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On Generators in Archimedean Copulas

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Abstract. This study after reviewing construction methods of generators in Archimedean copulas (AC), proposes several useful lemmas related with generators of AC. Then a new trigonometric Archimedean family will be shown which is based on cotangent function. The generated new family is able to model the low dependence structures.

1. Introduction

Sklar (1959) for the first time used the word copula, as a function which allows us to combine univariate distributions to obtain a joint distribution. Namely, copula is a function $C : [0,1]^2 \to [0,1]$ which satisfies

(a) for every u, v in [0, 1], $C(u,0) = 0 = C(0,v)$, and $C(u,1) = u$, $C(1,v) = v$;
(b) for every u_1, u_2, v_1, v_2 in [0, 1] such that $u_1 \leq u_2$, and, $v_1 \leq v_2$,

$$C(u_2,v_2) - C(u_2,v_1) - C(u_1,v_2) + C(u_1,v_1) \geq 0.$$

One of the most important classes of copulas is known as AC. Basic properties of AC are presented below. More information could be found in Nelsen [14].

Let φ be a convex continuous, strictly decreasing function from $[0,1]$ to $[0,\infty]$ such that $\varphi(1) = 0$. The pseudo-inverse of φ is function $\varphi^{[-1]}$ given by

$$\varphi^{[-1]}(t) = \begin{cases} \varphi^{(-1)}(t), & 0 \leq t \leq \varphi(0), \\ 0, & \varphi(0) \leq t \leq \infty. \end{cases} \tag{1.1}$$

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Functions of the form $C_\varphi(u,v) = \varphi^{[-1]}(\varphi(u)+\varphi(v))$, for every u, v in $[0,1]$, are called AC and the function φ is called a generator (additive generator) of the copula. If $\varphi[0] = \infty$, we say that φ is a strict generator. In this case, $\varphi^{[-1]} = \varphi^{(-1)}$ and $C_\varphi(u,v) = \varphi^{(-1)}(\varphi(u)+\varphi(v))$ is said a strict Archimedean copula.

This study reviews constructions of AC generators in the literature, then proposes a new trigonometric Archimedean family. Moreover, there are several useful lemmas to extend AC.

The rest of this paper constructed as follows. Section 2, reviews several construction methods of AC. Section 3 concerns with extra comments on generators. Section 4 proposes a trigonometric Archimedean family with related dependence measures and finally Section 5, summarizes the conclusion of our work.

2. A Review on Several Construction Methods of AC

AC find a wide range of applications mainly because of:

(1) the ease with which they can be constructed,
(2) the great variety of families of copulas which belong to this class,
(3) and the many nice properties possessed by the members of this class.

Hence, there are more efforts on construction of this class of copulas in the literature. In this section we review some of them. Let Φ be the set of all additive generators of binary AC.

Klement et al. [8] have mentioned a construction method for AC as below:
Let $f : [0,1] \to [0,1]$ be a concave increasing bijection. Then for any additive generator $\varphi \in \Phi$, also $(\varphi \circ f)(x) = \varphi(f(x))$, is an additive generator from Φ. Generalization of this construction (f need not to be a bijection) can be found in Durante et al. [4, 5]. Pekárová [15] proposed a new generator in the case, $f : [0,1] \to [0,1]$ is an absolutely monotone bijection (i.e., all derivatives of f on (0,1) exist and they are non-negative), then for any $\varphi \in \Phi$ also

$$f \circ \varphi \in \Phi. \tag{2.1}$$

Bacigál et al. [1], presented three construction methods for AC as below:

1. Let $f : [0,\infty] \to [0,\infty]$ be a concave increasing bijection. Then for any $\varphi \in \Phi$, also

$$f \circ \varphi \in \Phi. \tag{2.2}$$

2. Let $\varphi_1, \ldots, \varphi_k \in \Phi$ be additive generators, and $c_1, \ldots, c_k \in [0,1]$, such that $\sum_{i=1}^{k} c_i = 1$. Then

$$\varphi = \sum_{i=1}^{k} c_i \varphi_i \in \Phi. \tag{2.3}$$

Note that (2.3) also holds for any $\varphi_i \in \Phi$ and $c_i \in (0, \infty)$, $i = 1, \ldots, k$.

