Sahand Communications in Mathematical Analysis (SCMA) Vol. 13 No. 1 (2019), 141-152
http://scma.maragheh.ac.ir
DOI: 10.22130/scma.2017.29018

Some Fixed Point Results on Intuitionistic Fuzzy Metric Spaces with a Graph

Mohammad Esmael Samei

Abstract. In 2006, Espinola and Kirk made a useful contribution on combining fixed point theory and graph theory. Recently, Reich and Zaslavski studied a new inexact iterative scheme for fixed points of contractive and nonexpansive multifunctions. In this paper, by using the main idea of their work and the idea of combining fixed point theory on intuitionistic fuzzy metric spaces and graph theory, we present some iterative scheme results for G-fuzzy contractive and G-fuzzy nonexpansive mappings on graphs.

1. Introduction

the Banach contraction principle was proved on 1922, [6]. The fixed point results for different kinds of contractions are of great interest for fixed point theorists on some spaces such as quasi-metric spaces [9, 24], cone metric spaces [4, 35], partially ordered metric spaces [1, 8, 34], Menger spaces [29] and fuzzy metric spaces [20, 22, 28]. The concept of fuzzy sets introduced by Zadeh in 1965 [41]. In 1975, Kramosil and Michalek introduced the notion of fuzzy metric spaces [28] and George and Veeramani modified this concept in 1994 [21]. They also defined the notion of Hausdorff topology in fuzzy metric spaces [21]. This notion has very important applications in quantum particle physics particularly in connection with both string and E-infinity theory which introduced by El Naschie and Sigalotte [13–16, 18, 39]. Motivated by the potential applicability of fuzzy topology to quantum particle physics, Park introduced the notion of intuitionistic fuzzy metric spaces [31]. Actually, Park's notion is useful in modeling of some phenomena where it is

2010 *Mathematics Subject Classification.* 34B24, 34B27.
Key words and phrases. Fixed point, Intuitionistice fuzzy metric space, Connected graph, G-fuzzy contractive, G-fuzzy nonexpansive.
Received: 02 December 2016, Accepted: 15 June 2017.

necessary to study the relationship between two probability functions. Some authors have introduced and discussed several notions of intuitionistic fuzzy metric spaces in different ways [2, 5, 10]. Grabiec obtained a fuzzy version of the Banach contraction principle in fuzzy metric spaces in Kramosil and Michalek's sense [20] and since then many authors have proved fixed point theorems in fuzzy metric spaces [11, 27, 30, 32].

It is known that combining some branches is a typical activity in different fields of science specially in mathematics. In 2005, Echenique gave a short and constructive proof of an extension of Tarski's fixed point theorem which is important in the theory of games [12]. In 2006, Espinola and Kirk provided useful results on combining fixed pint theory and graph theory [19]. In 2008 and 2009, Jachymski continued this idea by using a different view [23, 25]. Then, Beg, Butt and Radojević obtained some results in 2010 [7] in the same direction. In this paper, we present some iterative scheme results for G-fuzzy contractive and G-fuzzy nonexpancive maps on graphs.

2. Preliminaries

In this section, we aim to present our main results.

At the first, we recall some basic notions.

Definition 2.1 ([38])**.** A binary operation $* : [0,1] \times [0,1] \to [0,1]$ is called a continuous t-norm whenever it satisfies the following conditions:

(a) $*$ is commutative and associative,
(b) $*$ is continuous,
(c) $a * 1 = a$ for all $a \in [0,1]$,
(d) $a * b \leq c * d$ for all $a, b, c, d \in [0,1]$ with $a \leq c$ and $c \leq d$.

For example, $a * b = ab$, $a * b = \min\{a, b\}$, $a * b = \max\{a + b - 1, 0\}$ and

$$a * b = \frac{ab}{\max\{a, b, \lambda\}},$$

for $0 < \lambda < 1$ are continuous t-norms.

