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Fuzzy e-regular spaces and strongly e-irresolute mappings

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ABSTRACT. The aim of this paper is to introduce fuzzy (e, almost) e^*-regular spaces in Šostak's fuzzy topological spaces. Using the r-fuzzy e-closed sets, we define r-(r-θ-, r-$e\theta$-) e-cluster points and their properties. Moreover, we investigate the relations among r-(r-θ-, r-$e\theta$-) e-cluster points, r-fuzzy (e, almost) e^*-regular spaces and their functions.

1. Introduction

Kubiak [10] and Šostak [15] introduced the fundamental concept of a fuzzy topological structure, as an extension of both crisp topology and fuzzy topology [1], in the sense that not only the objects are fuzzified, but also the axiomatics. In [13, 14], Šostak gave some rules and showed how such an extension can be realized. Chattopadhyay et al., [2] have redefined the same concept under the name gradation of openness. It has been developed in many directions [2–6, 9]. Kim et. al [4–6, 8, 9] investigate r-regulars closed sets, several operators and fuzzy (almost) regular spaces in Šostak's fuzzy topological spaces. In this paper, we introduce r-fuzzy e-closed sets in Šostak's fuzzy topological spaces. We study the notions of r-fuzzy (e, almost) e^*-regular spaces. We investigate some properties. In particular, we define r-(r-θ-, r-$e\theta$-) e-cluster points and their properties. Moreover, we investigate the relations among r-(r-θ-, r-$e\theta$-) e-cluster points, r-fuzzy (e, almost) e^*-regular spaces and their functions.

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2. Preliminaries

Throughout this paper, let X be a non-empty set, $I = [0,1]$, $I_0 = (0,1]$. A fuzzy set λ of X is a mapping $\lambda : X \to I$, and I^X be the family of all fuzzy sets on X. The complement of a fuzzy set λ is denoted by $\overline{1} - \lambda$. For $\lambda \in I^X, \overline{\lambda}(x) = \lambda$ for all $x \in X$. For each $x \in X$ and $t \in I_0$, a fuzzy point x_t is defined by

$$x_t(y) = \begin{cases} t & \text{if } y = x, \\ 0 & \text{if } y \neq x. \end{cases}$$

Let $Pt(X)$ be the family of all fuzzy points in X. For $\lambda, \mu \in I^X, \lambda$ is called quasi coincident with μ, denoted by $\lambda q \mu$, if there exists $x \in X$ such that $\lambda(x) + \mu(x) > 1$. Otherwise, we denote $\lambda \overline{q} \mu$. We define $x_t \in \lambda$ if $t \leq \lambda(x)$. All other notations and definitions are standard in the fuzzy set theory.

Definition 2.1 ([15])**.** A function $\tau : I^X \to I$ is called a fuzzy topology on X if it satisfies the following conditions:

(1) $\tau(\overline{0}) = \tau(\overline{1}) = 1$,
(2) $\tau(\bigvee_{i \in J} \mu_i) \geq \bigwedge_{i \in J} \tau(\mu_i)$, for any $\{\mu_i : i \in J\} \subseteq I^X$,
(3) $\tau(\mu_1 \wedge \mu_2) \geq \tau(\mu_1) \wedge \tau(\mu_2)$, for all $\mu_1, \mu_2 \in I^X$.

The pair (X, τ) is called a fuzzy topological space (for short, fts).

Definition 2.2 ([5])**.** Let (X, τ) be a fts, $\lambda, \mu \in I^X$ and $r \in I_0$. We define operators as follows:

$$C_\tau(\lambda, r) = \wedge\{\mu \in I^X | \lambda \leq \mu, \tau(\overline{1} - \mu) \geq r\},$$
$$I_\tau(\lambda, r) = \vee\{\mu \in I^X | \lambda \geq \mu, \tau(\mu) \geq r\}.$$

Definition 2.3 ([5])**.** Let (X, τ) be a fts. For $\lambda, \mu \in I^X$ and $r \in I_0$, λ is called r-fuzzy regular open (for short, r-fro) (resp. r-fuzzy regular closed (for short, r-frc)) if $\lambda = I_\tau(C_\tau(\lambda, r), r)$ (resp. $\lambda = C_\tau(I_\tau(\lambda, r), r)$).

Definition 2.4 ([12])**.** Let (X, τ) be a fts. $\lambda, \mu \in I^X$ and $r \in I_0$,

(i) δ-$I_\tau(\lambda, r) = \bigvee\{\mu \in I^X : \mu \leq \lambda, \mu$ is a r-fro set $\}$ is called the r-fuzzy δ-interior of λ.
(ii) δ-$C_\tau(\lambda, r) = \bigwedge\{\mu \in I^X : \mu \geq \lambda, \mu$ is a r-frc set $\}$ is called the r-fuzzy δ-closure of λ.

Definition 2.5 ([12])**.** Let (X, τ) be a fts. For $\lambda, \mu \in I^X$ and $r \in I_0$,

(i) λ is called an r-fuzzy δ-semiopen (resp. r-fuzzy δ-semiclosed) set if $\lambda \leq C_\tau(\delta\text{-}I_\tau(\lambda, r), r)$ (resp. $I_\tau(\delta\text{-}C_\tau(\lambda, r), r) \leq \lambda$).
(ii) λ is called an r-fuzzy δ-preopen (resp. r-fuzzy δ-preclosed) set if $\lambda \leq I_\tau(\delta\text{-}C_\tau(\lambda, r), r)$ (resp. $C_\tau(\delta\text{-}I_\tau(\lambda, r), r) \leq \lambda$).

(iii) λ is called an r-fuzzy semi δ-preopen (resp. r-fuzzy semi δ-preclosed) set if $\lambda \leq I_\tau(C_\tau(\delta\text{-}I_\tau(\lambda, r), r), r)$ (resp. $C_\tau(I_\tau(\delta\text{-}C_\tau(\lambda, r), r), r) \leq \lambda$).

(iv) λ is called an r-fuzzy e-open (resp. r-fuzzy e-closed) set if $\lambda \leq C_\tau(\delta\text{-}I_\tau(\lambda, r), r) \vee I_\tau(\delta\text{-}C_\tau(\lambda, r), r)$ (resp. $C_\tau(\delta\text{-}I_\tau(\lambda, r), r) \wedge I_\tau(\delta\text{-}C_\tau(\lambda, r), r) \leq \lambda$).

Definition 2.6 ([12])**.** Let (X, τ) be a fts. $\lambda, \mu \in I^X$ and $r \in I_0$,

(i) $eI_\tau(\lambda, r) = \bigvee\{\mu \in I^X : \mu \leq \lambda, \mu$ is a r-feo set $\}$ is called the r-fuzzy e-interior of λ.

(ii) $eC_\tau(\lambda, r) = \bigwedge\{\mu \in I^X : \mu \geq \lambda, \mu$ is a r-fec set $\}$ is called the r-fuzzy e-closure of λ.

Definition 2.7 ([8])**.** Let (X, τ) be a fts and $x_t \in Pt(X)$. We denote

$$\mathcal{Q}_\tau(x_t, r) = \{\mu \in I^X | x_t q \mu, \tau(\mu) \geq r\},$$
$$\mathcal{R}_\tau(x_t, r) = \{\mu \in I^X | x_t q \mu, \mu \text{ is } r\text{-fro}\}.$$

Definition 2.8 ([8])**.** Let (X, τ) be a fts, $\lambda \in I^X$, $x_t \in Pt(X)$ and $r \in I_0$. A fuzzy point x_t is called:

(i) an r-(resp. r-θ-) cluster point of λ if $\mu q \lambda$ (resp. $C_\tau(\mu, r) q \lambda$) for every $\mu \in \mathcal{Q}_\tau(x_t, r)$.

(ii) an r-(resp. r-θ-) regular cluster point of λ if $\mu q \lambda$ (resp. $C_\tau(\mu, r) q \lambda$) for every $\mu \in \mathcal{R}_\tau(x_t, r)$.

Also, we define operators RC_τ and RT_τ with respect to r-regular cluster and r-θ-regular cluster points respectively.

Theorem 2.9 ([7])**.** *Let (X, τ) be a fts. For each $\lambda, \mu, \rho \in I^X$ and $r \in I_0$ we have the following properties:*

(1) $C_\tau(\lambda, r) = \bigvee\{x_t \in Pt(X) | x_t \textit{ is an } r\textit{-cluster point of} \lambda\}$,
$RC_\tau(\lambda, r) = \bigwedge\{\mu \in I^X | \lambda \leq \mu, \mu \textit{ is } r\textit{-frc}\}$.

(2) $T_\tau(\lambda, r) = \bigwedge\{\mu \in I^X | \lambda \leq I_\tau(\mu, r), \tau(\bar{1} - \mu) \geq r\}$,
$RT_\tau(\lambda, r) = \bigwedge\{\mu \in I^X | \lambda \leq I_\tau(\mu, r), \mu \textit{ is } r\textit{-frc}\}$.

(3) *x_t is an r-θ-cluster point of λ iff $x_t \in T_\tau(\lambda, r)$,*
x_t is an r-θ-regular cluster point of λ iff $x_t \in RT_\tau(\lambda, r)$.

Definition 2.10 ([7])**.** Let (X, τ) be a fts. Then (X, τ) is called an r-fuzzy regular (resp. r-fuzzy almost regular) if for each $\tau(\mu) \geq r$ (resp. r-regular open μ), there exists a family $\{\nu_i \in I^X | \tau(\nu_i) \geq r\}$ such that $\mu = \bigvee_{i \in \Gamma} \nu_i$ with $C_\tau(\nu_i, r) \leq \mu$.

