Sahand Communications in Mathematical Analysis (SCMA) Vol. 10 No. 1 (2018), 85-96
http://scma.maragheh.ac.ir
DOI: 10.22130/scma.2017.28505

Some fixed point theorems for C-class functions in b-metric spaces

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Abstract. In this paper, via C-class functions, as a new class of functions, a fixed theorem in complete b-metric spaces is presented. Moreover, we study some results, which are direct consequences of the main results. In addition, as an application, the existence of a solution of an integral equation is given.

1. Introduction

In this exciting context, Bakhtin [5] and Czerwik [8, 9] developed the notion of b-metric spaces in connection with some problems concerning the convergence of measurable functions with respect to a measure. Moreover they proved some fixed point theorems for single-valued and multi-valued mappings in b-metric spaces. In addition, many authors studied the fixed point theory in this space such as [1–4, 6, 12]. Here, we study some fixed point theorems for a C-class functions. In order to do this, we recall some concepts as follows:

Definition 1.1 ([8]). Let X be a (nonempty) set and $s \geq 1$ be a given real number. A function $d : X \times X \to \mathbb{R}^+$ is a b-metric if for all $x, y, z \in X$ the following conditions are satisfied:

(b_1) $d(x, y) = 0$ iff $x = y$,
(b_2) $d(x, y) = d(y, x)$,
(b_3) $d(x, z) \leq s[d(x, y) + d(y, z)]$.

The pair (X, d) is called a b-metric space.

2010 *Mathematics Subject Classification.* 46N40, 47H10, 54H25, 46T99.
Key words and phrases. Fixed point, Complete metric space, b-metric space, C-class function.
Received: 25 February 2017, Accepted: 14 May 2017.
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Definition 1.2 ([6]). Let (X,d) be a $b-$metric space. Then a sequence $\{x_n\}$ in X is called: (a) $b-$convergent if and only if there exists $x \in X$ such that $d(x_n,x) \to 0$ as $n \to \infty$. In this case, we write $\lim\limits_{n\to\infty} x_n = x$. (b) $b-$Cauchy if and only if $d(x_n,x_m) \to 0$ as $n,m \to \infty$.

The b-metric space (X,d) is complete if every b-Cauchy sequence b-converges in X.

Definition 1.3 ([2]). A mapping $F:[0,\infty)^2 \to \mathbb{R}$ is called a C-class function if it is continuous and satisfies the following axioms:

I) $F(s,t) \le s$.
II) $F(s,t) = s$ implies that either $s=0$ or $t=0$; for all $s,t \in [0,\infty)$.

Note for some F, $F(0,0)=0$. Denote the set of C-class functions by $\mathcal{C}$.

Example 1.4 ([2]). The following functions $F:[0,\infty)^2 \to \mathbb{R}$ are elements of $\mathcal{C}$, for all $s,t \in [\infty)$:

1) $F(s,t) = s-t,\ F(s,t)=s \Rightarrow t=0$.
2) $F(s,t) = ms,\ 0<m<1,\ F(s,t)=s \Rightarrow s=0$.
3) $F(s,t) = (s+l)^{(1/(1+t)^r)} - l,\ l>1,\ r\in(0,\infty),\ F(s,t)=s \Rightarrow t=0$.
4) $F(s,t) = s\beta(s),\ \beta:[0,\infty)\to(0,1)$ and is continuous, $F(s,t) = s \Rightarrow s=0$.
5) $F(s,t) = s-\varphi(s), F(s,t)=s \Rightarrow s=0$, here $\varphi:[0,\infty)\to[0,\infty)$ is a continuous function such that $\varphi(t)=0 \Leftrightarrow t=0$.
6) $F(s,t) = sh(s,t),\ F(s,t)=s \Rightarrow s=0$, where $h:[0,\infty)\times[0,\infty)\to[0,\infty)$ is a continuous function such that $h(t,s)<1$ for all $t,s>0$.

Definition 1.5 ([13]). A function $\psi:[0,\infty)\to[0,\infty)$ is called an altering distance function if the following properties are satisfied:

1) ψ is non-decreasing and continuous.
2) $\psi(t)=0$ if and only if $t=0$.