Definition 2.2 ([38])**.** A binary operation $\Diamond : [0,1] \times [0,1] \to [0,1]$ is called a continuous t-conorm whenever it satisfies the following conditions:

(a) $\Diamond$ is commutative and associative,
(b) $\Diamond$ is continuous,
(c) $a \Diamond 0 = a$ for all $a \in [0,1]$,
(d) $a \Diamond b \leq c \Diamond d$ whenever $a \leq c$ and $c \leq d$, and $a, b, c, d \in [0,1]$.

For example, $a \Diamond b = \min\{a+b, 1\}$ and $a \Diamond b = \max\{a, b\}$ are continuous t-conorms.

Definition 2.3 ([31])**.** A 5-tuple $(X, M, N, *, \Diamond)$ is said to be an intuitionistic fuzzy metric space whenever X is a set, $*$ is a continuous t-norm, $\Diamond$ is a continuous t-conorm and M, N are fuzzy sets on $X^2 \times [0, \infty)$ satisfying the following conditions:

(i) $M(x, y, t) + N(x, y, t) \leq 1$,
(ii) $M(x, y, 0) = 0$,
(iii) $M(x, y, t) = 1$ for all $t > 0$ if and only if $x = y$,
(iv) $M(x, y, t) = M(y, x, t)$,
(v) $M(x, y, t) * M(y, z, s) \leq M(x, z, t+s)$ for all $x, y, z \in X$, $s, t > 0$,
(vi) $M(x, y, .) : [0, \infty) \to [0, 1]$ is continuous,
(vii) $\lim_{t\to\infty} M(x, y, t) = 1$ for all $x, y \in X$,
(viii) $N(x, y, 0) = 1$,
(ix) $N(x, y, t) = 0$ for all $t > 0$ if and only if $x = y$,
(x) $N(x, y, t) = N(y, x, t)$,
(xi) $N(x, y, t) \Diamond N(y, z, s) \geq N(x, z, t + s)$ for all $x, y, z \in X$, $s, t > 0$,
(xii) $N(x, y, .) : [0, \infty) \to [0, 1]$ is continuous,
(xiii) $\lim_{t\to\infty} N(x, y, t) = 0$ for all $x, y \in X$.

Then (M, N) is called an intuitionistic fuzzy metric on X. For an intuitionistic fuzzy metric space $(X, M, N, *, \Diamond)$, define

$$B(x, r, t) = \{y \in X \,:\, M(x, y, t) > 1 - r, N(x, y, t) < r\},$$

for all $t > 0$ and $0 < r < 1$. Denote the generated topology by the sets $B(x, r, t)$ by $\tau_{(M,N)}$. A sequence $\{x_n\}$ in $(X, M, N, *, \Diamond)$ is said to be Cauchy whenever for each $\varepsilon > 0$ and $t > 0$, there exists a natural number n_0 such that $M(x_n, x_m, t) > 1 - \varepsilon$ and $N(x_n, x_m, t) < \varepsilon$ for all $n, m \geq n_0$. Also, $(X, M, N, *, \Diamond)$ is called complete whenever every Cauchy sequence is convergent with respect $\tau_{(M,N)}$.

Definition 2.4 ([40])**.** Let A be a nonempty subset of an intuitionistic fuzzy metric space $(X, M, N, *, \Diamond)$. For $x \in X$ and $t > 0$, we define $M(x, A, t) = \sup\{M(x, a, t) : a \in A\}$.

Definition 2.5 ([11])**.** Let $(X, M, N, *, \Diamond)$ be an intuitionistic fuzzy metric space. The fuzzy metric (M, N) is triangular whenever

$$\frac{1}{M(x, y, t)} - 1 \leq \frac{1}{M(x, z, t)} - 1 + \frac{1}{M(z, y, t)} - 1,$$

and $N(x, y, t) \leq N(x, z, t) + N(z, y, t)$ for all $x, y, z \in X$ and $t > 0$.

