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Convergence of integro quartic and sextic B-spline interpolation

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Abstract. In this paper, quadratic and sextic B-splines are used to construct an approximating function based on the integral values instead of the function values at the knots. This process due to the type of used B-splines (fourth order or sixth order), called integro quadratic or sextic spline interpolation. After introducing the integro quartic and sextic B-spline interpolation, their convergence is discussed. The interpolation errors are studied. Numerical results illustrate the efficiency and effectiveness of the new interpolation method.

1. Introduction

Interpolation is one of the proposed issues in the computational mathematics field and has many practical applications in fields such as mechanics, statistics, climatology and numerical analysis. Spline functions are polynomials with special flexibility in node points, and they are particularly important in interpolation [1, 2].

Assume that $I = [a, b]$ is devided by the following $n+1$ equally spaced points

$$a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b,$$

in which $x_i = a + ih$ $(i = 0, 1, \ldots, n)$ and $h = \frac{b-a}{n}$. In the classical interpolation problems, the function values $y_i = y(x_i)$ at the knot points x_i are usually given over an interval $[a, b]$. Suppose that the function

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values $y_i = y(x_i)$ are not given and instead of them the integral values I_i of $y(x)$ are known on the subintervals $\Delta_i = [x_i, x_{i+1}]$ $(i = 0, 1, \ldots, n-1)$. Our work is to obtain an integro-interpolating function $S(x)$ such that:

$$\int_{x_i}^{x_{i+1}} S(x)dx = I_i = \int_{x_i}^{x_{i+1}} y(x)dx, \quad (i = 0, 1, \ldots, n-1).$$

Clearly, this is a generalization for the classical interpolation problem. In this new interpolation, the Spline functions can be used. Firstly, in 1996, Behforooz [3] presented a new method to construct integro cubic splines by using the integral values of $y(x)$, instead of the function values at the knots. After that, an integro quintic spline approach over a uniform partition was discussed by Behforooz [4]. These methods required several additional boundary conditions and their deduce process is very complicated. Besides, the error analysis were also not given. In this paper, we discuss the integro interpolation by using the quartic and sextic B-splines. The numerical results illustrate that the integro-interpolating quartic and sextic B-spline to reconstruct an approximating function are very effective and convergent, also the second order derivative value in the sextic is better.

2. DEFINITION AND PROBLEM STATE

2.1. **Construction of integro quartic spline.** Integro quartic spline interpolation problem (with four boundary conditions) is expressed as follows [5]:
with using the given integral values I_i of $y(x)$ on the n intervals $\Delta_i = [x_i, x_{i+1}], i = 0, 1, \ldots, n-1$ and four boundary conditions $y_0 = y(x_0), y_0 = y(x_1), y_{n-1} = y(x_{n-1}), y_n = y(x_n)$, we want to build a quartic spline $S(x) \in S_4(I)$ in a way that

$$\int_{x_i}^{x_{i+1}} S(x)dx = I_i = \int_{x_i}^{x_{i+1}} y(x)dx, \quad (i = 0, 1, \ldots, n-1), \tag{2.1}$$

and

$$s(x_0) = y_0, \qquad s(x_1) = y_1, \qquad s(x_{n-1}) = y_{n-1}, \qquad s(x_n) = y_n, \tag{2.2}$$

in which $S_4(I)$ is the univariate quartic spline space over the uniform partition and defined as

$$S_4(I) = \{s(x) \in C^3(I) \mid s_i(x) \in C^3(I), i = 0, 1, \ldots, n-1\}.$$

For each $s(x) \in S_4(I)$, it can be described as

$$s(x) = \sum_{j=-2}^{n+1} c_j B_j(x),$$

where $B_j(x), (i = -2, -1, \ldots, n-1)$ are the typical quadratic B-splines, which are defined as follows

$$B_i(x) = \frac{1}{24h^4} \begin{cases} (x - x_{i-2})^4, & \text{if } x \in [x_{i-2}, x_{i-1}], \\ (x - x_{i-2})^4 - 5(x - x_{i-1})^4, & \text{if } x \in [x_{i-1}, x]_i, \\ (x - x_{i-2})^4 - 5(x - x_{i-1})^4 + 10(x - x_i)^4, & \text{if } x \in [x_i, x_{i+1}], \\ (x - x_{i+3})^4 - 5(x - x_{i+2})^4, & \text{if } x \in [x_{i+1}, x_{i+2}], \\ (x - x_{i+3})^4, & \text{if } x \in [x_{i+2}, x_{i+3}], \\ 0, & \text{else.} \end{cases}$$