Definition 1.6 ([2]). An ultra altering distance function is a continuous, nondecreasing mapping $\varphi:[0,\infty)\to[0,\infty)$ such that $\varphi(t)>0,\ t>0$ and $\varphi(0)\ge 0$.

2. Main results

Theorem 2.1. *Let (X,d) be a complete b-metric space on X and $f: X\to X$ be a self mapping. Suppose*

$$\psi\left(sd\left(fx,fy\right)\right) \le F\left(\psi\left(M\left(x,y\right)\right),\varphi\left(M\left(x,y\right)\right)\right) + LN\left(x,y\right), \tag{2.1}$$

for all $x,y\in X$, where $L\ge 0$, $F:[0,\infty)^2\to\mathbb{R}$ is C-class function, $\psi:[0,\infty)\to[0,\infty)$ is an altering distance function, $\varphi:[0,\infty)\to[0,\infty)$

is an ultra altering distance function and

$$M(x,y)=\max\left\{d\left(x,y\right),\frac{d\left(x,fx\right)d\left(y,fy\right)}{1+d\left(fx,fy\right)}\right\},$$

and

$$N(x,y)=\min\left\{d\left(x,fx\right),d\left(x,fy\right),d\left(y,fx\right),d\left(y,fy\right)\right\}.$$

Then f has a unique fixed point.

Proof. Let $x_0\in X$. Define a sequence $\{x_n\}\subset X$ by $x_n=f^n(x_0)=fx_{n-1}$ for $n\in\mathbb{N}\cup\{0\}$. In order to show that $\{x_n\}$ is a Cauchy sequence, first we show $\lim\limits_{n\to\infty}d(x_n,x_{n+1})=0$. From (2.1) we have,

$$\begin{aligned}\psi\left(d(x_n,x_{n+1})\right)\leq&\psi\left(sd(x_n,x_{n+1})\right)\\=&\psi\left(sd(fx_{n-1},fx_n)\right)\\\leq&F\left(\psi\left(M(x_{n-1},x_n)\right),\varphi\left(M(x_{n-1},x_n)\right)\right)+LN(x_{n-1},x_n),\end{aligned}$$

where

$$\begin{aligned}M(x_{n-1},x_n)&=\max\left\{d(x_{n-1},x_n),\frac{d(x_{n-1},fx_{n-1})d(x_n,fx_n)}{1+d(fx_{n-1},fx_n)}\right\}\\&=\max\left\{d(x_{n-1},x_n),\frac{d(x_{n-1},x_n)d(x_n,x_{n+1})}{1+d(x_n,x_{n+1})}\right\}\\&=d(x_{n-1},x_n),\end{aligned}$$

and

$$\begin{aligned}N(x_{n-1},x_n)&=\min\left\{d(x_{n-1},fx_n),d(x_n,fx_n),d(x_{n-1},fx_{n-1}),d(x_n,fx_{n-1})\right\}\\&=\min\left\{d(x_{n-1},x_{n+1}),d(x_n,x_{n+1}),d(x_{n-1},x_n),d(x_n,x_n)\right\}\\&=0.\end{aligned}$$

Therefore

$$\psi\left(d(x_n,x_{n+1})\right)\leq F\left(\psi\left(d(x_{n-1},x_n)\right),\varphi\left(d(x_{n-1},x_n)\right)\right).\tag{2.2}$$

Thus

$$\psi\left(d(x_n,x_{n+1})\right)\leq F\left(\psi\left(d(x_n,x_{n-1})\right),\varphi\left(d(x_n,x_{n-1})\right)\right)\leq\psi\left(d(x_n,x_{n-1})\right).\tag{2.3}$$

Since ψ is non-decreasing, then $d(x_n,x_{n+1})\leq d(x_{n-1},x_n)$. This means $\{d(x_n,x_{n+1})\}$ is a decreasing sequence. Thus it converges and there exists $r\geq 0$ such that $\lim_{n\to\infty}d(x_n,x_{n+1})=r$. Let $n\to\infty$, then (2.3) implies

$$\psi(r)\leq F\left(\psi\left(r\right),\lim_{n\to\infty}\inf\varphi\left(d(x_{n-1},x_n)\right)\right)\leq F\left(\psi\left(r\right),\varphi\left(r\right)\right)\leq\psi\left(r\right).$$