Definition 2.6 ([22])**.** A sequence $\{x_n\}$ in an intuitionistic fuzzy metric space $(X, M, N, *, \Diamond)$ is called an intuitionistic fuzzy contractive sequence if there exists $0 < k < 1$ such that

$$\frac{1}{M(x_{n+1}, x_{n+2}, t)} - 1 \leq k \left(\frac{1}{M(x_n, x_{n+1}, t)} - 1 \right),$$

and $N(x_{n+1}, x_{n+2}, t) \leq kN(x_n, x_{n+1}, t)$ for all n and $t > 0$.

Lemma 2.7 ([27])**.** *Let $(X, M, N, *, \Diamond)$ be a triangular intuitionistic fuzzy metric space and $\{x_n\}$ an intuitionistic fuzzy contractive sequence in X. Then $\{x_n\}$ is a Cauchy sequence.*

Definition 2.8 ([33])**.** Let $(X, M, N, *, \Diamond)$ be an intuitionistic fuzzy metric space. A selfmap f on X is said to be intuitionistic fuzzy contractive whenever there exists $k \in (0, 1)$ such that $N(f(x), f(y), t) \leq kN(x, y, t)$ and

$$\frac{1}{M(f(x), f(y), t)} - 1 \leq k\left(\frac{1}{M(x, y, t)} - 1\right),$$

for all $x, y \in X$ and $t > 0$.

3. Main Results

Let $(X, M, N, *, \Diamond)$ be an intuitionistic fuzzy metric space and $\Delta = X \times X$. Consider a directed graph G such that the set $V(G)$ of its vertices coincides with X and the set $E(G)$ of its edges contains all loops, that is, $\Delta \subset E(G)$. We assume that G has no parallel edges. By G^{-1} we denote the conversion of the graph G, that is, the graph obtained from G by reversing the direction of the edges. Also, $\tilde{G}$ denotes the undirected graph obtained from G by ignoring the direction of the edges. We say that a mapping $f : X \to X$ preserves the edges of G whenever $(x, y) \in E(G)$ implies $(fx, fy) \in E(G)$ for all $x, y \in X$. Also, a mapping $f : X \to X$ is called a G-fuzzy contraction whenever f preserves the edges of G and there exists an $\alpha \in (0, 1)$ such that

$$\frac{1}{M(fx, fy, t)} - 1 \leq \alpha\left[\frac{1}{M(x, y, t)} - 1\right],$$

and f is called a G-fuzzy nonexpansive whenever in the above equation $\alpha = 1$ for all $(x, y) \in E(G)$. If f is a G-fuzzy contraction (resp. G-fuzzy nonexpansive), then f is a $\tilde{G}$-fuzzy contraction (resp. $\tilde{G}$-fuzzy nonexpansive) [25].

Example 3.1 ([25])**.** Each Banach contraction is a G_0-contraction, where the graph G_0 is defined by $E(G_0) = X \times X$.

Definition 3.2. Let x and y be two vertices in a graph G. A path in G from x to y of length n is a sequence $\{x_i\}_{i=0}^{n}$ of $n+1$ distinct vertices such that $x_0 = x$, $x_n = y$ and $(x_i, x_{i+1}) \in E(G)$ for all $i = 0, 1, \ldots, n-1$.

We denote by $r(x, y)$, the sum of the edges distances between x and y, that is,

$$r(x, y) = t\sum_{i=1}^{n}\left[\frac{1}{M(x_{i-1}, x_i, t)} - 1\right],$$

for all $t \geq 0$. A graph G is connected if there exists a path between any two vertices, and G is weakly connected if $\tilde{G}$ is connected. We denote by $[x]_G$ the set of all vertices in G so that there exists a path between x and those.

