|_H\\ &\leq N_1 N\left\|(L - zI)^{-1}f\right\|_H\|f\|_H\\ &\leq N_3\left\|(L - zI)^{-1}\right\|\|f\|_H^2,\end{aligned}$$

i.e.,

$$\left\|s^\alpha(t)\frac{\partial}{\partial t_i}(L - zI)^{-1}f\right\|_H^2 \leq N_3\left\|(L - zI)^{-1}\right\|\|f\|_H^2. \tag{2.16}$$

From (2.3) and (2.16), we get

$$\left\| s^{\alpha}(t)\frac{\partial}{\partial t_i}(L-zI)^{-1}f\right\|_H^2 \leq N_3 N_{2,\Phi_\theta}|z|^{-1}\|f\|_H^2. \tag{2.17}$$

It can be written in the form

$$\left\| s^{\alpha}(t)\frac{\partial}{\partial t_i}(L-zI)^{-1}f\right\|_H \leq N''_{\Phi_\theta}|z|^{-\frac{1}{2}}\|f\|_H,$$

which is equivalent to

$$\left\| s^{\alpha}(t)\frac{\partial}{\partial t_i}(L-zI)^{-1}\right\| \leq N''_{\Phi_\theta}|z|^{-\frac{1}{2}}.$$

Proceeding this procces, we get

$$\begin{aligned}
\left\| s^{\alpha}(t)d_i(L-zI)^{-1}f\right\|_H^2 &= \left\| s^{\alpha}(t)\frac{\partial^i}{\partial t_i \partial t_{i-1}\dots\partial t_1}(L-zI)^{-1}f\right\|_H^2 \\
&= \left\| s^{\alpha}(t)\frac{\partial}{\partial t_i}\left(\frac{\partial^{i-1}}{\partial t_{i-1}\partial t_{i-2}\dots\partial t_1}\right)(L-zI)^{-1}f\right\|_H^2 \\
&\leq N_1|z|^{-1}\left\| s^{\alpha}(t)\left(\frac{\partial^{i-1}}{\partial t_{i-1}\partial t_{i-2}\dots\partial t_1}\right)(L-zI)^{-1}f\right\|_H^2 \\
&\leq N_2|z|^{-2}\left\| s^{\alpha}(t)\left(\frac{\partial^{i-2}}{\partial t_{i-1}\partial t_{i-3}\dots\partial t_1}\right)(L-zI)^{-1}f\right\|_H^2 \\
&\ \ \vdots \\
&\leq N_i|z|^{-2^i}\|f\|_H^2 \\
&\leq N_i|z|^{-1}\|f\|_H^2.
\end{aligned}$$

The last inequality follows using the fact that $|z|^{-2^i} \leq |z|^{-1}$. Therefore,

$$\left\| s^{\alpha}(t)d_i(L-zI)^{-1}\right\| \leq N_{i,\Phi_\theta}|z|^{-\frac{1}{2}}.$$

So, the inequality (2.4) is proved. □

Theorem 2.2. *Assume that the operator L on the space $H = L_2(\Omega)$ is as above and let the complex-valued function $\mu(t)$ satisfying the condition* (2.1). *Then, for sufficiently large numbers in modulus $z \in \Phi_\theta$, the inverse operator $(L - zI)^{-1}$ exists and is continuous on the space $H = L_2(\Omega)$, and the following inequality holds:*

$$\left\|(L-zI)^{-1}\right\| \leq N_{2,\Phi_\theta}|z|^{-1}, \qquad (z \in \Phi_\theta, \quad |z| > C_{\Phi_\theta}), \tag{2.18}$$

where N_{2,Φ_θ} and C_{Φ_θ} are sufficiently large and positive numbers depending on Φ_θ.