Thus $\psi(r)=0$. Therefore $r=0$, that is

$$\lim_{n\to\infty}d(x_{n-1},x_n)=0.\tag{2.4}$$

Now, we prove that the sequence $\{x_n\}$ is a Cauchy sequence. Suppose that $\{x_n\}$ is not a Cauchy sequence. Then there exists an $\varepsilon > 0$ for which we can find two sequences of positive integers $\{m(k)\}$ and $\{n(k)\}$ such that for all positive integers k, $n(k) > m(k) > k$ and $d\left(x_{m(k)}, x_{n(k)}\right) \geq \varepsilon$. Let $n(k)$ be the smallest such positive integer $n(k) > m(k) > k$ such that

$$d\left(x_{m(k)}, x_{n(k)}\right) \geq \varepsilon, \qquad d\left(x_{m(k)}, x_{n(k)-1}\right) < \varepsilon. \tag{2.5}$$

By (2.4)

$$\frac{\varepsilon}{s} \leq \lim_{k\to\infty} \inf d\left(x_{m(k)}, x_{n(k)-1}\right), \tag{2.6}$$

and by (2.1)

(2.7)
$$\psi\left(sd\left(x_{m(k)+1}, x_{n(k)}\right)\right) \leq F\left(\psi\left(M(x_{m(k)}, x_{n(k)-1})\right), \varphi\left(M(x_{m(k)}, x_{n(k)-1})\right)\right) + LN(x_{m(k)}, x_{n(k)-1}),$$

where

$$\begin{aligned} &M\left(x_{m(k)}, x_{n(k)-1}\right) \\ &\quad = \max\left\{d\left(x_{m(k)}, x_{n(k)-1}\right), \frac{d\left(x_{n(k)-1}, f(x_{n(k)-1})\right) d\left(x_{m(k)}, f(x_{m(k)})\right)}{1 + d\left(fx_{n(k)-1}, fx_{m(k)}\right)}\right\} \\ &\quad = \max\left\{d\left(x_{m(k)}, x_{n(k)-1}\right), \frac{d\left(x_{n(k)-1}, x_{n(k)}\right) d\left(x_{m(k)}, x_{m(k)+1}\right)}{1 + d\left(x_{n(k)}, x_{m(k)+1}\right)}\right\}. \end{aligned}$$

Let $k \to \infty$ in the above inequalities and applying (2.4), (2.5) and (2.6), we get

(2.8)
$$\frac{\varepsilon}{s^2} \leq \liminf_{k\longrightarrow\infty} M\left(x_{m(k)}, x_{n(k)-1}\right) \leq \lim_{k\to\infty} \sup M\left(x_{m(k)}, x_{n(k)-1}\right) \leq \varepsilon.$$

Also

$$\begin{aligned} &\lim_{k\to\infty} N\left(x_{m(k)}, x_{n(k)-1}\right) \\ &\quad = \lim_{k\to\infty} \min\left\{d\left(x_{n(k)-1}, f(x_{n(k)-1})\right), d\left(x_{m(k)}, f(x_{m(k)})\right)\right. \\ &\qquad \left., d\left(x_{n(k)-1}, f(x_{m(k)})\right), d\left(x_{m(k)}, f(x_{n(k)-1})\right)\right\} \\ &\quad = \lim_{k\to\infty} \min\left\{d\left(x_{n(k)-1}, x_{n(k)}\right), d\left(x_{m(k)}, x_{m(k)+1}\right), d\left(x_{n(k)-1}, x_{m(k)+1}\right)\right. \\ &\qquad \left., d\left(x_{m(k)}, x_{n(k)}\right)\right\} \\ &\quad = 0. \end{aligned}$$

Note that

$$d\left(x_{m(k)}, x_{n(k)}\right) - sd\left(x_{m(k)}, x_{m(k)+1}\right) \leq sd\left(x_{m(k)+1}, x_{n(k)}\right). \tag{2.9}$$