3. Let $\varphi_1, \ldots, \varphi_k \in \Phi$ be additive generators, and $c_1, \ldots, c_k \in [0,1]$, such that $\sum_{i=1}^{k} c_i = 1$. Then $\varphi \in \Phi$ where

$$\varphi^{(-1)} = \sum_{i=1}^{k} c_i \varphi_i^{(-1)},$$

(i.e. pseudo-inverse of φ is a convex combination of pseudo-inverse of $\varphi_1, \ldots, \varphi_k$).

Michiels and De Schepper in [10], review existing transforms and several properties of AC generators as below,

(i) (left composition (LC)) If $\varphi \in \Phi$ and $f : [0, +\infty) \to [0, +\infty)$ be an increasing concave function with $f(0) = 0$, the function defined by

$$\varphi^T(t) = (f \circ \varphi)(t), \tag{2.4}$$

is a well defined Archimedean generator function. Furthermore, φ^T is strict iff, φ is strict. We recall that f needs to be increasing concave bijection (then it is same as the result (2.2) by Bacigál et al. [1]). Note that we have mentioned right composition in the study by Klement et al. [8].

(ii) If $\theta \in (0,1)$ and $\varphi \in \Phi$, the function defined by

$$\varphi^T(t) = \varphi(\theta t) - \varphi(\theta), \tag{2.5}$$

is a well defined Archimedean copulas generator function. Furthermore, φ^T is strict iff, φ is strict. Under the same supposing, Mesiar et al. in [9] show that, conditional copula $C_{(\theta)}$ $(= C_{[\theta]})$ of the Archimedean copula C with generator φ, is again Archimedean copula, and also it has a generator of the form (2.5), see also related study by Jágr et al. [6].

(iii) If φ_1, φ_2 be two generators with $(\varphi_2')^2 \leq \varphi_2''$, then function defined by

$$\varphi^T(t) = \varphi_1\left(e^{-\varphi_2(\theta)}\right),$$

is a well defined Archimedean generator function. Furthermore, φ^T is strict iff, φ_1 and φ_2 are strict.

(iv) Let α and β be two positive constants and φ_1, φ_2 are two generators, then function defined by

$$\varphi^T(t) = \alpha\varphi_1(t) + \beta\varphi_2(t), \tag{2.6}$$

is a well defined Archimedean generator function. Furthermore, φ^T is strict iff, φ_1 and φ_2 are strict. For a thorough discussion and analysis of the tail dependence behavior and limiting behavior of the four types of generator transforms, as well as various illustrations, we refer to Michiels and De Schepper [11]. Michiels et al. [12] proposed a new and advantageous method for constructing bivariate Archimedean copula families based on the function

$$\lambda(t) = \frac{\varphi(t)}{\varphi'(t)}, \quad t \in [0,1],$$

where φ is the generator of the Archimedean copula. See also Najjari and Rahimi [13].

Junker and May [7] has studied related with AC generators and they provided several examples of transformations that some of them have previously appeared in the literature. As the results which we have discussed above, totally conclude also results of Junker and May's study, hence we express two examples for it: let $\varphi \in \Phi$, then for any $a \in (1, \infty)$

$$\varphi^T(t) = a^{\varphi(t)} - 1, \tag{2.7}$$

and for any $a \in (0, 1)$

$$\varphi^T(t) = a^{-\varphi(t)} - 1, \tag{2.8}$$

are AC generators. It has been shown that, if $\varphi \in \Phi$, then for any $\lambda \in [1, \infty)$ also $\varphi^\lambda \in \Phi$, for details see [16].

Recently there are new comments on generators of copulas and aggregation functions by Bacigál et al. [2]. Moreover, Bacigál et al. [3] have summarized three general methods in generating generators of AC. These methods include almost all available generators. In Section 4, a new trigonometric Archimedean family will be shown which is based on one of the proposed methods in [3].

3. Extra Comments on Generators

Below we prepare the first lemma that is related with proposing new generators, by multiplication of other generators in AC. As proof is simple it is eliminated.