Proof. In order to prove (2.18), let us construct the non-negative functions $\varphi_{(1)}(t), \ldots, \varphi_{(m)}(t) \in C^{\infty}(\overline{\Omega})$ and the functions $\mu_{(1)}(t), \ldots, \mu_{(m)}(t) \in C^{\infty}(\overline{\Omega})$, satisfying the following properties:

$$\sum_{k=1}^{m} \varphi_{(k)}^{2}(t) \equiv 1, \left(\varphi_{(k)}(t)\right)' \in C_0^{\infty}(\Omega) \quad (\text{for } t \in \overline{\Omega} \text{ and } 1 \leq k \leq m);$$

$$\mu_{(k)}(t) \in \Phi_{\varphi,\psi}, \qquad (\text{for } t \in \overline{\Omega} \text{ and } 1 \leq k \leq m);$$

$$\left|\arg\left\{\mu_{(k)}(t_1)\mu_{(k)}^{-1}(t_2)\right\}\right| \leq \tfrac{\pi}{8}, \qquad (\text{for } t_1, t_2 \in \operatorname{supp}\varphi_{(k)}).$$

Taking $\left(L_{(k)}v\right)(t) = L(t)$, then we obtain the non-selfadjoint differential operator

$$\left(L_{(k)}v\right)(t) = \sum_{i,j=1}^{n} (-1)^j d_j \left(s^{2\alpha}(t) b_{ij}(t) \mu_{(k)}(t) d_i v(t)\right),$$

acting on the one-dimensional space $H = L_2(\Omega)$ with domain

$$D(L_{(k)}) = \left\{ v \in \overset{\circ}{H}_s \cap W_{2,loc}^{2n}(\Omega) : \sum_{i,j=1}^{n} (-1)^j d_j \left(s^{2\alpha}(t)\mu_{(k)}(t) b_{ij}(t) d_i v(t)\right) \in H \right\}.$$

According to Lemma 2.1, for $z \in \Phi_\theta$, the inverse operator $\left(L_{(k)} - zI\right)^{-1}$ exists and is continuous on the space $H = L_2(\Omega)$, satisfying the following properties:

$$\begin{aligned} &\left\|(L_{(k)} - zI)^{-1}\right\| \leq N_{2,\Phi_\theta} |z|^{-1}, \\ &\left\|s^{\alpha}(t) d_i (L_{(k)} - zI)^{-1}\right\| \leq N_{i,\Phi_\theta} |z|^{-\frac{1}{2}}. \end{aligned} \tag{2.19}$$

Let us introduce the operator $G(z)$ on the space $H = L_2(\Omega)$ as follows:

$$G(z) = \sum_{k=1}^{m} \varphi_{(k)} \left(L_{(k)} - zI\right)^{-1} \varphi_{(k)}. \tag{2.20}$$

Here, $\varphi_{(k)}$ is the multiplication operator by the function $\varphi_{(k)}(t)$. One can readily verify that

$$(L - zI)G(z) = I + F(z),$$

where

$$\begin{aligned} F(z) = {} & s^{2\alpha-1}(t)\sum_{k=1}^{m}\beta_k(t)\left(L_{(k)}-zI\right)^{-1}\varphi_{(k)} \\ & + s^{2\alpha}(t)\sum_{r=1}^{n}\sum_{k=1}^{m}\eta_{rk}^{1}(t)d_1\left(L_{(k)}-zI\right)^{-1}\varphi_{(k)} \\ & + s^{2\alpha}(t)\sum_{r=1}^{n}\sum_{k=1}^{m}\eta_{rk}^{2}(t)d_2\left(L_{(k)}-zI\right)^{-1}\varphi_{(k)} \\ & \vdots \\ & + s^{2\alpha}(t)\sum_{r=1}^{n}\sum_{k=1}^{m}\eta_{rk}^{r-1}(t)d_{r-1}\left(L_{(k)}-zI\right)^{-1}\varphi_{(k)} \\ & + s^{2\alpha}(t)\sum_{r=1}^{n}\sum_{k=1}^{m}\eta_{rk}^{r}(t)d_r\left(L_{(k)}-zI\right)^{-1}\varphi_{(k)}. \end{aligned}$$