By using the $B_i(x)$ integral values, we have

$$\begin{aligned} \int_{x_j}^{x_{j+1}} S(x)dx &= \int_{x_j}^{x_{j+1}} \sum_{i=-2}^{n+1} c_i B_i(x)dx \\ &= \sum_{i=-2}^{n+1} c_i \int_{x_j}^{x_{j+1}} B_i(x)dx \\ &= I_j, \end{aligned}$$

$$\frac{h}{120}(c_{j-2} + 26c_{j-1} + 66c_j + 26c_{j+1} + c_{j+2}) = I_j.$$

In addition to these four boundary conditions, it gives us the four linear equations below:

$$\begin{cases} S(x_0) = y_0 \\ S(x_1) = y_1 \end{cases} \Longrightarrow \begin{cases} c_{-2} + 11c_{-1} + 11c_0 + c_1 = 24y_0 \\ c_{-1} + 11c_0 + 11c_1 + c_2 = 24y_1 \end{cases}$$

$$\begin{cases} S(x_{n-1}) = y_{n-1} \\ S(x_n) = y_n \end{cases} \Longrightarrow \begin{cases} c_{-3} + 11c_{n-2} + 11c_{n-1} + c_n = 24y_{n-1} \\ c_{n-2} + 11c_{n-1} + 11c_n + c_{n+1} = 24y_n \end{cases}$$

Therefore we have a system of linear equations with $c_i(i = -2, -1, \ldots, n+1)$ which can be written in the matrix form

$$AC = F, \tag{2.3}$$

where

$$A = \begin{pmatrix} 1 & 11 & 11 & 1 & 0 & & & & \\ 0 & 1 & 11 & 11 & 1 & & & & \\ 1 & 26 & 66 & 26 & 1 & & & & \\ & 1 & 26 & 66 & 26 & 1 & & & \\ & & & \ddots & \ddots & \ddots & & & \\ & & 1 & 1 & 26 & 66 & 26 & 1 & \\ & & & & 1 & 26 & 66 & 26 & 1 \\ & & & & 1 & 11 & 11 & 1 & 0 \\ & & & & 0 & 1 & 11 & 11 & 1 \end{pmatrix}_{(n+4)\times(n+4)}$$

$$C = (c_{-2}, c_{-1}, c_0, c_1, \ldots, c_n, c_{n+1})^T,$$
$$F = \left(24y_0, 24y_1, \frac{120}{h}I_0, \ldots, \frac{120}{h}I_{n-1}, 24y_{n-1}, 24y_n\right)^T.$$

2.2. **Construction of integro sextic spline.** Integro sextic spline interpolation problem (with six boundary conditions) is expressed as follows [2]:
By using the integral values I_i of $y(x)$ on the n intervals $\Delta_i = [x_i, x_{i+1}], i = 0, 1, \ldots, n-1$ and six boundary conditions $y_0 = y(x_0)$, $y_0^{'} = y^{'}(x_0)$, $y_0^{''} = y^{''}(x_0)$, $y_n = y(x_n), y_n^{'} = y^{'}(x_n)$ and $y_n^{''} = y^{''}(x_n)$, construct a sextic spline $S(x) \in S_6^5(I)$ such that:

$$\int_{x_i}^{x_{i+1}} S(x)dx = I_i = \int_{x_i}^{x_{i+1}} y(x)dx, \quad i = 0, 1, \ldots, n-1$$

$$S(x_0) = y_0, S^{'}(x_0) = y_0^{'}, S^{''}(x_0) = y_0^{''},$$
$$S(x_n) = y_n, S^{'}(x_n) = y_n^{'}, S^{''}(x_n) = y_n^{''},$$

in which $S_6^5(I)$ is the univariate sextic spline space over the uniform partition and defined as

$$S_6^5(I) = \{S(x) \in C^5(I) \mid S_i(x) \in P_6[x], i = 0, 1, \ldots, n-1\},$$

where $S_i(x)$ and $P_6[x]$ are the restriction of $S(x)$ over Δ_i and the set of all univariate sextic polynomials, respectively. For each $S(x) \in S_6^5$, it can be written as $S(x) = \sum_{j=-6}^{n-1} c_j B_j(x)$, where $B_j(x), (i = -2, -1, \ldots, n-1)$ are the typical sextic uniform B-splines, which are defined as follows:

[1, 7, 8],

$$B_i(x) = \frac{1}{6!h^6}\begin{cases} (x-x_i)^6, & \text{if } x \in [x_i, x_{i+1}], \\ (x-x_i)^6 - C_7^1(x-x_{i+1})^6, & \text{if } x \in [x_{i+1}, x_{i+2}], \\ (x-x_i)^6 - C_7^1(x-x_{i+1})^6 + C_7^2(x-x_{i+2})^6, & \text{if } x \in [x_{i+2}, x_{i+3}], \\ (x-x_i)^6 - C_7^1(x-x_{i+1})^6 + C_7^2(x-x_{i+2})^6 + C_7^3(x-x_{i+3})^6, & \text{if } x \in [x_{i+3}, x_{i+4}], \\ (x-x_{i+7})^6 + C_7^1(x-x_{i+6})^6 + C_7^2(x-x_{i+5})^6, & \text{if } x \in [x_{i+4}, x_{i+5}], \\ (x-x_{i+7})^6 + C_7^1(x-x_{i+6})^6, & \text{if } x \in [x_{i+5}, x_{i+6}], \\ (x-x_{i+7})^6, & \text{if } x \in [x_{i+6}, x_{i+7}], \\ 0, & else. \end{cases}$$

Using the $B_i(x)$ integral value, we have

$$\begin{aligned} \int_{x_i}^{x_{i+1}} S(x)dx &= \int_{x_i}^{x_{i+1}} \sum_{j=-6}^{n-1} c_j B_j(x)dx \\ &= \sum_{j=-6}^{n-1} c_j \int_{x_i}^{x_{i+1}} B_j(x)dx \\ &= I_i, \end{aligned}$$

(2.4)

$$c_{i-6} + 120c_{i-5} + 1191c_{i-4} + 2416c_{i-3} + 1191c_{i-2} + 120c_{i-1} + c_i = \frac{5040}{h} I_i.$$

In addition, these six boundary conditions gives us the six linear equations as follow:

$$\begin{cases} S(x_0) = y_0 \\ S^{'}(x_0) = y_0^{'} \\ S^{''}(x_0) = y_0^{''} \\ S^{''}(x_n) = y_n^{''} \\ S^{'}(x_n) = y_n^{'} \\ S(x_n) = y_n \end{cases} \implies \begin{cases} c_{-6} + 57c_{-5} + 302c_{-4} + 302c_{-3} + 57c_{-2} + c_{-1} = 720y_0 \\ -c_{-6} - 25c_{-5} - 40c_{-4} + 40c_{-3} + 25c_{-2} + c_{-1} = 120hy_0^{'} \\ c_{-6} + 9c_{-5} - 10c_{-4} - 10c_{-3} + 9c_{-2} + c_{-1} = 24h^2 y_0 \\ c_{n-6} + 9c_{n-5} - 10c_{n-4} - 10c_{n-3} + 9c_{n-2} + c_{n-1} = 24h^2 y_n \\ -c_{n-6} - 25c_{n-5} - 40c_{n-4} + 40c_{n-3} + 25c_{n-2} + c_{n-1} = 120hy_n^{'} \\ c_{n-6} + 57c_{n-5} + 302c_{n-4} + 302c_{n-3} + 57c_{n-2} + c_{n-1} = 720y_n \end{cases}$$

Therefore the 3 and 4 provide us a system of linear equations with a seven-band coefficient matrix of order $(n + 6)$ by $(n + 6)$. It can be written in matrix form

$$AC = D, \tag{2.5}$$

in which

$$A = \begin{pmatrix} 1 & 57 & 302 & 302 & 57 & 1 & & & & & \\ -1 & -25 & -40 & 40 & 25 & 1 & & & & & \\ 1 & 9 & -10 & -10 & 9 & 1 & & & & & \\ 1 & 120 & 1191 & 2416 & 1191 & 120 & 1 & & & & \\ & 1 & 120 & 1191 & 2416 & 1191 & 120 & 1 & & & \\ & & & & \ddots & & & & & & \\ & & & 1 & 120 & 1191 & 2415 & 1191 & 120 & 1 \\ & & & & 1 & 9 & -10 & -10 & 9 & 1 \\ & & & & -1 & -25 & -40 & 40 & 25 & 1 \\ & & & & 1 & 57 & 302 & 302 & 57 & 1 \end{pmatrix}_{(n+6)\times(n+6)}$$

$$C = (c_{-6}, c_{-5}, \dots, c_{n-1})^T,$$

$$D = \left(720y_0, 120hy_0', 24h^2y_0, \frac{5040}{h}I_0, \frac{5040}{h}I_1, \dots, \frac{5040}{h}I_{n-1}, 24h^2y_0, 120hy_0', 720y_n\right)^T.$$