Lemma 3.1. *Let $\varphi_1, \ldots, \varphi_k \in \Phi$ be additive generators with parameters $\theta_i \in D_i$, $i = 1, \ldots, k$ respectively. Then $\prod_{i=1}^k \varphi_i \in \Phi$.*

Note that recently Bacigál et al. [2] investigated this case with aggregation functions. Let's see several examples,

Example 3.2. Let $\varphi_1(t) = -log(t)$ and $\varphi_2(t) = e^{t^{-\theta_2}} - e$, where $\theta_2 \in (0, \infty)$. Then $\varphi(t) = -log(t)\left(e^{t^{-\theta_2}} - e\right)$, for any $\theta_2 \in (0, \infty)$ is a generator for AC.

Example 3.3. Let $\varphi_1(t) = (1-t)^{\theta_1}$, is generator on $\theta_1 \in [1, \infty)$, $\varphi_2(t) = cot^{\theta_2}(\pi t/2)$, where $\theta_2 \in [1, \infty)$ and $\varphi_3(t) = e^{t^{-\theta_3}} - e$, where $\theta_3 \in (0, \infty)$. Then

$$\varphi(t) = (1-t)^{\theta_1} cot^{\theta_2}(\pi t/2)\left(e^{t^{-\theta_3}} - e\right),$$

for any $\theta_1, \theta_2 \in [1, \infty)$, and $\theta_3 \in (0, \infty)$ is a generator for AC.

Remark 3.4. Evidently it is difficult (and some times impossible) to find a closed form for AC by complicated generators. Therefore, usually it is not useful to apply Lemma 3.1 for multiplication of more than two generators. The Lemma 3.1 also has an interesting point of view. As the composed new generator is effected by other two (or more) generators, so we are able to construct an Archimedean copula with arbitrary behaviors in tails and dependency. We exemplify this fact. For instance, four Archimedean families with deferent structures in tails are selected to investigate their react in the Lemma 3.1. These Archimedean families are shown in Table 1.

TABLE 1. Details of the selected copulas in the Remark 3.4

Family	Generator	λ_L	λ_U	θ interval
Clayton	$\frac{1}{\theta}\left(\frac{1}{t^\theta} - 1\right)$	$2^{\frac{-1}{\theta}}$	0	$(0, \infty)$
4.2.2	$(1-t)^\theta$	0	$2 - 2^{\frac{1}{\theta}}$	$[1, \infty)$
4.2.16	$\left(\frac{\theta}{t} + 1\right)(1-t)$	$\frac{1}{2}$	0	$[0, \infty)$
4.2.18	$e^{\frac{\theta}{t-1}}$	0	1	$[2, \infty)$

Note: 4.2.2, 4.2.16, 4.2.18 are AC families that were numbered in Table 4.1, Nelsen's book [14].

Obviously Clayton and 4.2.2 families have converse behaviors in the tails. So, we are interested to know tails behaviors of the copula that is yielded with using these families generators via Lemma 3.1 as follows,

$$\varphi(t) = \frac{1}{\theta_1}\left(t^{-\theta_1} - 1\right)(1-t)^{\theta_2}, \tag{3.1}$$

where $\theta_1 \in (0, \infty)$ and $\theta_2 \in [1, \infty)$. Also, the same story is about families 4.2.2 and 4.2.16, therefore with using generators of them, we have a new

generator for AC as below,

$$\varphi(t) = \left(\frac{\theta_1}{t} + 1\right)(1-t)^{\theta_2+1}, \tag{3.2}$$

where $\theta_1 \in [0, \infty)$ and $\theta_2 \in [1, \infty)$. Since 4.2.2 and 4.2.18 families have similar behaviors in the tails, we are interested to know behaviors of the resulted copula by generators of them. With using generators of the 4.2.18 and 4.2.2 families, we have a new generator for AC as below,

$$\varphi(t) = e^{\frac{\theta_1}{t-1}}(1-t)^{\theta_2}, \tag{3.3}$$

where $\theta_1 \in [2, \infty)$ and $\theta_2 \in [1, \infty)$.