Here, β_k and η_{rk}^r are bounded functions with the following virtues:

$$\beta_k(t), \eta_{rk}^r(t) \in L_\infty(\Omega), \quad (\text{for } \ t \in \Omega \ \text{ and } 1 \le k \le m);$$

$$\text{supp}\beta_k, \text{supp}\eta_{rk}^r \subset \text{supp}\varphi_k.$$

By applying Hardy's inequality and estimates (2.3) and (2.4), one obtains

$$\begin{aligned} \left\|s^{2\alpha-1}(L_{(k)}-zI)^{-1}\right\| &\le N_i\sum_{i=1}^{n}\left\|s^{\alpha}d_i(L_{(k)}-zI)^{-1}\right\| \\ &\le N_i'|z|^{-\frac{1}{2}}. \end{aligned} \tag{2.21}$$

From (2.3), (2.11), (2.14), (2.15) and (2.21), we get

$$\|F(z)\| \le N|z|^{-\frac{1}{2}}. \tag{2.22}$$

By suitable choice z, we have $\|F(z)\| \le \frac{1}{2} < 1$. Making use of condition $\|F(z)\| < 1$ and the famous theorem in the operator theory, we find that the operator $I + F(z)$ and then the operator $(L - zI)G(z)$ are invertible. In result,

$$G(z)^{-1}(L-zI)^{-1} = (I \ + \ F(z))^{-1}. \tag{2.23}$$

Adding $+I$ and $-I$ to the right side of (2.23), it concludes that

$$(G(z))^{-1}(L-zI)^{-1} = (I + F(z))^{-1} - I + I.$$

Taking $Y(z) = (I + F(z))^{-1} - I$ in the latter equality, we get

$$(G(z))^{-1}(L-zI)^{-1} = Y(z) + I. \tag{2.24}$$

By (2.19), we have

$$\begin{aligned}\|G(z)\| &= \left\|\sum_{k=1}^{m}\varphi_{(k)}(L_{(k)}-zI)^{-1}\varphi_{(k)}\right\| \\ &\leq N_{\Phi_\theta}'' \left\|(L_{(k)}-zI)^{-1}\right\| \\ &\leq N_{\Phi_\theta}'' N_{2,\Phi_\theta}|z|^{-1} \\ &= N_{2,\Phi_\theta}'|z|^{-1}.\end{aligned} \tag{2.25}$$

By applying (2.3), the inequality $\|F(z)\| \leq \frac{1}{2} < 1$ is concluded and by using the geometric series for $Y(z)$, one obtains

$$\begin{aligned}\|Y(z)\| &\leq \sum_{k=1}^{+\infty}\left\|F^k(z)\right\| \\ &\leq \|F(z)\|\left(1+\|F^k(z)\|+\|F^k(z)\|^2+\cdots\right) \\ &\leq \|F(z)\|\left(1+\frac{1}{2}+\frac{1}{4}+\cdots\right) \\ &\leq 2N|z|^{-\frac{1}{2}}.\end{aligned}$$

Or

$$\|Y(z)\| \leq 2N|z|^{-\frac{1}{2}}. \tag{2.26}$$

From (2.24),-(2.26), it follows that

$$\begin{aligned}\left\|(L-zI)^{-1}\right\| &\leq \|G(z)\|\|(Y(z)+I)\| \\ &\leq N_{2,\Phi_\theta}'|z|^{-1}\left(1+2N|z|^{-\frac{1}{2}}\right) \\ &\leq N_{2,\Phi_\theta}'|z|^{-1}+N_{2,\Phi_\theta}'2N|z|^{-1} \\ &= N_{2,\Phi_\theta}|z|^{-1}.\end{aligned}$$

Thus,

$$\left\|(L-zI)^{-1}\right\| \leq N_{2,\Phi_\theta}|z|^{-1}.$$

This ends the proof of (2.18). □

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