By (2.7), (2.8) and (2.9) we obtain

$$\begin{aligned}\psi(\varepsilon) &\le \psi\left(\lim_{k\to\infty}\sup d\left(x_{m(k)},x_{n(k)}\right)\right)\\ &= \psi\left(\lim_{k\to\infty}\sup sd\left(x_{m(k)+1},x_{n(k)}\right)\right)\\ &\le F\left(\psi\left(\lim_{k\to\infty}\sup M\left(x_{m(k)},x_{n(k)-1}\right)\right)\right.\\ &\quad \left.,\varphi\left(\lim_{k\to\infty}\inf M\left(x_{m(k)},x_{n(k)-1}\right)\right)\right)\\ &\le F\left(\psi(\varepsilon),\varphi\left(\lim_{k\to\infty}\inf M\left(x_{m(k)},x_{n(k)-1}\right)\right)\right)\\ &\le \psi(\varepsilon),\end{aligned}$$

so

$$\psi(\varepsilon)=0, \qquad \varphi\left(\lim_{k\to\infty}\inf M\left(x_{m(k)},x_{n(k)-1}\right)\right)=0,$$

that is $\varepsilon=0$ or $\lim_{k\to\infty}\inf M\left(x_{m(k)},x_{n(k)-1}\right)=0$ which is a contradiction. Thus $\{x_n\}$ is a b-Cauchy sequence in X. Since (X,d) is a complete b-metric space, there exists $u\in X$ such that $\lim_{n\to\infty}x_n\to u$.

Now, we show u is a fixed point of f. We know that

$$\psi(d(u,fu))\le\psi(sd(u,fu)),$$

where $s\ge 1$. Inequality (2.1) implies

$$\psi(sd(u,fu))\le F(\psi(M(u,fu)),\varphi(M(u,fu)))+LN(u,fu).$$

But it is easy to see that $M(u,fu)=d(u,fu)$ and $N(u,fu)=0$. Thus

$$\psi(d(u,fu))\le\psi(d(u,fu)),$$

and so $\psi(d(u,fu))=0$, or $\varphi(d(u,fu))=0$ so $d(u,fu)=0$ and $fu=u$. Moreover, u is a unique fixed point of f. Let $v\ne u$ be another fixed point of f. From (2.1) we have

$$\begin{aligned}\psi(d(u,v)) &\le\psi(sd(u,v))\\ &=\psi(sd(fu,fv))\\ &\le F(\psi(M(u,v)),\varphi(M(u,v)))+LN(u,v),\end{aligned}$$

where $M(u,v)=d(u,v)$ and $N(u,v)=0$. Therefore

$$\begin{aligned}\psi(d(u,v)) &\le F(\psi(M(u,v)),\varphi(M(u,v)))+LN(u,v)\\ &= F(\psi(d(u,v)),\varphi(d(u,v)))\\ &\le\psi(d(u,v)).\end{aligned}$$

So $\psi(d(u,v)) = 0$ or $\varphi(d(u,v)) = 0$, thus $d(u,v) = 0$ that is $u = v$. This shows f has a unique fixed point. $\square$

Here as an application of Theorem 2.1, the existence of a solution of an integral equation is proved.

Consider the following integral equation

$$x(t) = p(t) + \int_0^T s(t,r)\, f(r, x(r))\, dr, \quad t \in [0,T], \tag{2.10}$$

where $T > 0$, $f : [0,T] \times \mathbb{R} \to \mathbb{R}$, $p : [0,T] \to \mathbb{R}$ and $s : [0,T] \times \mathbb{R} \to [0,\infty)$ are continuous. Assume for all $x, y \in X$

$$0 \le f(r,y) - f(r,x) \le \sqrt[q]{\frac{F\left(|y-x|^q, \varphi\left(|y-x|^q\right)\right)}{2^{q-1}}},$$

where $F : [0,\infty)^2 \to \mathbb{R}$ is a C-class function and $\varphi : [0,\infty) \to [0,\infty)$ is an ultra altering distance function. Suppose $\max_{t\in I}\left(\int_0^T \mid s(t,r) \mid dr\right)^q \le 1$. Then (2.10) has a solution in $C([0,T],\mathbb{R})$, where $C([0,T],\mathbb{R})$ is the set of continuous real functions defined on $[0,T]$. Note that this space with the b-metric given by $d(x,y) = \max_{t\in I} \mid x(t) - y(t) \mid^q$ for all $x, y \in C([0,T],\mathbb{R})$ is a complete b-metric space with $s = 2^{q-1}$ and $q \ge 1$.