Definition 2.11. Let (X, τ) and (Y, η) be fts's, a function $f : (X, \tau) \to (Y, \eta)$ is called:

(i) fuzzy continuous [11] iff $\tau(f^{-1}(\mu)) \geq \eta(\mu)$,

(ii) fuzzy open (resp. fuzzy closed) [11] iff $\eta(f(\lambda)) \geq \tau(\lambda)$ (resp. $\eta(\overline{1} - f(\lambda)) \geq \tau(\overline{1} - \lambda)$),
(iii) fuzzy e-irresolute [12] iff $f^{-1}(\mu)$ is r-feo for each r-feo $\mu \in I^Y$.
(iv) fuzzy e-continuous [12] (resp. fuzzy weakly e-continuous) iff for each $\mu \in \mathcal{Q}_\eta(f(x)_t, r)$, there exists $\lambda \in e_\tau(x_t, r)$ such that $f(\lambda) \leq \mu$ (resp. $f(\lambda) \leq eC_\eta(\mu, r)$),
(v) f is called fuzzy δ-semiopen [12] (resp. fuzzy δ-preopen, fuzzy semi δ-preopen and fuzzy e-open) iff $f(\lambda)$ is an r-fδso (resp. r-fδpo, r-fsδpo and r-feo) set of Y for each $\lambda \in I^X, r \in I_0$ with $\tau_1(\lambda) \geq r$.
(vi) f is called fuzzy δ-semiclosed [12] (resp. fuzzy δ-preclosed, fuzzy semi δ-preclosed and fuzzy e-closed) iff $f(\lambda)$ is an r-fδsc (resp. r-fδpc, r-fsδpc and r-fγc) set of Y for each $\lambda \in I^X, r \in I_0$ with $\tau_1(\overline{1} - \lambda) \geq r$.

Theorem 2.12 ([12])**.** *Let (X, τ) be a fts and $r \in I_o$.*
(i) *Any union of r-feo sets is an r-feo set.*
(ii) *Any intersection of r-fec sets is an r-fec set.*

Definition 2.13 ([7])**.** Let (X, τ) and (Y, η) be fts's a function $f : (X, \tau) \to (Y, \eta)$ is called a supercontinuous iff for each $\mu \in \mathcal{Q}_\eta(f(x)_t, r)$, there exists $\lambda \in \mathcal{R}_\tau(x_t, r)$ such that $f(\lambda) \leq \mu$.

3. Fuzzy e-regular spaces

Definition 3.1. Let (X, τ) be a fts and $x_t \in Pt(X)$. We denote
$$\mathcal{E}_\tau(x_t, r) = \{\mu \in I^X | x_t q \mu, \mu \text{is} r\text{-feo}\}.$$

Definition 3.2. Let (X, τ) be a fts, $\lambda \in I^X$, $x_t \in Pt(X)$ and $r \in I_0$. A fuzzy point x_t is called:
(i) an r-$e\theta$-cluster point of λ if $eC_\tau(\mu, r) q \lambda$ for every $\mu \in \mathcal{Q}_\tau(x_t, r)$,
(ii) an r-(resp. r-θ-, r-$e\theta$-) e-cluster point of λ if $\mu q \lambda$ (resp. $C_\tau(\mu, r) q \lambda$, $eC_\tau(\mu, r) q \lambda$) for every $\mu \in \mathcal{E}_\tau(x_t, r)$,
(iii) an r-$e\theta$-regular cluster point of λ if $eC_\tau(\mu, r) q \lambda$ for every $\mu \in \mathcal{R}_\tau(x_t, r)$.

We define operators $eT_\tau, eeT_\tau : I^X \times I_0 \to I^X$ as follows:
$$eT_\tau(\lambda, r) = \bigvee \{x_t \in Pt(X) | x_t \text{ is an } r\text{-}\theta\text{-}e\text{cluster point of } \lambda\},$$
$$eeT_\tau(\lambda, r) = \bigvee \{x_t \in Pt(X) | x_t \text{ is a } r\text{-}e\theta\text{-}e\text{cluster point of } \lambda\}.$$

Also, we define operators ReT_τ and CeT_τ with respect to r-$e\theta$-regular cluster and r-$e\theta$-cluster points respectively.

Theorem 3.3. *Let (X, τ) be a fts. For $\lambda, \mu \in I^X, r \in I_0$, it holds the following properties*

(1) $eC_\tau(\overline{1}-\lambda, r) = \overline{1} - eI_\tau(\lambda, r)$,
(2) $\lambda \le eC_\tau(\lambda, r) \le C_\tau(\lambda, r)$,
(3) *If* $\tau(\lambda) \ge r$ *and* $\tau(\overline{1}-\lambda) \ge r$, *then* $eC_\tau(\lambda, r) = C_\tau(\lambda, r)$,
(4) $eC_\tau(eC_\tau(\lambda, r), r) = eC_\tau(\lambda, r)$.

Proof. (1) For each $\lambda \in I^X, r \in I_0$, we have

$$\begin{aligned} eI_\tau(\overline{1}-\lambda, r) &= \bigvee \{\mu \in I^X | \mu \le \overline{1}-\lambda, \mu \text{is} r\text{-feo}\} \\ &= \overline{1} - \bigwedge \{\overline{1}-\mu | \overline{1}-\mu \ge \lambda, \overline{1}-\mu \text{is} r\text{-fec}\} \\ &= \overline{1} - eC_\tau(\lambda, r). \end{aligned}$$

(2) Since $\tau(\overline{1}-\lambda) \ge r$, then μ is r-fec. Thus the result holds.
(3) Suppose $eC_\tau(\lambda, r)(x) < t < C_\tau(\lambda, r)(x)$. There exists an r-fec set μ with $\lambda \le \mu$ such that

$$eC_\tau(\lambda, r)(x) < \mu(x) < t < C_\tau(\lambda, r)(x).$$

Since μ is r-fec,

$$C_\tau(\delta\text{-}I_\tau(\mu, r), r) \wedge I_\tau(\delta\text{-}C_\tau(\mu, r), r) \le \mu.$$

Since $\tau(\lambda) \ge r$ and $\tau(\overline{1}-\lambda) \ge r$, $I_\tau(\lambda, r) = \lambda$ and $C_\tau(\lambda, r) = \lambda$. So

$$\begin{aligned} C_\tau(\lambda, r)(x) &= C_\tau(\delta\text{-}I_\tau(\lambda, r), r)(x) \wedge I_\tau(\delta\text{-}C_\tau(\lambda, r), r)(x) \\ &\le C_\tau(\delta\text{-}I_\tau(\mu, r), r)(x) \wedge I_\tau(\delta\text{-}C_\tau(\mu, r), r)(x) \\ &\le \mu(x) \\ &< t. \end{aligned}$$

It is a contradiction.
(4) Since $eC_\tau(\lambda, r)$ is r-fec from Theorem 2.12 (2), it is trivial. □

Theorem 3.4. *Let* (X, τ) *be a fts. The following statements hold:*

r-e cluster	$\Rightarrow$	r-$e\theta$-e cluster	$\Rightarrow$	r-θ-e cluster
$\Downarrow$		$\Downarrow$		$\Downarrow$
r-cluster	$\Rightarrow$	r-$e\theta$ cluster	$\Leftrightarrow$	r-θ cluster
$\Downarrow$		$\Downarrow$		$\Downarrow$
r-regular cluster	$\Rightarrow$	r-$e\theta$ regular cluster	$\Leftrightarrow$	r-θ regular cluster

Proof. By Theorem 3.3 (3), since $eC_\tau(\mu, r) = C_\tau(\mu, r)$ for $\tau(\mu) \ge r$, x_t is an r-$e\theta$ (resp. r-$e\theta$ regular) cluster point iff x_t is an r-θ (resp. r-θ regular) cluster point. Other implications follow from the definitions. □

Theorem 3.5. *Let* (X, τ) *be a fts. For each* $\lambda, \mu, \rho \in I^X$ *and* $r \in I_0$ *we have the following properties:*

(1) $\mathcal{R}_\tau(x_t,r) \subset \mathcal{Q}_\tau(x_t,r) \subset \mathcal{E}_\tau(x_t,r)$.
(2) $eC_\tau(\lambda,r) = \bigvee\{x_t \in Pt(X) | x_t$ *is an r-ecluster point of*$\lambda\}$.
(3) $eT_\tau(\lambda,r) = \bigwedge\{\mu \in I^X | \lambda \le I_\tau(\mu,r), \mu$ *is r-fec*$\}$.
(4) $eeT_\tau(\lambda,r) = \bigwedge\{\mu \in I^X | \lambda \le eI_\tau(\mu,r), \mu$ *is r-fec*$\}$,

$$CeT_\tau(\lambda,r) = \bigwedge\{\mu \in I^X | \lambda \le eI_\tau(\mu,r), \tau(\overline{1}-\mu) \ge r\},$$

$$ReT_\tau(\lambda,r) = \bigwedge\{\mu \in I^X | \lambda \le eI_\tau(\mu,r), \mu \text{ is } r\text{-frc}\}.$$

(5) x_t *is an e-cluster point of* λ *iff* $x_t \in eC_\tau(\lambda,r)$,
x_t *is an r-θ- (resp. r-eθ-) e cluster point of* λ *iff* $x_t \in eT_\tau(\lambda,r)$ *(resp.* $x_t \in eeT_\tau(\lambda,r)$*)*,
x_t *is an r-eθ-regular cluster point of* λ *iff* $x_t \in ReT_\tau(\lambda,r)$.
(6) $CeT_\tau(\lambda,r) = T_\tau(\lambda,r)$ *and* $ReT_\tau(\lambda,r) = RT_\tau(\lambda,r)$.
(7) $eC_\tau(\lambda,r) \le eeT_\tau(\lambda,r) \le eT_\tau(\lambda,r) \le T_\tau(\lambda,r) \le RT_\tau(\lambda,r)$.
(8) $eC_\tau(\lambda,r) \le C_\tau(\lambda,r) \le RC_\tau(\lambda,r) \le T_\tau(\lambda,r) \le RT_\tau(\lambda,r)$.
(9) *If* ρ *is r-feo, then*

$$eC_\tau(\rho,r) = eeT_\tau(\rho,r),$$

and

$$\begin{aligned} C_\tau(\rho,r) &= RC_\tau(\rho,r) \\ &= T_\tau(\rho,r) \\ &= RT_\tau(\rho,r). \end{aligned}$$