Now, we are ready to state and prove our main results. We say that G is a (C)-graph whenever for each sequence $\{x_n\}_{n\geq 0}$ in X with $x_n \to x$ and $(x_n, x_{n+1}) \in E(G)$ for all $n \geq 0$, there exists a subsequence $\{x_{n_k}\}_{k\geq 0}$ such that $(x_{n_k}, x) \in E(G)$ for all $k \geq 0$ [25]. We say that G is a (P)-graph whenever $\{x_n\}_{n\geq 0}$ is a convergent sequence to a point x and $x_n \in [x]_G$ for all $n \geq 0$, we have $r(x_n, x) \to 0$. Now by providing the following examples, we show that the notions are independent on infinite graphs. Of course, it is normal that we deal infinite graphs because we have the notion of the limit here.

Example 3.3 ([3])**.** Let $X = \left\{\frac{1}{n} : n \geq 1\right\} \cup \mathbb{N} \cup \{0\}$ via the Euclidean metric. Define the undirected graph G_2 by $V(G_2) = X$ and $E(G_2) = \left\{\left(\frac{1}{n}\right) : n \geq 1\right\} \cup \{(n, 0) : n \geq 1\}$. Let $x_n = \frac{1}{n}$ for all $n \geq 1$. Then $x_n \to 0$, $x_n \in [0]_{G_2}$ while $r(x_n, 0) = 2n - \frac{1}{n} \nrightarrow 0$. Hence G_2 is not a (P)-graph, but it G_2 is a (C)-graph.

Example 3.4 ([3])**.** Let $X = \left\{\frac{1}{n} : n \geq 1\right\} \cup \left\{\frac{1}{\sqrt{2}+n} : n \geq 1\right\} \cup \{0\}$ via the Euclidean metric. Define the undirected graph G_3 by $V(G_3) = X$ and

$$E(G_3) = \left\{\left(\frac{1}{n}, \frac{1}{n+1}\right) : n \geq 1\right\} \cup \left\{\left(\frac{1}{n}, \frac{1}{\sqrt{2}+n}\right) : n \geq 1\right\} \cup \left\{\left(\frac{1}{\sqrt{2}+n}, 0\right) : n \geq 1\right\}.$$

Let $x_n = \frac{1}{n}$ for all $n \geq 1$. Then $x_n \to 0$ and $(x_n, x_{n+1}) \in E(G_3)$ for all $n \geq 1$ while there is no subsequence $\{x_{n_k}\}_{k\geq 1}$ such that $(x_{n_k}, x) \in E(G_3)$ for all $k \geq 1$. Thus, G_3 is not a (C)-graph, but it G_3 is a (P)-graph.

Theorem 3.5. *Let G be a weakly connected (C)-graph of a (P)-graph. Suppose that T is a G-fuzzy contraction selfmap on X with constant α and $\{\varepsilon_i\}_{i\geq 0}$ is a sequence in $(0, \infty)$ satisfying*

$$\sum_{i=0}^{\infty} \varepsilon_i < \infty.$$

Assume that $\{T_i\}_{i\geq 0}$ is a sequence of selfmaps on X such that T_i preserves the edges of G, i.e.,

$$\frac{1}{M(T_i x, Tx, t)} - 1 < \frac{\varepsilon_i}{t},$$

and $(T_ix, T_{i+1}y) \in E(G)$ *for all* $x \in G$*,* $i \geq 0$*,* $t \geq 0$ *and* $(x, y) \in E(G)$*. Let* $x_0 \in X$ *and* $x_{i+1} = T_ix_i$ *for all* $i \geq 0$*. Then* $\{x_i\}_{i\geq 0}$ *converges to a fixed point of* T*.*