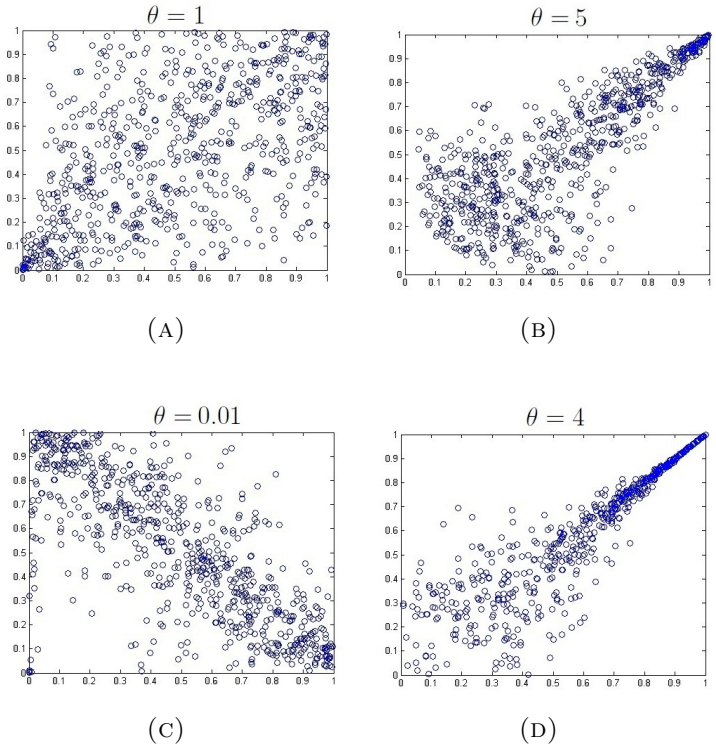


FIGURE 1. Scatterplots of 1a Clayton family $\theta = 1$, 1b 4.2.2 family for $\theta = 5$, 1c 4.2.16 family, $\theta = 0.01$ and 1d 4.2.18 family $\theta = 4$, $n = 700$.

Figure 1 displays the scaterplots of the discussed families in Remark 3.4 for several parameter values, and Figure 2 displays the scaterplots of the generated new families by using Lemma 3.1 and the yielded generators in (3.1)-(3.3).

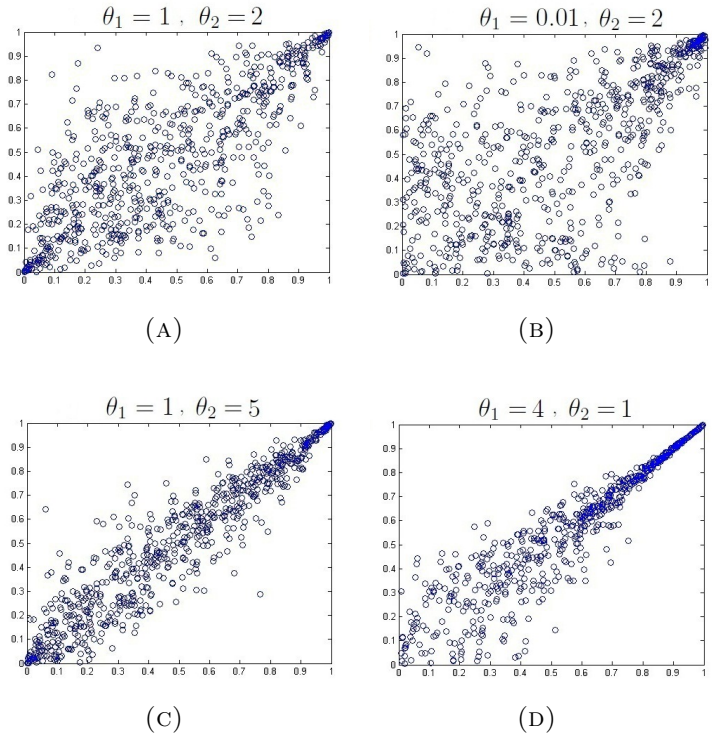


FIGURE 2. Scatterplots of the generated new family, 2a by generators of 4.2.16 with $\theta = 1$ and 4.2.2 $\theta = 2$, 2b by generators of 4.2.16 with $\theta = 0.01$ and 4.2.2, $\theta = 2$, 2c Clayton $\theta = 1$ with generator of 4.2.2, $\theta = 5$ and 2d generators of 4.2.18 with $\theta = 4$ and 4.2.2 $\theta = 1$, $n = 700$.