(10) *If* $\tau(\rho) \ge r$, *then*

$$\begin{aligned} eC_\tau(\rho,r) &= eeT_\tau(\rho,r) \\ &= eT_\tau(\rho,r) \\ &= C_\tau(\rho,r) \\ &= RC_\tau(\rho,r) \\ &= T_\tau(\rho,r) \\ &= RT_\tau(\rho,r). \end{aligned}$$

Proof. (1) It follows from the definitions.
(2) Put $\rho = \vee\{x_t \in Pt(X) | x_t$ is an r-e cluster point of $\lambda\}$. Suppose $eC_\tau(\lambda,r) \not\le \rho$. Then there exist $x \in X$ and $t \in (0,1)$ such that $eC_\tau(\lambda,r)(x) > t > \rho(x)$. Then x_t is not an r-e cluster point of λ. So, there exists $\mu \in \mathcal{E}_\tau(x_t,r)$, $\lambda \le \overline{1}-\mu$ and $\overline{1}-\mu$ is r-fec. By the definition of eC_τ, in Theorem 3.3

$$eC_\tau(\lambda,r)(x) \le (\overline{1}-\mu)(x) < t.$$

It is a contradiction. Thus $eC_\tau(\lambda,r) \le \rho$.

Suppose $eC_\tau(\lambda,r) \not\ge \rho$. Then there exists an r-e cluster point $y_s \in Pt(X)$ of λ such that $eC_\tau(\lambda,r)(y) < s \le \rho(y)$. By the definition of eC_τ, there exists an r-fec set μ with $\lambda \leq \mu$ such that $eC_\tau(\lambda,r)(y) \leq \mu(y) < s < \rho(y)$. Then, $\overline{1}-\mu \in \mathcal{E}_\tau(y_s,r)$ and $\lambda \overline{q} \overline{1}-\mu$. Hence, y_s is not an r-e cluster point of λ. It is a contradiction. So $eC_\tau(\lambda,r) \geq \rho$.

(3) Put

$$\delta = \bigwedge\{\mu \in I^X | \lambda \leq I_\tau(\mu,r), \mu \text{is} r\text{-fec}\}.$$

Suppose $eT_\tau(\lambda,r) \not\geq \delta$. Then there exist $x \in X$ and $t \in (0,1)$ such that $eT_\tau(\lambda,r)(x) < t < \delta(x)$. Then x_t is not an r-θ-e cluster point of λ. So, there exists $\mu \in \mathcal{E}_\tau(x_t,r)$ and $C_\tau(\mu,r) \leq \overline{1}-\lambda$. Thus $\overline{1}-\mu$ is r-fec and

$$\lambda \leq \overline{1} - C_\tau(\mu,r) = I_\tau(\overline{1}-\mu,r).$$

Hence $\delta(x) \leq (\overline{1}-\mu)(x) < t$. It is a contradiction. Thus $eT_\tau(\lambda,r) \geq \delta$.

Suppose $eT_\tau(\lambda,r) \not\leq \delta$. Then there exists an r-θ-e cluster point y_s of λ such that $eT_\tau(\lambda,r)(y) \geq s > \delta(y)$. By the definition of δ, there exists μ with $\lambda \leq I_\tau(\mu,r)$ and μ is r-fec such that

$$eT_\tau(\lambda,r)(y) \geq s > \mu(y) \geq \delta(y).$$

Then, μ is r-fec and $\overline{1}-\mu \in \mathcal{E}_\tau(y_s,r)$. So

$$\lambda \leq I_\tau(\mu,r) = \overline{1} - C_\tau(\overline{1}-\mu,r),$$

implies $\lambda \overline{q} C_\tau(\overline{1}-\mu,r)$. Hence, y_s is not an r-θe cluster point of λ. It is a contradiction. Thus $eT_\tau(\lambda,r) \leq \delta$.

(4) Put

$$\gamma = \bigwedge\{\mu \in I^X | \lambda \leq eI_\tau(\mu,r), C_\tau(I_\tau(\mu,r),r) = \mu\}.$$

Suppose $ReT_\tau(\lambda,r) \not\geq \gamma$. There exist $x \in X$ and $t \in (0,1)$ such that $ReT_\tau(\lambda,r)(x) < t < \gamma(x)$. Then x_t is not an r-$e\theta$ regular cluster point of λ. So, there exists $\mu \in R_\tau(x_t,r)$, $eC_\tau(\mu,r) \leq \overline{1}-\lambda$. Thus

$$\begin{aligned}\lambda &\leq \overline{1} - eC_\tau(\mu,r)\\ &= eI_\tau(\overline{1}-\mu,r), C_\tau(I_\tau(\mu,r),r)\\ &= \overline{1}-\mu.\end{aligned}$$

Hence $\gamma(x) \leq (\overline{1}-\mu)(x) < t$. It is a contradiction. Thus $ReT_\tau(\lambda,r) \geq \gamma$.

Suppose $ReT_\tau(\lambda,r) \not\leq \gamma$. Then there exists an r-$e\theta$ regular cluster point y_s of λ such that $ReT_\tau(\lambda,r)(y) \geq s > \gamma(y)$. By the definition of γ, there exists μ with $\lambda \leq eI_\tau(\mu,r)$, $C_\tau(I_\tau(\mu,r),r) = \mu$ such that $RT_\tau(\lambda,r)(y) \geq s > \mu(y) \geq \gamma(y)$. Then, μ is r-frc and $\overline{1}-\mu \in \mathcal{R}_\tau(y_s,r)$. Furthermore, $\lambda \leq eI_\tau(\mu,r) =$

$\overline{1} - eC_\tau(\overline{1} - \mu, r)$ implies $\lambda \overline{q} eC_\tau(\overline{1} - \mu, r)$. Hence, y_s is not an r-$e\theta$ regular cluster point of λ. It is a contradiction. Thus $ReT_\tau(\lambda, r) \leq \gamma$. Other cases are similarly proved.

(5) We show that x_t is an r-$e\theta$-e cluster point of λ iff $x_t \in eeT_\tau(\lambda, r)$.

($\Rightarrow$) It is trivial.

($\Leftarrow$) Suppose that x_t is not an r-$e\theta$-e cluster point of λ. Then there exists $\mu \in \mathcal{E}_\tau(x_t, r)$ such that $eC_\tau(\mu, r) \leq \overline{1} - \lambda$. Thus,

$$\lambda \leq \overline{1} - eC_\tau(\mu, r) = eI_\tau(\mu, r).$$

By (3), we have $eeT_\tau(\lambda, r)(x) \leq (\overline{1} - \mu)(x) < t$. Hence $x_t \notin eeT_\tau(\lambda, r)$. Other cases are similarly proved.

(6-8) Are easily proved from Theorem 3.4.

(9) For each r-feo set ρ, we will show that $eC_\tau(\rho, r) = eeT_\tau(\rho, r)$. Then there exist $x \in X$ and $t \in I_0$ such that

$$eC_\tau(\rho, r)(x) < t < eeT_\tau(\rho, r)(x).$$

Thus, x_t is not an r-e cluster point of ρ. So, there exists $\lambda \in \mathcal{E}_\tau(x_t, r)$ such that $\lambda \leq \overline{1} - \rho$. It implies $eC_\tau(\lambda, r) \leq \overline{1} - \rho$. Thus, x_t is not an r-$e\theta$-e-cluster point of ρ. Hence $eC_\tau(\rho, r) = eeT_\tau(\rho, r)$. Let $\rho \leq I_\tau(C_\tau(\rho, r), r)$ be given. Since $C_\tau(\rho, r)$ is r-fec, by (3), $eT_\tau(\rho, r) \leq C_\tau(\rho, r)$. Moreover, since

$$\begin{aligned} C_\tau(\rho, r) &\leq C_\tau(I_\tau(C_\tau(\rho, r), r) \\ &\leq C_\tau(\rho, r), \end{aligned}$$

then $C_\tau(\rho, r)$ is r-frc. Since $\rho \leq I_\tau(C_\tau(\rho, r), r)$ and $C_\tau(\rho, r)$ is r-frc, by (3), $RT_\tau(\rho, r) = C_\tau(\rho, r)$. From (8), we have

$$\begin{aligned} C_\tau(\rho, r) &= RC_\tau(\rho, r) \\ &= T_\tau(\rho, r) \\ &= RT_\tau(\rho, r). \end{aligned}$$

(10) There exist $\rho \in I^X$ with $\tau(\rho) \geq r$ such that

$$eC_\tau(\rho, r) \ngeq eT_\tau(\rho, r).$$

Then there exists $x \in X$ and $t \in I$ such that

$$eC_\tau(\rho, r)(x) < t < eT_\tau(\rho, r)(x).$$

Thus, x_t is not an r-e cluster point of ρ. So, there exists $\lambda \in \mathcal{E}_\tau(x_t, r)$ such that $\lambda \leq \overline{1} - \rho$. It implies $C_\tau(\lambda, r) \leq \overline{1} - \rho$. Thus, x_t is not an r-θ-e cluster point of ρ. Hence

$$\begin{aligned} eC_\tau(\rho, r) &= eeT_\tau(\rho, r) \\ &= eT_\tau(\rho, r). \end{aligned}$$

By (7-9), we have

$$\begin{aligned} eC_\tau(\rho, r) &= C_\tau(\rho, r) \\ &= RC_\tau(\rho, r) \\ &= T_\tau(\rho, r) \\ &= RT_\tau(\rho, r). \end{aligned}$$