Set $X := C([0,T],\mathbb{R})$. Define $G : X \to X$ by

$$G(x(t)) = p(t) + \int_0^T s(t,r)\, f(r, x(r))\, dr.$$

For $x, y \in X$ we have

$$\begin{aligned} 2^{q-1} \mid Gx(t) - Gy(t) \mid^q &= 2^{q-1} \left| \int_0^T s(t,r)\left[f(r,x(r)) - f(r,y(r))\right] dr \right|^q \\ &\le 2^{q-1} \left(\int_0^T \mid s(t,r)\left[f(r,x(r)) - f(r,y(r))\right] \mid dr \right)^q \\ &\le \max_{r\in I} F\left(|y-x|^q, \varphi\left(|y-x|^q\right)\right) \left(\int_0^T \mid s(t,r) \mid dr \right)^q \\ &\le F\left(d(x,y), \varphi(d(x,y))\right) \\ &\le F\left(M(x,y), \varphi(M(x,y))\right), \end{aligned}$$

where

$$M(x,y) = \max\left\{ d(x,y), \frac{d(x,fx)\, d(y,fy)}{1 + d(fx,fy)} \right\}.$$

For all $x, y \in X$

$$sd(G(x), G(y)) \le F\left(M(x,y), \varphi(M(x,y))\right).$$

By Theorem 2.1, G has a unique fixed point. This means (2.10) has a solution.

Some corollaries are presented as follows:

Corollary 2.2. *Let* (X, d) *be a complete* b*-metric space on* X *and* $f : X \to X$ *be a self mapping. Suppose*

$$\psi\left(sd\left(fx, fy\right)\right) \leq \psi\left(M\left(x, y\right)\right) - \varphi\left(M\left(x, y\right)\right) + LN\left(x, y\right),$$

for all $x, y \in X$*, where* $L \geq 0, \psi : [0, \infty) \to [0, \infty)$ *is an altering distance function* $\varphi : [0, \infty) \to [0, \infty)$ *is an ultra altering distance function and*

$$M\left(x, y\right) = \max\left\{d\left(x, y\right), \frac{d\left(x, fx\right) d\left(y, fy\right)}{1 + d\left(fx, fy\right)}\right\},$$

and

$$N\left(x, y\right) = \min\left\{d\left(x, fx\right), d\left(x, fy\right), d\left(y, fx\right), d\left(y, fy\right)\right\}.$$

Then f *has a unique fixed point.*

Proof. Set $F(s, t) = s - t$ in Theorem 2.1. □

Corollary 2.3. *Let* (X, d) *be a complete* b*-metric space* $f : X \to X$ *be a self mapping. Suppose*

$$\psi\left(sd\left(fx, fy\right)\right) \leq \psi\left(M\left(x, y\right)\right) \beta\psi\left(\left(M\left(x, y\right)\right)\right) + LN\left(x, y\right),$$

for all $x, y \in X$*, where* $L \geq 0,\ \beta : [0, \infty) \to [0, 1)$ *is continuous and*

$$M\left(x, y\right) = \max\left\{d\left(x, y\right), \frac{d\left(x, fx\right) d\left(y, fy\right)}{1 + d\left(fx, fy\right)}\right\},$$

and

$$N\left(x, y\right) = \min\left\{d\left(x, fx\right), d\left(x, fy\right), d\left(y, fx\right), d\left(y, fy\right)\right\}.$$

Then f *has a unique fixed point.*

Proof. Let $F\left(s, t\right) = s\beta\left(s\right)$ in Theorem 2.1, where $\beta : [0, \infty) \to [0, 1)$ and is continuous. □

Setting $\psi\left(t\right) = t$ in Corollary 2.3, we have the following corollary.

Corollary 2.4. *Let* (X, d) *be a complete* b*-metric space and* $f : X \to X$ *be a self mapping. Suppose*

$$d\left(fx, fy\right) \leq \left[\frac{\beta\left(M\left(x, y\right)\right)}{s}\right] M\left(x, y\right) + LN\left(x, y\right),$$

for all $x, y \in X$*, where* $L \geq 0,\ \beta : [0, \infty) \to [0, 1)$ *is continuous and*

$$M\left(x, y\right) = \max\left\{d\left(x, y\right), \frac{d\left(x, fx\right) d\left(y, fy\right)}{1 + d\left(fx, fy\right)}\right\},$$

and

$$N\left(x, y\right) = \min\left\{d\left(x, fx\right), d\left(x, fy\right), d\left(y, fx\right), d\left(y, fy\right)\right\}.$$

Then f *has a unique fixed point.*

Let $F(s,t) = \phi(s)$ in Theorem 2.1, where $\phi : [0,\infty) \to [0,\infty)$ is a continuous function such that $\phi(0) = 0$, and $\phi(t) < t$ for $t > 0$ and $\psi(t) = t$. Then we have the following corollary.