Proof. Since $[x_0]_{\tilde{G}} = X$, $T_0x_0 \in [x_0]_{\tilde{G}}$, so there exists a path $\{\gamma_i\}_{i=0}^N$ from x_0 to x_1. Now, we show that there exists a path from x_i to x_{i+1} for all i. Put $x = x_0$ and $y = \gamma_1$ and take $i = 0$. By using the assumption, we get $(T_1T_0x_0, T_2T_1\gamma_1) \in E(G)$. By continuing this process and using induction, we conclude that

$$(T_{i-1}\cdots T_1T_0x_0, T_i\cdots T_2T_1\gamma_1) \in E(G),$$

for all $i \geq 1$. Thus, $(x_i, T_i\cdots T_2T_1\gamma_1) \in E(G)$ for all i. Now, it is easy to see that

$$x_i, T_iT_{i-1}\cdots T_1\gamma_1, T_iT_{i-1}\cdots T_1\gamma_2, \cdots, T_iT_{i-1}\cdots T_1\gamma_{N-1}, x_{i+1},$$

is a path from x_i to x_{i+1} for all i. Since T is a $\tilde{G}$-fuzzy contraction, for each $i \geq 1$ we have

$$\begin{aligned}
r(x_i, x_{i+1}) = t\Bigg[&\left(\frac{1}{M(x_i, T_{i-1}T_{i-2}\cdots T_0\gamma_1, t)} - 1\right) \\
&+ \left(\frac{1}{M(T_{i-1}T_{i-2}\cdots T_0\gamma_1, T_iT_{i-1}\cdots T_0\gamma_1, t)} - 1\right) \\
&+ \left(\frac{1}{M(T_iT_{i-1}\cdots T_1\gamma_1, T_iT_{i-1}\cdots T_1\gamma_2, t)} - 1\right) \\
&+ \cdots + \left(\frac{1}{M(T_iT_{i-1}\cdots T_1\gamma_{n-1}, x_{i+1}, t)} - 1\right)\Bigg] \\
\leq t\Bigg[&2\varepsilon_{i-1} + \alpha\left(\frac{1}{M(x_{i-1}, T_{i-2}\cdots T_0\gamma_1, t)} - 1\right) + \varepsilon_{i-1} \\
&+ \varepsilon_i + \alpha\left(\frac{1}{M(T_{i-2}\cdots T_0\gamma_1, T_{i-1}\cdots T_0\gamma_1, t)} - 1\right) + 2\varepsilon_i \\
&+ \alpha\left(\frac{1}{M(T_{i-1}\cdots T_1\gamma_1, T_{i-1}\cdots T_1\gamma_2, t)} - 1\right) + 2\varepsilon_i \\
&\left(\frac{1}{M(T_{i-1}\cdots T_1\gamma_{n-1}, x_i, t)} - 1\right)\Bigg] \\
= 3&\varepsilon_{i-1} + (2N-1)\varepsilon_i + \alpha r(x_{i-1}, x_i).
\end{aligned}$$

Thus, $r(x_1, x_2) \leq \alpha r(x_0, x_1) + 3\varepsilon_0 + (2N-1)\varepsilon_1$ and so

$$\begin{aligned}
r(x_2, x_3) &\leq 3\varepsilon_1 + (2N-1)\varepsilon_2 + \alpha r(x_1, x_2) \\
&\leq \alpha^2 r(x_0, x_1) + \alpha(3\varepsilon_0 + (2N-1)\varepsilon_1) + 3\varepsilon_1 + (2N-1)\varepsilon_2.
\end{aligned}$$

Also, by induction for each $n \geq 1$ we have

$$r(x_n, x_{n+1}) \leq \alpha^n r(x_0, x_1) + \sum_{i=0}^{n-1} \alpha^i (3\varepsilon_{n-i-1} + (2N-1)\varepsilon_{n-i}).$$