□

Example 3.6. Let $X = \{a, b, c\}$, $\alpha, \beta, \gamma, \delta \in I^X$ are defined as

$$\begin{array}{llll} \alpha(a) = 0.3, & \beta(a) = 0.6, & \gamma(a) = 0.6, & \delta(a) = 0.3, \\ \alpha(b) = 0.4, & \beta(b) = 0.5, & \gamma(b) = 0.5, & \delta(b) = 0.4, \\ \alpha(c) = 0.5, & \beta(c) = 0.5, & \gamma(c) = 0.4, & \delta(c) = 0.4. \end{array}$$

We define the smooth topology $\tau : I^X \to I$ as follows:

$$\tau(\lambda) = \begin{cases} 1 & \text{if } \lambda \in \{\overline{0}, \overline{1}\}, \\ \frac{1}{2} & \text{if } \lambda = \alpha, \\ \frac{1}{2} & \text{if } \lambda = \beta, \\ \frac{1}{2} & \text{if } \lambda = \gamma, \\ \frac{1}{2} & \text{if } \lambda = \delta, \\ 0 & \text{otherwise.} \end{cases}$$

For $r = \frac{1}{2}$, then the fuzzy sets $\alpha, \beta, \gamma, \delta$ are r-feo sets, $\overline{1}-\alpha, \overline{1}-\beta, \overline{1}-\gamma, \overline{1}-\delta$ are r-fec sets. Let $\lambda(a) = 0.4, \lambda(b) = 0.5, \lambda(c) = 0.5$, $eC_\tau(\lambda, r) = \lambda$, and clearly, $eeT_\tau(\lambda, r) \leq eT_\tau(\lambda, r) = T_\tau(\lambda, r) = RT_\tau(\lambda, r)$.

Definition 3.7. Let (X, τ) be a fts. Then (X, τ) is called:

(1) r-fuzzy e-regular if for each r-feo μ there exists a family $\{\nu_i \in I^X | \tau(\nu_i) \geq r\}$ such that $\mu = \bigvee_{i\in\Gamma} \nu_i$ with $C_\tau(\nu_i, r) \leq \mu$.

(2) r-fuzzy e^*-regular (resp. r-fuzzy ee^*-regular, r-fuzzy almost e^*-regular) if for each $\tau(\mu) \geq r$ (resp. r-feo μ, r-fro μ), there exists a family $\{\nu_i \in I^X | \nu_i \text{ is } r\text{-feo} \}$ such that $\mu = \bigvee_{i\in\Gamma} \nu_i$ with $eC_\tau(\nu_i, r) \leq \mu$.

(3) fuzzy (e, almost) (e^*)-regular if (X, τ) is r-fuzzy (e, almost)(e^*-) regular, for each $r \in I_0$.

We easily prove the following Lemma.

Lemma 3.8. *For $\lambda, \lambda_i, \mu \in I^X$ and $x_t \in Pt(X)$, we have*

(1) *$\lambda \leq \mu$ iff $x_t q \lambda$ implies $x_t q \mu$.*

(2) *$x_t q \bigvee_{i\in\Lambda} \lambda_i$ iff there exists $i \in \Lambda$ such that $x_t q \lambda_i$.*

Theorem 3.9. *Let (X, τ) be a fts and $r \in I_0$. Then the following statements are equivalent:*

(1) *(X, τ) is r-fuzzy almost e^*-regular.*

(2) *For all* $\mu \in R_\tau(x_t, r)$, *there exists* $\nu \in \mathcal{E}_\tau(x_t, r)$ *with* $eC_\tau(\nu, r) \leq \mu$.
(3) *For each* $x_t \in Pt(X)$ *and each* r*-frc* $\lambda \in I^X$ *with* $x_t \notin \lambda$, *there exist* $\nu \in \mathcal{E}_\tau(x_t, r)$ *and* r*-feo* $\mu \in I^X$ *such that* $\lambda \leq \mu$ *and* $\mu\bar{q}\nu$.
(4) *For each* r*-frc* $\lambda \in I^X$, $\lambda = \bigwedge\{eC_\tau(\nu, r) | \lambda \leq \nu,\ \nu$ *is* r*-feo* $\}$.
(5) *For each* r*-frc* $\lambda \in I^X$ *with* $\rho \nleq \lambda$, *there exist* $\nu \in \mathcal{E}_\tau(x_t, r)$ *and* r*-feo* μ *such that* $\lambda \leq \mu$, $\rho q \nu$ *and* $\mu\bar{q}\nu$.

Proof. (1) $\Rightarrow$ (2): Let $\mu \in \mathcal{R}_\tau(x_t, r)$ be given. Since (X, τ) is r-fuzzy almost e^*-regular, there exists a family $\{\nu_i | \nu_i$ is r-feo $\}$ such that $\mu = \bigvee_{i\in\Gamma} \nu_i$ with $eC_\tau(\nu_i, r) \leq \mu$. Since $x_t q\ (\mu = \bigvee_{i\in\Gamma} \nu_i)$, by Lemma 3.8 (2), there exists $i \in \Gamma$ such that $\nu_i \in \mathcal{E}_\tau(x_t, r)$ with $eC_\tau(\nu_t, r) \leq \mu$.

(2) $\Rightarrow$ (1): For each $\mu \in \mathcal{R}_\tau(x_t, r)$, there exists $\nu_i \in \mathcal{E}_\tau(x_t, r)$ such that $eC_\tau(\nu_i, r) \leq \mu$. Let $\{\nu_i \in \mathcal{E}_\tau(x_t, r) | i \in \Lambda, eC_\tau(\nu_i, r) \leq \mu\}$ be the family satisfying the above condition. Trivially, $\bigvee_{i\in\Lambda} \nu_i \leq \mu$. We only show that, by Lemma 3.8 (1), $x_t q \bigvee_{i\in\Lambda} \nu_i$ for each $x_t q\mu$. For each $\mu \in \mathcal{R}_\tau(x_t, r)$, by (2), there exists $\nu_i \in \mathcal{E}_\tau(x_t, r)$ such that $eC_\tau(\nu_i, r) \leq \mu$. So, $x_t q \nu_i$ implies $x_t q \bigvee_{i\in\Lambda} \nu_i$. Then $\mu = \bigvee_{i\in\Lambda} \nu_i$ such that $eC_\tau(\nu_i, r) \leq \mu$.

(2) $\Rightarrow$ (3): Let $x_t \notin \lambda$ with r-frc λ. Then $\bar{1} - \lambda \in \mathcal{R}_\tau(x_t, r)$. By (2), there exists $\nu \in \mathcal{E}_\tau(x_t, r)$ such that $eC_\tau(\nu, r) \leq \bar{1} - \lambda$. Put $\mu = \bar{1} - eC_\tau(\nu, r)$. By Theorem 2.12 (1), μ is r-feo such that $\lambda \leq \mu$ and $\mu\bar{q}\nu$.

(3) $\Rightarrow$ (4): Suppose there exists r-frc $\lambda \in I^X$ such that

$$\lambda \nleq \bigwedge\{eC_\tau(\nu, r) | \lambda \leq \nu, \nu \text{ is } r\text{-feo}\}.$$

Then there exist $x \in X$ and $t \in I_0$ such that

$$\lambda(x) < t < \bigwedge\{eC_\tau(\nu, r)(x) | \lambda \leq \nu, \nu \text{ is } r\text{-feo}\}. \tag{3.1}$$

Since $x_t \notin \lambda$, by (4), there exist $\mu \in \mathcal{E}_\tau(x_t, r)$ and r-feo ν such that $\lambda \leq \nu$ and $\mu\bar{q}\nu$. Since ν is r-feo, $\lambda \leq eI_\tau(\nu, r)$ and $eI_\tau(\nu, r)$ is r-feo. Hence

$$\lambda(x) < t < eC_\tau(eI_\tau(\nu, r), r)(x).$$

By the definition of eC_τ, we have

$$\begin{aligned} eC_\tau(eI_\tau(\nu, r), r)(x) &\leq eC_\tau(\nu, r)(x) \\ &\leq \bar{1} - \mu(x) < t. \end{aligned}$$

It is contradiction for (3.1). Thus

$$\lambda = \bigwedge\{eC_\tau(\nu, r) | \lambda \leq \nu, \nu \text{ is } r\text{-feo}\}.$$

(4) $\Rightarrow$ (5): Let $\lambda \in I^X$ be r-frc with $\rho \nleq \lambda$. Then $x_t \in Pt(X)$ such that $x_t \in \rho$ and $t > \lambda(x)$. By (4), there exist r-feo μ such that $\lambda \leq \mu$

and $eC_\tau(\mu,r)(x) < t$. Put $\nu = \overline{1} - eC_\tau(\mu,r)$. By Theorem 2.12 (1), ν is r-feo, that is, $\nu \in \mathcal{E}_\tau(x_t,r)$ such that $\lambda \le \mu$, $\rho q \nu$ and $\mu \bar{q} \nu$.

(5) $\Rightarrow$ (2): For all $\mu \in \mathcal{R}_\tau(x_t,r)$, $t > \overline{1} - \mu(x)$. So, $x_t \not\leq \overline{1} - \mu$ and $\overline{1} - \mu$ is r-frc, by (5), there exist $\nu \in \mathcal{E}_\tau(x_t,r)$ and r-feo ρ such that $\overline{1} - \mu \le \rho$ and $\rho \bar{q} \nu$. Thus, $\nu \le \overline{1} - \rho \le \mu$. Since $\overline{1} - \rho$ is r-fec and μ is r-fro, $eC_\tau(\overline{1} - \rho, r) \le \mu$. It implies $\nu \in \mathcal{E}_\tau(x_t,r)$ such that $eC_\tau(\nu,r) \le \mu$.