Corollary 2.5. *Let (X,d) be a complete b-metric space and $f : X \to X$ be a self mapping. Suppose*

$$sd(fx, fy) \leq \phi(M(x,y,z)) + LN(x,y),$$

for all $x, y \in X$, where $L \geq 0$, $\phi : [0,\infty) \to [0,\infty)$ is a continuous function such that $\phi(0) = 0$ and $\phi(t) < t$ for $t > 0$ and

$$M(x,y) = \max\left\{d(x,y), \frac{d(x,fx)\,d(y,fy)}{1 + d(fx,fy)}\right\},$$

and

$$N(x,y) = \min\{d(x,fx), d(x,fy), d(y,fx), d(y,fy)\}.$$

Then f has a unique fixed point.

Suppose $\phi(t) = lt,\ 0 < l < 1$ in Corollary 2.5 or with choice $F(s,t) = ls,\ 0 < l < 1$ and $\psi(t) = t,\ L = 0$ in Theorem 2.1, we have the following corollary .

Corollary 2.6. *Let (X,d) be a complete b-metric space and $f : X \to X$ be a self mapping. Suppose*

$$sd(fx, fy) \leq lM(x,y), \tag{2.11}$$

for all $x, y \in X$, where $0 < l < 1$, and

$$M(x,y) = \max\left\{d(x,y), \frac{d(x,fx)\,d(y,fy)}{1 + d(fx,fy)}\right\},$$

Then f has a unique fixed point.

Example 2.7. Let $X = [0,\infty)$. Suppose the function $f : X \to X$ is given by

$$f(x) = \frac{3x + 8}{5}.$$

If we define the b-metric d on X by $d(x,y) = |x - y|^2$, then (X,d) is a complete b-metric space with $s = 2$. We obtain

$$\begin{aligned} sd(fx, fy) &= 2d\left(\frac{3x+8}{5}, \frac{3y+8}{5}\right) = \frac{18}{25}|x-y|^2 \\ &\leq \frac{18}{25}M(x,y). \end{aligned}$$

Hence by Corollary 2.6, f has a unique fixed point (which is $x = 4$).

Example 2.8. Let $X = \{1, 2, 4\}$. Assume the function $f : X \to X$ is given by

$$f = \begin{pmatrix} 1 & 2 & 4 \\ 4 & 2 & 2 \end{pmatrix}.$$

First, define the b-metric d on X by $d(1,2) = 7$, $d(1,4) = 10$, $d(2,4) = \frac{1}{5}$ and $d(x,x) = 0$. Then (X,d) is a complete b-metric space with $s = \frac{25}{18}$. It is easy to see that $sd(fx, fy) \leq \frac{1}{36} M(x,y)$, so by Corollary 2.6, f has a unique fixed point (which is $x = 2$).

Corollary 2.9. *Let (X,d) be a complete b-metric space and $f : X \to X$ be a self mapping. Suppose*

$$sd(fx, fy) \leq ad(x,y) + b\frac{d(x,fx)\,d(y,fy)}{1 + d(fx,fy)},$$

for all $x, y \in X$, where $a + b < 1$. Then f has a unique fixed point.

Letting $\phi(t) = \frac{t}{1+t}$ in Corollary 2.5 or with choice $F(s,t) = \frac{s}{1+s}$, $\psi(t) = t$, $L = 0$ in Theorem 2.1, we have the following corollary.

Corollary 2.10. *Let (X,d) be a complete b-metric space and $f : X \to X$ be a self mapping. Suppose*

$$sd(fx, fy) \leq \frac{M(x,y)}{1 + M(x,y)},$$

for all $x, y \in X$, where

$$M(x,y) = \max\left\{d(x,y), \frac{d(x,fx)\,d(y,fy)}{1 + d(fx,fy)}\right\}.$$

Then f has a unique fixed point.