2.3. **Interpolation convergence.**

Quartic. After solving the system (2.3), the quartic spline

$$S(x) = \sum_{j=-2}^{n+1} C_j B_j(x)$$

has been obtained that it is called integro quartic spline. After that, $s(x)$, $s''(x)$ are applied to approximate $y(x)$, $y''(x)$ at knot points.The values of $s(x)$ and $s''(x)$ at knots are [3]

$$\begin{aligned} S_j &= S(x_j) \\ &= \sum_{i=-6}^{n-1} C_i B_i(x_j) \\ &= \frac{1}{720} c_{j-6} + 57c_{j-5} + 302c_{j-4} + 302c_{j-3} + 57c_{j-2} + c_{j-1}, \\ M_j &= S''(x_j) \\ &= \sum_{i=-6}^{n-1} C_i B_i''(x_j) \\ &= \frac{1}{24h^2}(c_{j-6} + 9c_{j-5} - 10c_{j-4} - 10c_{j-3} + 9c_{j-2} + c_{j-1}). \end{aligned} \tag{2.6}$$

For a given step size h and an infinitely differentiable $y(x)$, we define the shift operator with h, differential operator and identity operator as follows

$$Ey(x) = y(x+h), Dy(x) = y^{'}, Iy(x) = y(x).$$

Similarly, for a positive integer m, we have

$$E^m y(x) = y(x+mh), D^m y(x) = y^{'}, I^m y(x) = y(x).$$

It implies $E = e^{hD}$. Similarly we have $E^m = e^{mhD}$.

Lemma 2.1. *Suppose $y_i = y(x_i)$, then we have [5]*

$$I_i = \frac{E-I}{D} y_i = \sum_{k=0}^{\inf} \frac{h^k+1}{(k+1)!} y^{(k)}(x_i).$$

Lemma 2.2. *Let $s(x)$ be the integro-interpolating quartic spline obtained by (2.1) and (2.2) for $y(x)$. For $j = 0, 1, \dots, n$ we have [5]*

$$s_j = \frac{5}{h}\left(\frac{E^{-2}+11E^{-1}+11I+E}{E^{-2}+26E^{-1}+66I+26E+E^2}\right) I_j, \tag{2.7}$$

$$m_j = \frac{20}{h^2}\left(\frac{-E^{-2}-3E^{-1}+3I+E}{E^{-2}+26E^{-1}+66I+26E+E^2}\right) I_j.$$

Theorem 2.3. *Assume that $y(x)$ be a function that is infinitely differentiable and $S(x)$ be the integro quartic spline obtained from (2.1) and (2.2), then for $i = 0, 1, \dots, n$ we have [5]*

$$s_j = y(x_j) + \frac{1}{5040} h^6 y^6(x_i) + O(h^8),$$

$$m_j = y^{'}(x_j) + \frac{1}{720} h^4 y^5(x_j) + O(h^6).$$

Proof. Here, we give a brief proof for (2.7), the proof for the others are similar and omitted. By Lemma 2.1 and (2.7), we have

$$s_j = \frac{5}{hD}\left(\frac{-E^{-2}-10E^{-1}+10E+E^2}{E^{-2}+26E^{-1}+66I+26E+E^2}\right) y_j.$$

Let $u = hD$, we have

$$\begin{aligned}
\frac{5}{u}\left(\frac{-e^{-2u}-10e^{-u}+10e^{u}+e^{2u}}{e^{-2u}+26e^{-u}+66+26e^{u}+e^{2u}}\right) &= \frac{120+30u^2+\frac{7}{2}u^4+\frac{23}{84}u^6+\cdots}{120+30u^2+\frac{7}{2}u^4+\frac{1}{4}u^6+\cdots}, \\
&= 1 + \frac{\frac{23}{84}-\frac{1}{4}}{120} u^6 + cu^8 + \cdots, \\
&= 1 + \frac{1}{5040} u^6 + cu^8 + \cdots,
\end{aligned}$$

where c is a certain constant. Hence,

$$s_j = \left(1 + \frac{(hD)^6}{5040} + c(hD)^8 + \cdots\right) = y(x_j) + \frac{1}{5040} h^6 y^6(x_i) + O(h^8).$$