□

Corollary 3.10. *Let (X,τ) be a fts and $r \in I_0$. Then the following statements are equivalent:*

(i) *(X,τ) is r-fuzzy ee^*-regular (resp, r-fuzzy e^*-regular).*

(ii) *For all $\mu \in \mathcal{E}_\tau(x_t,r)$ (resp. $\mu \in Q_\tau(x_t,r)$), there exists $\nu \in \mathcal{E}_\tau(x_t,r)$ with $eC_\tau(\nu,r) \le \mu$.*

(iii) *For each $x_t \in Pt(X)$ and each r-fec $\lambda \in I^X$ (resp. $\tau(\overline{1}-\lambda) \ge r$) with $x_t \notin \lambda$, there exist $\nu \in \mathcal{E}_\tau(x_t,r)$ and r-feo $\mu \in I^X$ such that $\lambda \le \mu$ and $\mu \bar{q} \nu$.*

(iv) *For each r-fec $\lambda \in I^X$ (resp. $\tau(\overline{1} - \lambda) \ge r$),*

$$\lambda = \bigwedge\{eC_\tau(\nu,r) | \lambda \le \nu, \nu \text{ is } r-\text{feo}\}.$$

(v) *For each r-fec $\lambda \in I^X$ (resp. $\tau(\overline{1} - \lambda) \ge r$) with $\rho \not\leq \lambda$, there exist $\nu \in \mathcal{E}_\tau(x_t,r)$ and r-feo μ such that $\lambda \le \mu$, $\rho q \nu$ and $\mu \bar{q} \nu$.*

Corollary 3.11. *Let (X,τ) be a fts and $r \in I_0$. Then the following statements are equivalent:*

(i) *(X,τ) is r-fuzzy e-regular (resp, r-fuzzy regular, r-fuzzy almost regular).*

(ii) *For all $\mu \in \mathcal{E}_\tau(x_t,r)$ (resp. $\mu \in \mathcal{Q}_\tau(x_t,r), \mu \in \mathcal{R}_\tau(x_t,r)$), there exists $\nu \in \mathcal{Q}_\tau(x_t,r)$ with $C_\tau(\nu,r) \le \mu$.*

(iii) *For each $x_t \in Pt(X)$ and each r-fec $\lambda \in I^X$ (resp. $\tau(\overline{1} - \lambda) \ge r, r$-frc) with $x_t \notin \lambda$, there exist $\nu \in \mathcal{Q}_\tau(x_t,r)$ and $\tau(\mu) \ge r$ such that $\lambda \le \mu$ and $\mu \bar{q} \nu$.*

(iv) *For each r-fec $\lambda \in I^X$ (resp. $\tau(\overline{1} - \lambda) \ge r$, r-frc),*

$$\lambda = \bigwedge\{C_\tau(\nu,r) | \lambda \le \nu, \tau(\nu) \ge r\}.$$

(v) *For each r-fec $\lambda \in I^X$ (resp. $\tau(\overline{1} - \lambda) \ge r$, r-frc) with $\rho \not\leq \lambda$, there exist $\nu \in \mathcal{Q}_\tau(x_t,r)$ and $\tau(\mu) \ge r$ such that $\lambda \le \mu$, $\rho q \nu$ and $\mu \bar{q} \nu$.*

Lemma 3.12. *Let (X,τ) be a fts.*

(i) *For each $x_t q \lambda$, there exists $\mu \in \mathcal{Q}_\tau(x_t,r)$ such that*

$$C_\tau(\mu,r) \le \lambda \quad \text{iff} \quad \overline{1} - \lambda = T_\tau(\overline{1} - \lambda, r).$$

(ii) *For each* $x_t q\lambda$, *there exists* $\mu \in \mathcal{E}_\tau(x_t, r)$ *such that*

$$eC_\tau(\mu, r) \leq \lambda \quad \text{iff} \quad \overline{1} - \lambda = eeT_\tau(\overline{1} - \lambda, r).$$

Proof. (i) It is similarly proved as the following (ii).

(ii) ($\Rightarrow$) We only show that $\overline{1} - \lambda \geq eeT_\tau(\overline{1} - \lambda, r)$. Let $x_t \not\leq \overline{1} - \lambda$. Then $x_t q\lambda$. By hypothesis, there exists $\mu \in \mathcal{E}_\tau(x_t, r)$ such that $eC_\tau(\mu, r) \leq \lambda$. Thus, $x_t \notin eeT_\tau(\overline{1} - \lambda, r)$.
($\Leftarrow$) For each $x_t q\lambda$, since $\overline{1} - \lambda = eeT_\tau(\overline{1} - \lambda, r)$, x_t is not r-$e\theta$-e cluster point of $\overline{1} - \lambda$. There exists $\mu \in \mathcal{E}_\tau(x_t, r)$ such that

$$eC_\tau(\mu, r) \leq \overline{1} - \lambda.$$

□

Theorem 3.13. *Let* (X, τ) *be a fts and* $r \in I_0$. *The following statements are equivalent:*

(1) (X, τ) *is* r*-fuzzy ee***-regular (resp.* r*-fuzzy e***-regular,* r*-fuzzy almost e***-regular).*
(2) *For each* r*-feo* μ *(resp.* $\tau(\mu) \geq r$, r*-fro* μ*),* $\overline{1} - \mu = eeT_\tau(\overline{1} - \mu, r)$.
(3) *For each* $\lambda \in I^X$, $eC_\tau(\lambda, r) = eeT_\tau(\lambda, r)$ *(resp.* $C_\tau(\lambda, r) = eeT_\tau(\lambda, r)$, $RC_\tau(\lambda, r) = eeT_\tau(\lambda, r)$*).*

Proof. (1) $\Leftrightarrow$ (2) It is easy from Lemma 3.12 (2).

(2) $\Rightarrow$ (3) Suppose there exists $\lambda \in I^X$ with $eC_\tau(\lambda, r) \not\geq eeT_\tau(\lambda, r)$. Then there exist $x \in X$ and $t \in I_0$ such that

$$eC_\tau(\lambda, r)(x) < t < eeT_\tau(\lambda, r)(x).$$

By the definition of eC_τ, there exists r-fec set $\rho \in I^X$ with $\lambda \leq \rho$ such that

$$eC_\tau(\lambda, r)(x) \leq \rho(x) < t < eeT_\tau(\lambda, r)(x).$$

By (2), since $eeT_\tau(\rho, r) = \rho$, we have

$$eeT_\tau(\lambda, r)(x) \leq eeT_\tau(\rho, r)(x) = \rho(x) < t.$$

It is a contradiction.

(3) $\Rightarrow$ (2) It is easy.

□

Corollary 3.14. *Let* (X, τ) *be a fts and* $r \in I_0$. *The following statements are equivalent:*

(i) (X, τ) *is* r*-fuzzy regular (resp.* r*-fuzzy e-regular,* r*-fuzzy almost regular)*
(ii) *For each* $\tau(\mu) \geq r$ *(resp.* r*-feo* μ, r*-fro* μ*),* $\overline{1} - \mu = T(\overline{1} - \mu, r)$.
(iii) *For each* $\lambda \in I^X$, $C_\tau(\lambda, r) = T_\tau(\lambda, r)$ *(resp.* $eC_\tau(\lambda, r) = T_\tau(\lambda, r)$, $RC_\tau(\lambda, r) = T_\tau(\lambda, r)$*).*

Remark 3.15. Let (X, τ) be a fts. We have:

$$\begin{array}{ccccc} r\text{-fuzzy } e\text{-regular} & \Rightarrow & r\text{-fuzzy regular} & \Rightarrow & r\text{-fuzzy almost regular} \\ \Downarrow & & \Downarrow & & \Downarrow \\ r\text{-fuzzy } ee^*\text{-regular} & \Rightarrow & r\text{-fuzzy } e^*\text{-regular} & \Rightarrow & r\text{-fuzzy almost } e^*\text{-regular.} \end{array}$$

Example 3.16. Let $X = \{a, b, c\}$ be a set and $a_{0.6} \in Pt(X)$. We define the fuzzy topology $\tau : I^X \to I$ as follows:

$$\tau(\lambda) = \begin{cases} 1 & \text{if } \lambda \in \{\overline{0}, \overline{1}\}, \\ \frac{1}{2} & \text{if } \lambda \in \{\chi_{\{a\}}, \chi_{\{b,c\}}\}, \\ \frac{1}{2} & \text{if } \lambda \in \{a_{0.6}, a_{o.6} \vee \chi_{\{b,c\}}\}, \\ 0 & \text{otherwise.} \end{cases}$$

(1) For $0 < r \leq \frac{1}{2}$, since $\chi_{\{a\}}$ and $\chi_{\{b,c\}}$ are r-fro and r-frc sets, $C_\tau(\chi_{\{a\}}, r) = \chi_{\{a\}}$ and $C_\tau(\chi_{\{b,c\}}, r) = \chi_{\{a,c\}}$, then (X, τ) is r-fuzzy almost regular.

(2) For $a_{0.6} \in \mathcal{Q}_\tau(a_{0.7}, 1/2)$, for all $\mu \in \mathcal{Q}_\tau(a_{o.7}, 1/2)$ we have $C_\tau(\mu, 1/2) \not\leq a_{0.6}$. So, (X, τ) is not a 1/2-fuzzy regular. Moreover, for $a_{0.9} \in \mathcal{E}_\tau(a_{0.2}, 1/2)$ and for all $\mu \in \mathcal{E}_\tau(a_{0.2}, 1/2)$, we have $eC_\tau(\mu, 1/2) \not\leq a_{0.9}$. So, (X, τ) is not a 1/2-fuzzy ee^*-regular.