Based on investigation of several examples and simulations, we conclude our result in Lemma 3.5. Of course this result is based on simulation, and proofing accuracy/inaccuracy of it, is left as open problem to readers. For simplicity we see the case $k = 2$.

Lemma 3.5. *Let Archimedean copula C is generated by using Lemma 3.1 and the generators of Archimedean copulas C_i, where λ_{L_i}, λ_{U_i}, $i = 1, 2$ are their lower and upper tail dependence respectively. Then for the generated new family, we have, $\lambda_L = \max\{\lambda_{L_1}, \lambda_{L_2}\}$, and $\lambda_U = \max\{\lambda_{U_1}, \lambda_{U_2}\}$.*

Lemma 3.1 also holds in the case that at least one of the $\varphi_i, i = 1, \ldots, k$ is AC generator and others are arbitrary convex continuous, strictly decreasing function on $[0, 1]$. We restate this fact in the following proposition:

Proposition 3.6. *Let $\varphi_1 \in \Phi$ be an additive generator with parameter $\theta_1 \in D_1$, and for any $i = 2, \ldots, k$, $g_i(t)$ is an arbitrary convex continuous, strictly decreasing function on $[0,1]$, with parameter $\theta_i \in D_i$, where for all $t \in [0,1]$, $g_i(t) \geq 0$. Then $\varphi(t) = \varphi_1(t) \prod_{i=2}^{k} g_i(t) \in \Phi$, where for any $i = 1, \ldots, k$, $\theta_i \in D_i$.*

Proof. As proof is straightforward, it is omitted. □

Example 3.7. Let $\varphi_1(t) = (1-t)^{\theta_1}$, clearly $\varphi_1 \in \Phi$ for any $\theta_1 \in [1, \infty)$ (this family has mentioned 4.2.2 in Nelsen [14]), and $g(t) = \frac{\theta_2}{t}$, for any $\theta_2 \in (0, \infty)$ is a convex and decreasing function on $[0, \infty)$. Therefore $\varphi(t) = \frac{\theta_2}{t}(1-t)^{\theta_1}$ for any $\theta_1 \in [1, \infty)$, $\theta_2 \in (0, \infty)$, generates an Archimedean family. Evidently for $\theta_1 = 1$ this is Clayton family. Figure 3 displays its scaterplots for several values of its parameters.

Example 3.8. Let $\varphi_1(t) = \frac{1}{\theta_1}\left(t^{-\theta_1} - 1\right)$, for any $\theta_1 \in (0, \infty)$ (Clayton family), and $g(t) = \frac{\theta_2}{t}$, for any $\theta_2 \in (0, \infty)$. It is a convex and decreasing function on $[0, \infty)$. Therefore $\varphi(t) = \frac{\theta_2}{\theta_1 t}\left(t^{-\theta_1} - 1\right)$ for any $\theta_1, \theta_2 \in (0, \infty)$, generates an Archimedean family. Figure 3 displays the scaterplots for several values of its parameters.

Example 3.9. Let $\varphi_1(t) = 1 - t$, clearly $\varphi_1 \in \Phi$, and $g(t) = \left(\frac{\theta_2}{t} + 1\right)$, where for any $\theta_2 \in [0, \infty)$, g(t) is a convex and decreasing function on $[0,1]$. Therefore $\varphi(t) = (1-t)\left(\frac{\theta}{t} + 1\right)$ for any $\theta \in [0, \infty)$, generates an Archimedean family. This family had been cited in Nelsen [14] as 4.2.16 family.

Example 3.10. Let $\varphi_1(t) = 1 - t$, clearly $\varphi_1 \in \Phi$, and $g(t) = 1/(1+\theta t)$, where for any $\theta \in [0, \infty)$, g(t) is a convex and decreasing function on $[0,1]$. Therefore $\varphi(t) = (1-t)/(1+\theta t)$ for any $\theta \in [0, \infty)$, generates an Archimedean family. This family had been cited in Nelsen [14] as 4.2.8 family.