□

This theorem shows clearly that

$$\max_{0\le j\le n} |s_j - y_j| = O(h^6),$$
$$\max_{0\le j\le n} |M_j - y_j^{''}| = O(h^4).$$

Sextic. After solving the system (2.5), the sextic spline

$$S(x) = \sum_{j=-6}^{n-1} C_j B_j(x)$$

has been obtained that it is called integro sextic spline. After that, $s(x)$, $s^{''}(x)$ are applied to approximate $y(x)$, $y^{''}(x)$ at knots. The values of $s(x)$ and $s^{''}(x)$ at knots are [6]

$$\begin{aligned} S_j &= S(x_j) \\ &= \sum_{i=-6}^{n-1} C_i B_i\left(x_j\right), \\ &= \frac{1}{720} c_{j-6} + 57c_{j-5} + 302c_{j-4} + 302c_{j-3} + 57c_{j-2} + c_{j-1}, \\ M_j &= S^{''}(x_j) \\ &= \sum_{i=-6}^{n-1} C_i B_i^{''}(x_j), \\ &= \frac{1}{24h^2}(c_{j-6} + 9c_{j-5} - 10c_{j-4} - 10c_{j-3} + 9c_{j-2} + c_{j-1}). \end{aligned} \tag{2.8}$$

Theorem 2.4. *Suppose that $y(x)$ be an infinitely differentiable function and $s(x)$ be the integro sextic spline obtained by, then for $i = 0, 1, \ldots, n$ we have [6]*

$$\begin{aligned} S_i &= y(x_i) - \frac{h^8}{151200} y^8(x_i) + O(h^{10}), \\ M_i &= y^{''}(x_i) + \frac{h^6}{6048} y^8(x_i) + O(h^8). \end{aligned} \tag{2.9}$$

Proof. We firstly give the proof for (2.9). By using (2.4) and (2.6), we have

$$S_i + 120S_{i+1} + 1191S_{i+2} + 2416S_{i+3} + 1191S_{i+4}$$
$$+ 120S_{i+5} + S_{i+6} = \frac{7}{h}(I_i + 57I_{i+1} + 302I_{i+2} + 302I_{i+3} + 57I_{i+4} + I_{i+5}).$$

Using the above operators, Lemma 2.1, and the Taylor formula of e^x, we have

$$\begin{aligned} S_j &= \frac{7}{h}\left(\frac{I + 57E + 302E^2 + 302E^3 + 57E^4 + E^5}{I + 120E + 1991E^2 + 2416E^3 + 1991E^4 + 120E^5 + E^6}\right) I_j \\ &= \frac{7}{hD}\frac{\left(I + 57E + 302E^2 + 302E^3 + 57E^4 + E^5\right)(E - I)}{I + 120E + 1991E^2 + 2416E^3 + 1991E^4 + 120E^5 + E^6} y_j \\ &= \frac{7}{hD}\frac{\left(I + 57e^{hD} + 302e^{2hD} + 302e^{3hD} + 57e^{4hD} + e^{5hD}\right)(e^{hD} - I)}{I + 120e^{hD} + 1191e^{2hD} + 2416e^{3hD} + 1191e^{4hD} + 120e^{5hD} + e^{6hD}} \\ &= \left(I - \frac{(hD)^8}{151200} + \frac{(hD)^{10}}{399168} + \cdots\right) y_i \\ &= y(x_i) - \frac{h^8}{151200} y^{(8)}(x_i) + O(h^{10}). \end{aligned}$$

Similarly, by using (2.4) and (2.8), we easily have

$$M_i + 120M_{i+1} + 1191M_{i+2} + 2416M_{i+3} + 1191M_{i+4}$$
$$+ 120M_{i+5} + M_{i+6} = \frac{210}{h^3}(I_i + 9I_{i+1} - 10I_{i+2} - 10I_{i+3}9I_{i+4} + I_{i+5}).$$