(3) For $0 < r \leq 1/2$, we have the following (a) and (b).
   (a) If $a_{0.4} < a_s < a_{0.6}$, then a_s is r-feo and r-fec. For $a_{0.6} \in \mathcal{Q}_\tau(a_t, r)$, there exists $a_s \in \mathcal{E}_\tau(a_t, r)$ with $eC_\tau(a_s \leq a_{0.6}$.
   (b) Let $a_{0.6} \vee \chi_{\{b,c\}} \in \mathcal{Q}_\tau(x_t, r)$. If $(x = a)_t$, by (a), there exist $a_s \in \mathcal{E}(a_t, r)$ such that $eC_\tau(a_s, r) = a_s \leq a_{0.6} \vee \chi_{\{b,c\}}$. If $(x = b)_t$ or $(x = c)_t$, there exists x_s with $s + t > 1$ and $a_s \in \mathcal{E}_\tau(x_t, r)$ such that $eC_\tau(x_t, r) = x_s \leq a_{0.6} \vee \chi_{\{b,c\}}$.

Hence, (X, τ) is r-fuzzy e^*-regular.

Example 3.17. Let X be a set containing at least three points. We define the fuzzy topology $\tau : I^X \to I$ as follows:

$$\tau(\lambda) = \begin{cases} 1 & \text{if } \lambda \in \{\overline{0}, \overline{1}\}, \\ \frac{1}{2} & \text{if } \lambda = a_{0.6}, \\ 0 & \text{otherwise.} \end{cases}$$

For $0 < r \leq 1/2$, if $\lambda \not\leq \overline{0.4}$, λ is r-feo and if $\mu \not\geq \overline{0.6}$, μ is r-fec. Let $\lambda \in \mathcal{E}_\tau(a_t, r)$. Since $\lambda \not\leq \overline{0.4}$, there exists $y \in X$ such that $\lambda(y) > 0.4$, $\lambda(a) + t > 1$. Put $\mu \in I^X$ as

$$\mu(x) = \begin{cases} \lambda(x) & \text{if } x \in \{a, y\}, \\ \min\{0.5, \lambda(x)\} & \text{otherwise.} \end{cases}$$

So, μ is r-fec and $\mu \in \mathcal{E}_\tau(a_t, r)$ such that $eC_\tau(\mu, r) = \mu \leq \lambda$. Hence (X, τ) is r-fuzzy ee^*-regular. But it is neither r-fuzzy e-regular nor r-fuzzy regular because $\overline{0.6} \in \mathcal{Q}_\tau(a_t, r)$ and for all $\lambda \in \mathcal{Q}_\tau(a_t, r)$, $C_\tau(\lambda, r) \nleq \overline{0.6}$.

4. Strongly e-irresolute mappings

Definition 4.1. Let (X, τ) and (Y, η) be fts's, a function $f : (X, \tau) \to (Y, \eta)$ is called:

(i) strongly θ-e-continuous (resp. strongly e-irresolute) iff for each $\mu \in \mathcal{Q}_\eta(f(x)_t, r)$ (resp. $\mu \in \mathcal{E}_\eta(f(x)_t, r)$), there exists $\lambda \in \mathcal{E}_\tau(x_t, r)$ such that $f(eC_\tau(\lambda, r)) \leq \mu$,

(ii) θ-e-irresolute (resp. quasi e-irresolute) iff for each $\mu \in \mathcal{E}_\eta(f(x)_t, r)$, there exists $\lambda \in \mathcal{E}_\tau(x_t, r)$ such that $f(eC_\tau(\lambda, r)) \leq eC_\eta(\mu, r)$ (resp. $f(\lambda) \leq eC_\eta(\mu, r)$).

Theorem 4.2. *Let (X, τ) and (Y, η) be fts's and $f : X \to Y$ a function. Then the following statements are equivalent:*

(1) *f is e-irresolute.*
(2) *For each $\mu \in \mathcal{E}_\eta(f(x)_t, r)$, there exists $\lambda \in \mathcal{E}_\tau(x_t, r)$ such that $f(\lambda) \leq \mu$.*
(3) *$f(eC_\tau(\lambda, r)) \leq eC_\eta(f(\lambda, r))$ for each $\lambda \in I^X$.*
(4) *$eC_\tau(f^{-1}(\mu), r) \leq f^{-1}(eC_\eta(\mu, r))$ for each $\mu \in I^Y$.*

Proof. (1) $\Rightarrow$ (2) For $\mu \in \mathcal{E}_\eta(f(x)_t, r)$, by (1), there exists $f^{-1}(\mu) \in \mathcal{E}_\tau(x_t, r)$ such that $f(f^{-1}(\mu)) \leq \mu$.

(2) $\Rightarrow$ (1) For each r-feo μ, we only show that

$$f^{-1}(\mu) = \bigvee\{\lambda | \lambda \leq f^{-1}(\mu), \lambda \text{ is } r\text{-feo}\}.$$

Suppose there exist $x \in X$ and $t \in I_0$ such that

$$\begin{aligned} f^{-1}(\mu)(x) &= \mu(f(x)) \\ &> 1 - t \\ &> \bigvee\{\lambda(x) | \lambda \leq f^{-1}(\mu), \lambda \text{ is } r\text{-feo}\}. \end{aligned}$$

For each $\mu \in \mathcal{E}_\eta(f(x)_t, r)$, by (2), there exists $\lambda \in \mathcal{E}_\tau(x_t, r)$ such that $f(\lambda) \leq \mu$. Thus $\lambda \leq f^{-1}(\mu)$ and $\lambda q x_t$ implies $1 - t < \lambda(x)$. It is a contradiction. Hence $f^{-1}(\mu)$ is r-feo.

(1) ⇒ (3)

$$\begin{aligned} eC_\eta(f(\lambda),r) &= \wedge\{\mu | f(\lambda)\le \mu, \mu \text{is} r\text{-fec}\} \\ &\ge \bigwedge\{\mu | f(\lambda)\le \mu, f^{-1}(\mu) isr-fec\} \\ &\ge \bigwedge \left\{f(f^{-1}(\mu)) | \lambda \le f^{-1}(\mu), f^{-1}(\mu)\text{is}r\text{-fec}\right\} \\ &\ge f\left(\bigwedge\{(f^{-1}(\mu)) | \lambda \le f^{-1}(\mu), f^{-1}(\mu)\text{is}r\text{-fec}\}\right) \\ &\ge f(eC_\tau(\lambda,r)). \end{aligned}$$

(3) ⇒ (4) Put $\lambda = f^{-1}(\mu)$. Then

$$\begin{aligned} eC_\tau(f^{-1}(\mu),r) &\le f^{-1}(f(eC_\tau(f^{-1}(\mu),r))) \\ &\le f^{-1}(eC_\eta(\mu,r)). \end{aligned}$$

(4) ⇒ (1) For each r-feo $\mu \in I^Y$, we have $eC_\eta(\overline{1}-\mu,r) = \overline{1}-\mu$. By (4),

$$\begin{aligned} eC_\tau(\overline{1}-f^{-1}(\mu),r) &\le f^{-1}(eC_\eta(\overline{1}-\mu,r)) \\ &= \overline{1}-f^{-1}(\mu). \end{aligned}$$

So, $eC_\tau(\overline{1}-f^{-1}(\mu),r) = \overline{1}-f^{-1}(\mu)$. By Theorem 2.12 (2), $f^{-1}(\mu)$ is r-feo.

□

Corollary 4.3. *Let (X,τ) and (Y,η) be fts's and $f: X\to Y$ a function. Then the following statements are equivalent:*

(1) *f is e-continuous (resp. supercontinuous).*
(2) *$f(eC_\tau(\lambda,r)) \le C_\eta(f(\lambda,r))$ (resp. $f(RC_\tau(\lambda,r)) \le C_\eta(f(\lambda),r)$), for each $\lambda \in I^X$.*
(3) *$eC_\tau(f^{-1}(\mu),r) \le f^{-1}(C_\eta(\mu,r))$ (resp. $RC_\tau(f^{-1}(\mu),r) \le f^{-1}$ $(C_\eta(\mu,r))$), for each $\mu\in I^Y$.*

Theorem 4.4. *The following implications hold:*

$$\begin{aligned} stronglye-irresolute &\Rightarrow strongly\theta-e-continuous, \\ stronglye-irresolute &\Rightarrow e-irresolute, \\ \theta-e-irresolute &\Rightarrow quasie-irresolute. \end{aligned}$$

Proof. We show that e-irresolute ⇒ θ-e-irresolute. Let (X,τ) and (Y,η) be fts's and $f: X\to Y$ a function. For each $\mu\in\mathcal{E}_\eta(f(x)_t,r)$, by e-irresolutity and Theorem 4.2 (4), $f^{-1}(\mu)\in\mathcal{E}_\tau(x_t,r)$ such that

$$eC_\tau(f^{-1}(\mu),r) \le f^{-1}(eC_\eta(\mu,r)).$$

It implies

$$f(eC_\tau(f^{-1}(\mu),r)) \le eC_\eta(\mu,r).$$

□

Example 4.5. Let $X = \{a\}$ be a set. We define the fuzzy topologies $\tau, \eta : I^X \to I$ as follows:

$$\tau(\lambda) = \begin{cases} 1 & \text{if } \lambda \in \{\overline{0}, \overline{1}\}, \\ \frac{1}{2} & \text{if } \lambda = \overline{0.6}, \\ 0 & \text{otherwise.} \end{cases} \qquad \eta(\lambda) = \begin{cases} 1 & \text{if } \lambda \in \{\overline{0}, \overline{1}\}, \\ \frac{1}{2} & \text{if } \lambda = \overline{0.6}, \\ 0 & \text{otherwise.} \end{cases}$$

For $r = 1/2$, $\lambda = \overline{0.6}$ is r-feo in (Y, η) and λ is r-feo in (X, τ). Hence the identity function $id_X : (X, \tau) \to (X, \eta)$ is fuzzy e-irresolute and strongly θ-e-continuous because for $\overline{0.6} \in \mathcal{Q}_\eta(a_t, r)$, there exists $\overline{0.6} \in \mathcal{E}_\tau(a_t, r)$ such that $eC_\tau(\overline{0.6}, r) = \overline{0.6} \leq \overline{0.6}$.