Hence,

$$\begin{aligned}
\frac{1}{M(x_{n+1}, x_{n+2}, t)} - 1 &\leq \frac{\varepsilon_n}{t} + \frac{\varepsilon_{n+1}}{t} + \alpha \left(\frac{1}{M(x_n, x_{n+1}, t} - 1 \right) \\
&\leq \frac{\varepsilon_n}{t} + \frac{\varepsilon_{n+1}}{t} + \alpha r(x_n, x_{n+1}) \\
&\leq \frac{\varepsilon_n}{t} + \frac{\varepsilon_{n+1}}{t} + \alpha^{n+1} r(x_0, x_1) \\
&\quad + \sum_{i=0}^{n-1} \alpha^{i+1} (3\varepsilon_{n-i-1} + (2N-1)\varepsilon_{n-i}).
\end{aligned}$$

Therefore we obtain

$$\begin{aligned}
\sum_{n=0}^{\infty} \left(\frac{1}{M(x_{n+1}, x_{n+2}, t)} - 1 \right) &\leq \sum_{n=0}^{\infty} (\frac{\varepsilon_n}{t} + \frac{\varepsilon_{n+1}}{t} + \alpha^{n+1} r(x_0, x_1)) \\
&\quad + \sum_{i=0}^{n-1} \alpha^{i+1} (3\varepsilon_{n-i-1} + (2N-1)\varepsilon_{n-i}) \\
&= \sum_{n=0}^{\infty} (\frac{\varepsilon_n}{t} + \frac{\varepsilon_{n+1}}{t} + \alpha^{n+1} r(x_0, x_1)) \\
&\quad + \sum_{i=1}^{n} \alpha^{n-i+1} (3\varepsilon_{i-1} + (2N-1)\varepsilon_i) \\
&\leq \sum_{n=0}^{\infty} \frac{\varepsilon_n}{t} + \sum_{n=0}^{\infty} \frac{\varepsilon_{n+1}}{t} + \sum_{n=0}^{\infty} \alpha^{n+1} r(x_0, x_1) \\
&\quad + \sum_{i=1}^{\infty} \left(\sum_{j=1}^{\infty} \alpha^j (3\varepsilon_{i-1} + (2N-1)\varepsilon_i) \right) \\
&\leq \sum_{n=0}^{\infty} \frac{\varepsilon_n}{t} + \sum_{n=0}^{\infty} \frac{\varepsilon_{n+1}}{t} \\
&\quad + \sum_{n=0}^{\infty} \alpha^{n+1} \left[r(x_0, x_1) + \sum_{n=1}^{\infty} (3\varepsilon_{n-1} + (2N-1)\varepsilon_n) \right] \\
&< \infty.
\end{aligned}$$

Thus, $\{x_n\}$ is a Cauchy sequence and so there exists $x_* \in X$ such that $x_n \to x_*$. If G is a (P)-graph, then we have

$$\frac{1}{M(x_*, Tx_*, t)} - 1 \leq \left(\frac{1}{M(x_*, x_{n+1}, t)} - 1 \right) + \left(\frac{1}{M(x_{n+1}, Tx_*, t)} - 1 \right)$$

$$\leq \left(\frac{1}{M(x_*, x_{n+1}, t)} - 1\right) + \left(\frac{1}{M(x_{n+1}, Tx_n, t)} - 1\right) + \left(\frac{1}{M(Tx_n, Tx_*, t)} - 1\right)$$

$$\leq \left(\frac{1}{M(x_*, x_{n+1}, t)} - 1\right) + \frac{\varepsilon_n}{t} + \alpha\left(\frac{1}{M(x_n, x_*, t)} - 1\right).$$

In this case, we get $Tx_* = x_*$ because $x_n \in [x_*]_{\tilde{G}}$ for all n. Now, suppose that G is a (C)-graph. Since $(x_n, x_{n+1}) \in E(G)$ for all n, there exists a subsequence $\{x_{n_k}\}_{k\geq 0}$ such that $(x_{n_k}, x_*) \in E(G)$ for all $k \geq 0$. Thus,