Then

$$\begin{aligned} M_j &= \frac{210}{h^3}\left(\frac{I + 9E - 10E^2 - 10E^3 + 9E^4 + E^5}{I + 120E + 1991E^2 + 2416E^3 + 1991E^4 + 120E^5 + E^6}\right) I_j \\ &= D^2 \frac{210}{(hD)^3}\frac{\left(I + 9E - 10E^2 - 10E^3 + 9E^4 + E^5\right)(E - I)}{I + 120E + 1991E^2 + 2416E^3 + 1991E^4 + 120E^5 + E^6}) y_j \\ &= D^2 \frac{210}{(hD)^3}\frac{\left(I + 9e^{hD} - 10e^{2hD} - 10e^{3hD} + 9e^{4hD} + e^{5hD}\right)(e^{hD} - I)}{I + 120e^{hD} + 1191e^{2hD} + 2416e^{3hD} + 1191e^{4hD} + 120e^{5hD} + e^{6hD}} y_i \\ &= D^2\left(I - \frac{(hD)^6}{6048} - \frac{(hD)^8}{33600} + \cdots\right) y_i \\ &= y''(x_i) - \frac{h^6}{6048} y^{(8)}(x_i) + O(h^8). \end{aligned}$$

□

This theorem shows clearly that:

$$\max_{0 \le j \le n} |S_j - y(x_j)| = O(h^8), \tag{2.10}$$

$$\max_{0 \le j \le n} |M_j - y''(x_j)| = O(h^6). \tag{2.11}$$

2.4. **Numerical experiments.** In this section, the numerical results are presented to illustrate the accuracy, efficiency and super convergence properties of integro quartic and sextic splines. The numerical examples are performed with using the MATLAB software. Assume that $y(x) \in C^{\infty}[0,1]$ and consider the following function [4–6]

$$y(x) = \frac{1}{x+2},$$

Let $S(x)$ be the integro-interpolating quartic spline with four boundary obtained by (2.10) and (2.11) and $P(x)$ be the integro-interpolating sextic spline with six boundary conditions obtained by (2.10) and (2.11) for $y(x)$. The maximum absolute error between the approximated function $y(x)$ and its corresponding integro quartic and sextic spline at knots are defined as

$$\begin{aligned}
E_s(n) &= \max_{0\le j\le n} |S_j - y(x_j)|,\\
E_p(n) &= \max_{0\le j\le n} |P_j - y(x_j)|,\\
E_s^{''}(n) &= \max_{0\le j\le n} |S_j^{''} - y^{''}(x_j)|,\\
E_p^{''}(n) &= \max_{0\le j\le n} |P_j^{''} - y^{''}(x_j)|,
\end{aligned}$$

where $E_s(n)$ and $E_p(n)$ are the maximum absolute errors for $S(x)$ and $P(x)$, respectively. TABLE 1. shows the maximum errors $E_s(n)$ and $E_s^{''}(n)$ between the approximated function $y(x)$ and its corresponding integro quartic spline $S(x)$ and TABLE 2. shows the maximum errors $E_p(n)$ and $E_p^{''}(n)$ between the approximated function $y(x)$ and its corresponding integro sextic spline $P(x)$. From these tables, it concludes that the integro quartic and sextic splines certainly have super convergence order in approximating function values and second-order derivative values at the knots. These approximation behaviors are very satisfactory and it is an interesting result. According to the TABLE 2. the second order derivative value in the sextic is better, because it's absolute error is much lower.

TABLE 1. The maximum errors of $S(x)$ for $y(x)$

n	$E_s(n)$	$E_s^{''}(n)$
10	9.427×10^{-10}	1.127×10^{-5}
20	1.952×10^{-11}	7.899×10^{-7}
40	3.538×10^{-13}	5.204×10^{-8}

TABLE 2. The maximum errors of $P(x)$ for $y(x)$

n	$E_p(n)$	$E''_p(n)$
10	1.01×10^{-11}	1.32×10^{-8}
20	8.46×10^{-14}	4.07×10^{-10}
40	7.64×10^{-16}	1.511×10^{-11}

3. Conclusion

In this paper, the integro interpolation problem was studied by using quadratic and sextic splines. This new interpolation is very useful in numerous fields. Integro spline interpolation rebuild the approximated function by using the integral values on arbitrary successive intervals instead of the function values at nodes. By using quartic and sextic B-splines, we study a concise method to make the integro-interpolating quartic and sextic splines. The integro spline interpolation can approximate the function values and also it can approximate the first and second order derivative values.

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