Let's investigate the cases which for any $\varphi_1 \in \Phi$, $\varphi(t) = \varphi_1(t) + \alpha t + \beta$ is a generator in AC. Evidently generator properties tends us to $\alpha < 0$ and $\alpha = -\beta$. So we have $\varphi(t) = \varphi_1(t) + \beta(1-t)$, where $\beta > 0$. As $\varphi_2(t) = (1-t)$ is an AC generator, therefore, $\varphi(t) = \varphi_1(t) + \beta\varphi_2(t)$, where $\beta > 0$. This result is a case of the result by Michiels and De Schepper in (2.6).

4. New Family of Archimedean Copulas

Any generator φ of an Archimedean copula is convex and strictly decreasing function on $[0,1]$, which means for any $t \in (0,1)$, it satisfies

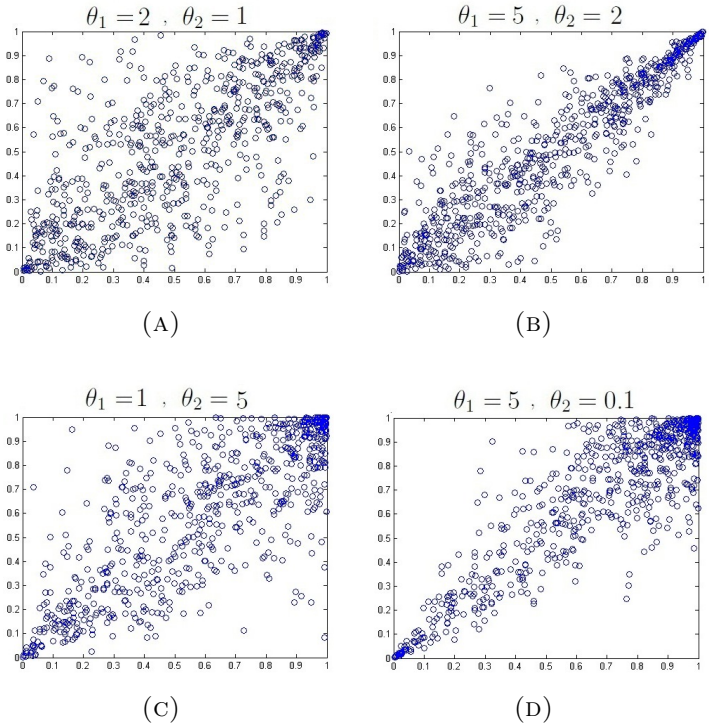


FIGURE 3. Scatterplots of the extention of family 4.2.2 in Example 3.7 (a-b), and Clayton family in Example 3.8, (c-d).

the following properties:

$$\varphi(1) = 0, \qquad \varphi^{'}(t) < 0, \qquad \varphi^{''}(t) > 0. \tag{4.1}$$

These are sufficient conditions for any generator of bivariate AC. Let

$$\varphi(t) = \cot(\theta t) - \cot(\theta) \tag{4.2}$$

be our newly proposed generator[1]. It can be easily check that $\varphi(t)$ has two continuous derivatives on $(0, 1)$ and satisfies (4.1) for all $\theta \in (0, \frac{\pi}{2}]$. Since $\lim_{t\to 0^+} \varphi(t) = \infty$, φ generates a family of strict Archimedean copulas and its inverse exists in the form $\varphi^{-1}(t) = \operatorname{arccot}(t + \cot(\theta))/\theta$,

[1]Generalization of this proposed generator is discussed in Bacigál et al. [3] as follows:
Let $h : [a, b] \to]-\infty, \infty]$ be an arbitrary convex continuous, strictly decreasing function on D where $[0, 1] \subseteq D = [a, b]$. Then, $\varphi(t) = g(\theta t) - g(\theta)$, for any $\theta \in D - \{0\}$, is a generator in AC.