Letting $\phi(t) = t - \frac{t}{n+t}$, $n \geq 1$ in Corollary 2.5, we have the following corollary.

Corollary 2.11. *Let (X,d) be a complete b-metric space and $f : X \to X$ be a self mapping. Suppose*

$$sd(fx, fy) \leq M(x,y) - \frac{M(x,y)}{n + M(x,y)},$$

for all $x, y \in X$, where

$$M(x,y) = \max\left\{d(x,y), \frac{d(x,fx)\,d(y,fy)}{1 + d(fx,fy)}\right\}.$$

Then f has a unique fixed point.

Assume $\psi(t) = \ln(1+t)$ in Corollary 2.5 or with choice $F(s,t) = \ln(1+s)$, $\psi(t) = t$, $L = 0$ in Theorem 2.1, we have the following corollary.

Corollary 2.12. *Let (X,d) be a complete b-metric space and $f : X \to X$ be a self mapping. Suppose*

$$sd(fx,fy) \leq \ln(1+M(x,y)),$$

for all $x,y \in X$, where

$$M(x,y) = \max\left\{d(x,y), \frac{d(x,fx)\,d(y,fy)}{1+d(fx,fy)}\right\}.$$

Then f has a unique fixed point.

We recall the following lemma.

Lemma 2.13 ([11])**.** *Let X be a nonempty set and $f : X \to X$ be a function. Then there exists a subset $E \subseteq X$ such that $f(E) = f(X)$ and $f : E \to f(X)$ is one-to-one.*

Theorem 2.14. *Let (X,d) be a complete b-metric space, $f,T : X \to X$ be such that $f(X) \subseteq T(X)$. Suppose that (T,f) satisfy the following condition:*

$$\psi(sd(fx,fy)) \leq F(\psi(M(d(Tx,Ty)), \varphi(M(Tx,Ty)))) + LN(Tx,Ty),$$

for all $x,y \in X$, where $L \geq 0$, $F : [0,\infty)^2 \to \mathbb{R}$ is a C-class function, $\psi : [0,\infty) \to [0,\infty)$ is an altering distance function, $\varphi : [0,\infty) \to [0,\infty)$ is an ultra altering distance function and

$$M(Tx,Ty) = \max\left\{d(Tx,Ty), \frac{d(Tx,fx)\,d(Ty,fy)}{1+d(fx,fy)}\right\},$$

and

$$N(Tx,Ty) = \min\{d(Tx,fx), d(Tx,fy), d(Ty,fx), d(Ty,fy)\}.$$

Then (f,T) has a unique coincidence point.

Proof. Let $f : X \to X$. By Lemma 2.13 there exists $E \subset X$ such that $T(E) = T(X)$ and $T|_E$ is one-to-one. Since $T(E) \subseteq T(X) \subseteq X$, one can define the mapping $\mathcal{A} : T(E) \to X$ by $\mathcal{A}(Tx) = fx$ for all $x \in E$. Since $T|_E$ is one-to-one, then $\mathcal{A}$ is well-defined. Now

$$\begin{aligned}\psi(sd(\mathcal{A}(Tx), \mathcal{A}(Ty))) &= \psi(sd(fx,fy)) \\ &\leq F(\psi(M(d(Tx,Ty)), \varphi(M(Tx,Ty)))),\end{aligned}$$

for all $x,y \in X$, where

$$M(Tx,Ty) = \max\left\{d(Tx,Ty), \frac{d(Tx,\mathcal{A}(Tx))\,d(Ty,\mathcal{A}(Ty))}{1+d(\mathcal{A}(Tx),\mathcal{A}(Ty))}\right\},$$

and

$$\begin{aligned}&N(Tx,Ty) \\ &= \min\{d(Tx,\mathcal{A}(Tx)), d(Tx,\mathcal{A}(Ty)), d(Ty,\mathcal{A}(Tx)), d(Ty,\mathcal{A}(Ty))\}.\end{aligned}$$

Thus by Theorem 2.1, there exists a unique fixed point $u \in T(E)$ of $\mathcal{A}$, i.e. $\mathcal{A}u = u$. Since $u \in T(E)$, there exists $w \in E$ such that

$$fw = \mathcal{A}(Tw) = \mathcal{A}u = u = Tw.$$

Thus w is a unique coincidence point of f and T. □

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