But the identity function $id_X : (X, \tau) \to (X, \eta)$ is not strongly e-irresolute because for $\overline{0.75} \in \mathcal{E}_\eta(a_{0.3}, r)$, and for all $a_s \in \mathcal{E}_\tau(a_{0.3}, r)$ we have $eC_\tau(a_s, r) = \overline{1} \not\leq \overline{0.75}$. Moreover, id_X is quasi e-irresolute but not θ-e-irresolute.

Theorem 4.6. *Let (X, τ) and (Y, η) be fts's and $f : X \to Y$ a function. If f is e-irresolute, then $f^{-1}(\mu) = eeT_\tau(f^{-1}(\mu), r)$ for each $\mu = eeT_\eta(\mu, r)$.*

Proof. Let $\mu = eeT_\eta(\mu, r)$. For each $x_t q \overline{1} - f^{-1}(\mu)$, we have $f(x)_t q(\overline{1} - \mu)$. By Lemma 3.12 (2), there exists $\rho \in e_\eta(f(x)_t, r)$ such that $eC_\eta(\rho, r) \leq \overline{1} - \mu$. Since f is e-irresolute, by Theorem 4.2 (4), there exists $f^{-1}(\rho) \in \mathcal{E}_\tau(x_t, r)$ such that

$$\begin{aligned} eC_\tau(f^{-1}(\rho), r) &\leq f^{-1}(eC_\tau(\rho, r)) \\ &\leq \overline{1} - f^{-1}(\mu). \end{aligned}$$

By Lemma 3.12 (2), $f^{-1}(\mu) = eeT_\tau(f^{-1}(\mu), r)$. □

Theorem 4.7. *Let (X, τ) and (Y, η) be fts's and $f : X \to Y$ a function. Then the following statements are equivalent:*

1. *f is θ-e-irresolute.*
2. *$f(eeT_\tau(\lambda, r)) \leq eeT_\eta f(\lambda, r)$ for each $\lambda \in I^X$.*
3. *$eeT_\tau(f^{-1}(\mu), r) \leq f^{-1}(eeT_\eta(\mu, r))$ for each $\mu \in I^Y$.*
4. *$eeT_\tau(f^{-1}(\mu), r) \leq f^{-1}(eC_\eta(\mu, r))$ for each r-feo $\mu \in I^Y$.*

Proof. (1) ⇒ (2) Suppose there exist $\lambda \in I^Y$ and $r \in I_0$ such that

$$f(eeT_\tau(\lambda, r)) \not\leq eeT_\eta(f(\lambda), r).$$

Then there exist $x \in X$ and $t \in I_0$ such that

$$\begin{aligned} f(eeT_\tau(\lambda, r))(f(x)) &\geq eeT_\tau(\lambda, r)(x) \\ &> t \\ &> eeT_\eta(f(\lambda), r)(f(x)). \end{aligned}$$

Let $f(x)_t \notin eeT_\eta(f(\lambda, r)$. Then there exists $\rho \in \mathcal{E}_\eta(f(x)_t, r)$ such that $eC_\eta(\rho, r) \leq \overline{1} - f(\lambda)$. Since f is θ-e-irresolute, for $\rho \in \mathcal{E}_\eta(f(x)_t, r)$, there exists $\mu \in \mathcal{E}_\tau(x_t, r)$ such that

$$\begin{aligned} f(eC_\tau(\mu, r)) &\leq eC_\eta(\rho, r) \\ &\leq \overline{1} - f(\lambda). \end{aligned}$$

It implies

$$\begin{aligned} eC_\tau(\mu, r) &\leq f^{-1}(f(eC_\tau(\mu, r))) \\ &\leq f^{-1}(eC_\eta(\rho, r)) \\ &\leq \overline{1} - f^{-1}(f(\lambda)) \\ &\leq \overline{1} - \lambda. \end{aligned}$$

Hence x_t is not an r-$e\theta$-e cluster point of λ. It is a contradiction. Thus (2) holds.

(2) ⇒ (3) Put $\lambda = f^{-1}(\mu)$. It is easy.

(3) ⇒ (4) Since $eeT_\eta(\mu, r) = eC_\eta(\mu, r)$ for each r-feo $\mu \in I^Y$ from Theorem 3.5 (9), it is trivial.

(4) ⇒ (1) Let $\mu \in e_\eta(f(x)_t, r)$. Then $eC_\eta(\mu, r)\overline{q}(\overline{1} - eC_\eta(\mu, r))$. Hence $f(x)_t$ is not an r-$e\theta$-e cluster point of $\overline{1} - eC_\eta(\mu, r)$. By Theorem, $f(x)_t$ is not an r-e-cluster point of $\overline{1} - eC_\eta(\mu, r)$. Thus,

$$\begin{aligned} t &> eC_\eta(\overline{1} - eC_\eta(\mu, r), r)(f(x)) \\ &= f^{-1}(eC_\eta(\overline{1} - eC_\eta(\mu, r), r))(x). \end{aligned}$$

Since $\overline{1} - eC_\eta(\mu, r)$ is r-feo, by (4),

$$f^{-1}(eC_\eta(\overline{1} - eC_\eta(\mu, r), r)) \geq eeT_\tau(f^{-1}(\overline{1} - eC_\eta(\mu, r)), r).$$

It implies

$$t > eeT_\tau\left(f^{-1}(\overline{1} - eC_\eta(\mu, r)), r\right)(x).$$

Hence x_t is not an r-$e\theta$-e-cluster point of $f^{-1}(\overline{1} - eC_\eta(\mu, r))$. There exists $\rho \in \mathcal{E}_\tau(x_t, r)$ such that

$$\begin{aligned} eC_\tau(\rho, r) &\leq \overline{1} - f^{-1}(\overline{1} - eC_\eta(\mu, r)) \\ &= f^{-1}(eC_\tau(\mu.r)). \end{aligned}$$

Thus, $f(eC_\tau(\rho, r)) \leq eC_\eta(\mu, r)$.

□

Theorem 4.8. *Let (X, τ) and (Y, η) be fts's and $f : X \to Y$ a function. Then the following statements are equivalent:*

(1) *f is strongly θ-e-continuous.*
(2) *$\overline{1} - f^{-1}(\mu) = eeT_\tau(\overline{1} - f^{-1}(\mu), r)$ for each $\eta(\mu) \geq r$.*
(3) *$f^{-1}(\mu) = eeT_\tau(f^{-1}(\mu), r)$ for each $\eta(\overline{1} - \mu) \geq r$.*

(4) $f(eeT_\tau(\lambda, r)) \leq C_\eta(f(\lambda), r)$ *for each* $\lambda \in I^X$.
(5) $eeT_\eta(f^{-1}(\mu), r) \leq f^{-1}(C_\tau(\mu, r))$ *for each* $\mu \in I^Y$.

Proof. (1) $\Rightarrow$ (2) Suppose there exists $\mu \in I^Y$ with $\eta(\mu) \geq r$ such that

$$\overline{1} - f^{-1}(\mu) \neq eeT_\tau(\overline{1} - f^{-1}(\mu), r).$$

Then there exist $x \in X$ and $t \in I_0$ such that

$$(\overline{1} - f^{-1}(\mu))(x) < t < eeT_\tau(\overline{1} - f^{-1}(\mu), r)(x). \tag{4.1}$$

Since $x_t q f^{-1}(\mu)$ implies $f(x)_t q \mu$; we have $\mu \in \mathcal{Q}_\eta(f(x)_t, r)$. Since f is strongly θ-e-continuous, for $\mu \in \mathcal{Q}_\eta(f(x)_t, r)$, there exists $\lambda \in \mathcal{E}_\tau(x_t, r)$ such that $f(eC_\tau(\lambda, r)) \leq \mu$. It implies $eC_\tau(\lambda, r) \leq f^{-1}(\mu)$. Thus, $eC_\tau(\lambda, r)\overline{q}(\overline{1} - f^{-1}(\mu))$. Hence x_t is not an r-$e\theta$-e cluster point of $\overline{1} - f^{-1}(\mu)$. Hence $eeT_\tau(\overline{1} - f^{-1}(\mu), r)(x) < t$. It is a contradiction. Hence (2) holds.

(2) and (3) are equivalent.

(3) $\Rightarrow$ (4) Suppose there exist $\lambda \in I^X$ and $t \in I_0$ such that

$$f(eeT_\tau(\lambda, r)) \not\leq C_\eta(f(\lambda), r).$$

Then there exist $y \in Y$ and $t \in I_0$ such that

$$f(eeT_\tau(\lambda, r))(y) > t > C_\eta(f(\lambda), r)(y).$$

By the definition of $f(eeT_\tau(\lambda, r))$, there exists $x \in X$ with $f(x) = y$ such that

$$\begin{aligned} f(eeT_\tau(\lambda, r))(f(x)) &\geq eeT_\tau(\lambda, r)(x) \\ &> t \\ &> C_\eta(f(\lambda), r)(f(x)). \end{aligned}$$

By the definition of $C_\eta(f(\lambda), r)$, there exists $\mu \in I^Y$ with $f(\lambda) \leq \mu$, $\eta(\overline{1} - \mu) \geq r$ such that

$$\begin{aligned} f(eeT_\tau(\lambda, r))(f(x)) &\geq eeT_\tau(\lambda, r)(x) \\ &> t \\ &> \mu(f(x)). \end{aligned} \tag{4.2}$$

On the other hand, by (3), $f^{-1}(\mu) = eeT_\tau(f^{-1}(\mu), r)$ for each $\eta(\overline{1} - \mu) \geq r$. Then $\lambda \leq \mu$ implies

$$\begin{aligned} eeT_\tau(\lambda, r)(x) &\leq eeT_\tau(f^{-1}(\mu), r)(x) \\ &= \mu(f(x)) \\ &< t. \end{aligned}$$

It is a contradiction. Hence (4) holds.