so that

$$\begin{aligned} C_\theta(u,v) &= \varphi^{-1}\left(\varphi(u)+\varphi(v)\right) \\ &= \frac{1}{\theta}\text{arccot}\left(\cot(\theta u)+\cot(\theta v)-\cot(\theta)\right). \end{aligned}$$

Kendall's tau for this new family is calculated as follows:

$$\begin{aligned} \tau_C &= 4\iint\limits_{[0,1]^2} C_\theta(u,v)dC_\theta(u,v)-1 \\ &= 1+4\int_0^1 \frac{\varphi(t)}{\varphi'(t)}dt \\ &= 1-\frac{2}{\theta^2}+\frac{2cot(\theta)}{\theta}. \end{aligned}$$

Evidently, $\tau_C \in [1-\frac{8}{\pi^2}, \frac{1}{3})$. So the generated new family is able to model the low dependence structures. For details see Figure 4 and Figure 5.

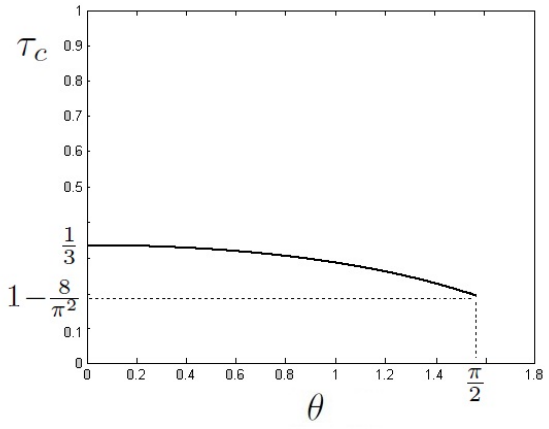


FIGURE 4. The graph of τ_C, for $\theta \in (0, \frac{\pi}{2}]$

For tails behaviors, by investigation of two commonly-used tail dependence coefficients defined by

$$\begin{aligned} \lambda_L &= \lim_{t\to 0^+} P\left\{F(X)\le t | G(X)\le t\right\} = \lim_{t\to 0^+}\frac{C(t,t)}{t}, \\ \lambda_U &= \lim_{t\to 1^-} P\left\{F(X)> t | G(X)> t\right\} = \lim_{t\to 1^-}\frac{1-2t+C(t,t)}{1-t}, \end{aligned}$$

we get

$$\lambda_L = \frac{1}{2}, \qquad \lambda_U = 0,$$

for the whole copula parameter range. The scaterplots of simulated pairs in Figure 5 demonstrate the dependence structure captured by the new copula for several parameter values.

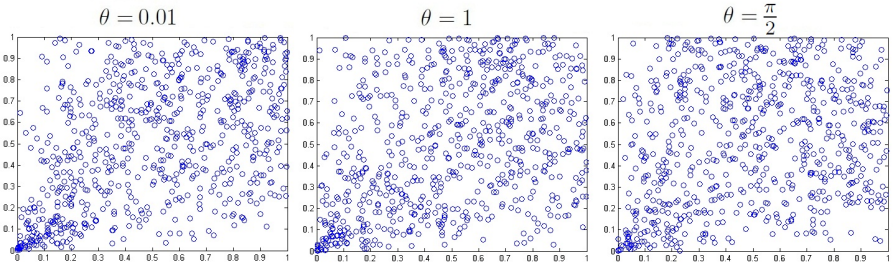


FIGURE 5. Scatterplots of the new family for $\theta = 0.01$ (left), $\theta = \frac{\pi}{2}$ (right), n=500.

5. Conclusion

In this study, after reviewing construction methods of AC generators, we propose a new trigonometric Archimedean family which its details are summarized in the following:

$$C_\theta(u,v) = \frac{1}{\theta}\text{arccot}\left(\cot(\theta u) + \cot(\theta v) - \cot(\theta)\right)$$
$$\varphi(t) = \cot(\theta t) - \cot(\theta)$$

where $\theta \in (0, \frac{\pi}{2}]$, $\lambda_L = \frac{1}{2}, \lambda_U = 0$ and $\tau_C \in [1 - \frac{8}{\pi^2}, \frac{1}{3})$. So, the generated new family is able to model low dependence structures. Moreover, several lemmas are given to extend AC.

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