(4) $\Rightarrow$ (5) Put $\lambda = f^{-1}(\mu)$. It is easy.

(5) $\Rightarrow$ (1) For each $\mu \in \mathcal{Q}_\eta(f(x)_t, r)$, $C_\eta(\overline{1} - \mu, r) = \overline{1} - \mu$. By (5),

$$eeT_\eta(\overline{1} - f^{-1}(\mu), r) = \overline{1} - f^{-1}(\mu).$$

Since $f(x)_t q\mu$ implies $x_t q f^{-1}(\mu)$, by Lemma 3.8 (2), there exists $\lambda \in \mathcal{E}_\tau(x_t, r)$ such that $eC_\tau(\lambda, r) \leq f^{-1}(\mu)$. It implies $f(eC_\tau(\lambda, r)) \leq \mu$. Hence f is strongly θ-e-continuous.

□

Theorem 4.9. *Let (X, τ) and (Y, η) be fts's and $f : X \to Y$ a function. Let (Y, η) be a fuzzy regular space. Then the following statements are equivalent:*

(1) *f is weakly e-continuous.*
(2) *f is e-continuous.*
(3) *f is strongly θ-e-continuous.*

Proof. (1) $\Rightarrow$ (2) For $\mu \in \mathcal{Q}_\eta(f(x)_t, r)$, since (Y, η) is a fuzzy regular space, there exists $\omega \in \mathcal{Q}_\eta(x_t, r)$ such that $\mu \leq C_\eta(\omega, r) \leq \mu$. Since f is weakly e-continuous, there exists $\lambda \in \mathcal{E}_\tau(x_t, r)$ such that $f(\lambda) \leq C_\eta(\omega, r) \leq \mu$.

(2) $\Rightarrow$ (3) For $\mu \in \mathcal{Q}_\eta(f(x)_t, r)$, since (Y, η) is a fuzzy regular space, there exists $\mu \in \mathcal{Q}_\eta(x_t, r)$ such that $\mu \leq C_\eta(\mu, r) \leq \nu$. Since f is e-continuous, there exists $\lambda \in \mathcal{E}_\tau(x_t, r)$ such that $f(\lambda) \leq \mu$. We will show that $f(eC_\tau(\lambda, r)) \leq C_\eta(\mu, r)$. Suppose

$$f(eC_\tau(\lambda, r))(y) > t > C_\eta(\mu, r)(y).$$

Then there exist $x \in X$ with $f(x) = y$ and $\rho \in I^Y$, $\mu \leq \rho$ with $\eta(\overline{1} - \rho) \geq r$ such that

$$\begin{aligned} f(eC_\tau(\lambda, r))(y) &\geq eC_\tau(\lambda, r)(x) \\ &> t \\ &> \rho(f(x)) \\ &\geq C_\eta(\mu, r)(y). \end{aligned}$$

On the other hand, since f is e-continuous, for $\eta(\overline{1} - \rho) \geq r$, there exists $\omega \in \mathcal{E}_\tau(x_t, r)$ such that $f(\omega) \leq \overline{1} - \rho$. Thus

$$\lambda \leq f^{-1}(\mu) \leq f^{-1}(\rho) \leq \overline{1} - \omega.$$

So, $eC_\tau(\lambda, r)(x) \leq (\overline{1} - \omega)(x) < t$. It is a contradiction. Thus, $f(eC_\tau(\lambda, r)) \leq C_\eta(\mu, r)$. Hence f is strongly θ-e-continuous.

(3) $\Rightarrow$ (1) It is trivial.

□

Theorem 4.10. *Let (X, τ) and (Y, η) be fts's.*

(1) *Every fuzzy continuous function $f : X \to Y$ is strongly θ-e-continuous iff (X, τ) is fuzzy e-regular.*

(2) *Every e-continuous function $f : X \to Y$ is strongly θ-e-continuous iff (X, τ) is fuzzy ee^*-regular.*
(3) *Every supercontinuous function $f : X \to Y$ is strongly θ-e-continuous iff (X, τ) is fuzzy almost e^*-regular.*

Proof. (1) ($\Rightarrow$) For an identity function $f : (X, \tau) \to (Y, \sigma)$, by hypothesis, f is fuzzy continuous and strongly θ-e-continuous. For $\mu \in \mathcal{Q}_\eta(f(x)_t, r)$, there exists $\lambda \in \mathcal{E}_\tau(x_t, r)$ such that

$$f\left(eC_\tau(\lambda, r)\right) \leq \mu.$$

Since $eC_\tau(\lambda, r) \leq C_\tau(\lambda, r)$ then

$$f\left(eC_\tau(\lambda, r)\right) \leq f\left(C_\tau(\lambda, r)\right) \leq \mu.$$

We have

$$f^{-1}(\mu) = f\left(\mathcal{Q}_\eta(f(x)_t, r)\right) = \mathcal{Q}_\tau\left(f^{-1}f(x)_t, r\right) = \mathcal{Q}_\tau(x_t, r).$$

Since f is fuzzy continuous. Then we have

$$\begin{aligned} f\left(eC_\tau(\lambda, r)\right) \leq \mu \quad &\Rightarrow \quad eC_\tau(\lambda, r) \leq f^{-1}(\mu) \\ &\Rightarrow \quad eC_\tau(\lambda, r) \leq C_\tau(\lambda, r) \leq f^{-1}(\mu). \end{aligned}$$

By Corollary 3.11 (2), (X, τ) is fuzzy e-regular.

($\Leftarrow$) Let f be fuzzy continuous. For each $\nu \in \mathcal{Q}_\eta(f(x)_t, r)$, $f^{-1}(\nu) \in \mathcal{Q}_\tau(x_t, r)$. Since (X, τ) is fuzzy e-regular, there exists $\mu \in \mathcal{E}_\tau(x_t, r)$ such that $\mu \leq eC_\tau(\mu, r) \leq f^{-1}(\nu)$. Thus, $f(eC_\tau(\mu, r)) \leq \nu$. Hence f is strongly θ-e-continuous.

(2) ($\Rightarrow$) Since every fuzzy continuous function is fuzzy e-continuous then the proof followed by the necessary part of (1).

($\Leftarrow$) By Remark 3.15, since every fuzzy e-regular space is fuzzy ee^*-regular. Then the proof followed by the sufficiency part of (1).

(3) Proof is similar from the above (1) and (2).

□

Theorem 4.11. *Let (X, τ) and (Y, η) be fts's and $f : X \to Y$ a function. Let (Y, η) be a ee^*-regular space. Then the following statements are equivalent:*

(1) *f is strongly e-irresolute.*
(2) *f is e-irresolute.*
(3) *f is θ-e-irresolute.*
(4) *f is quasi-e-irresolute.*

Proof. (1) $\Rightarrow$ (2), (2)$\Rightarrow$ (3) and (3)$\Rightarrow$(4) are trivial from Theorem 4.2

(3) $\Rightarrow$ (1) For each $\nu \in \mathcal{E}_\eta(f(x)_t, r)$, since (X, η) is fuzzy ee^*-regular, there exists $\mu \in e_\tau(f(x)_t, r)$ such that $\mu \leq eC_\eta(\mu_t, r) \leq \nu$. for

$\mu \in e_\tau(f(x)_t, r)$, by (3), there exists $\lambda \in \mathcal{E}_\tau(x_t, r)$ such that $f(eC_\tau(\lambda, r)) \leq eC_\eta(\mu, r) \leq \nu$. Hence f is strongly e-irresolute.

(4) $\Rightarrow$ (2) For each r-feo ν, we only show that

$$f^{-1}(\nu) = \vee\{\lambda | \lambda \leq f^{-1}(\nu), \lambda \text{ is } r\text{-feo}\}.$$

Then there exist $x \in X$ and $t \in I_0$ such that

$$\begin{aligned} f^{-1}(\nu)(x) &= \nu(f(x)) \\ &> 1 - t \\ &> \vee\{\lambda | \lambda \leq f^{-1}(\nu), \lambda \mathrm{is} r - \mathrm{feo}\}. \end{aligned}$$

For each $\nu \in \mathcal{E}_\eta(f(x)_t, r)$, since (Y, η) is fuzzy ee^*-regular, there exists $\mu \in \mathcal{E}_\eta(f(x)_t, r)$ such that $\mu \leq eC_\eta(\mu, r) \leq \nu$. By (4), there exists $\lambda \in \mathcal{E}_\tau(x_t, r)$ such that $f(\lambda) \leq eC_\eta(\mu, r) \leq \nu$. Thus $\lambda \leq f^{-1}(\nu)$ and $\lambda q x_t$ implies $1 - t < \lambda(x)$. It is a contradiction. Thus $f^{-1}(\nu)$ is r-feo.

□

Theorem 4.12. *Let (X, τ) and (Y, η) be fts's and $f : X \to Y$ a function. Let (X, τ) be a fuzzy ee^*-regular space. Then f is θ-e-irresolute iff f is quasi e-irresolute.*

Proof. Let f be quasi-e-irresolute. For each $\nu \in \mathcal{E}_\eta(f(x)_t, r)$, there exists $\lambda \in \mathcal{E}_\tau(x_t, r)$ such that $f(\lambda) \leq eC_\eta(\nu, r)$. Since (X, τ) is fuzzy e^*-regular, there exists $\mu \in \mathcal{E}_\tau(x_t, r)$ such that $\mu \leq eC_\tau(\mu, r) \leq \lambda$. Hence $f(eC_\tau(\mu, r)) \leq eC_\eta(\nu, r)$. Then f is θ-e-irresolute. □

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