$$\begin{aligned}\frac{1}{M(x_*, Tx_*, t)} - 1 &\leq \left(\frac{1}{M(x_*, x_{n_k+1}, t)} - 1\right) + \left(\frac{1}{M(x_{n_k+1}, Tx_*, t)} - 1\right)\\ &= \left(\frac{1}{M(x_*, x_{n_k+1}, t)} - 1\right) + \left(\frac{1}{M(T_{n_k}x_{n_k}, Tx_*, t)} - 1\right)\\ &\leq \left(\frac{1}{M(x_*, x_{n_k+1}, t)} - 1\right) + \left(\frac{1}{M(T_{n_k}x_{n_k}, Tx_{n_k}, t)} - 1\right)\\ &\quad + \left(\frac{1}{M(Tx_{n_k}, Tx_*, t)} - 1\right)\\ &\leq \left(\frac{1}{M(x_*, x_{n_k+1}, t)} - 1\right) + \frac{\varepsilon_{n_k}}{t}\\ &\quad + \alpha\left(\frac{1}{M(x_{n_k}, x_*, t)} - 1\right).\end{aligned}$$

In this case, we conclude that $Tx_* = x_*$. □

Theorem 3.6. *Let G be a weakly connected graph such that for each x, $y \in X$ we have*

$$\frac{1}{M(x, y, t)} - 1 = \frac{r(x, y)}{t}.$$

Let $T : X \to X$ be a G-fuzzy nonexpansive mapping such that T preserves the edges of G, $\{\varepsilon_i\}_{i=0}^{\infty}$ is a sequence in $(0, \infty)$ with

$$\sum_{i=0}^{\infty} \varepsilon_i < \infty,$$

and F is a nonempty subset of X. Assume that $\{T_i\}_{i\geq 0}$ is a sequence of selfmaps on X such that

$$\frac{1}{M(T_i x, Tx, t)} - 1 < \frac{\varepsilon_i}{t},$$

and $(T_i x, T_y) \in E(G)$ *for all* $x \in G$, $i \geq 0$ *and* $(x, y) \in E(G)$. *Let* x_0, $t_0 \in X$. *If the sequence* $\{x_{i+1} = Tx_i\}_{i=0}^{\infty}$ *satisfies*

$$\lim_{i\to\infty}\left[\frac{1}{M(x_i, F, t)} - 1\right] = 0,$$

then

$$\frac{1}{M(t_i, F, t)} - 1 \to 0,$$

where $\{t_{i+1} = T_i t_i\}_{i=0}^{\infty}$.

Proof. Let $\delta > 0$ be given. Choose a natural number p such that

$$\sum_{i=p}^{\infty} \varepsilon_i < \frac{\delta}{2}.$$

Since $[t_{p+1}]_{\tilde{G}} = X$, $x_{p+1} = Tx_p \in [t_{p+1}]_{\tilde{G}}$. Thus, there exists a path between t_{p+1} and x_{p+1}. Let $k \geq p$. If $\{\gamma_i\}_{i=0}^{N}$ be a path from t_{p+1} to x_{p+1}, then it is easy to see that for each $k \geq p+1$, t_k, $T^{k-(p+1)}\gamma_1$, $T^{k-(p+1)}\gamma_2, \ldots, T^{k-(p+1)}\gamma_{N-1}$, x_k is a path between t_k and x_k. But

$$\frac{1}{M(T_k\lambda_j, T^{k-(p+1)}\lambda_{j+1}, t)} - 1 \leq \frac{\varepsilon_k}{t} + \left(\frac{1}{M(\lambda_j, T^{k-(p+1)}\lambda_{j+1}, t)} - 1\right),$$

for all $(\lambda_j, \lambda_{j+1}) \in E(G)$ and $k \geq p+1$. Thus,

$$\begin{aligned}
r(t_k, x_k) &= t\left[\left(\frac{1}{M(t_k, T^{k-(p+1)}\gamma_1, t)} - 1\right)\right.\\
&\quad + \left(\frac{1}{M(T^{k-(p+1)}\gamma_1, T^{k-(p+1)}\gamma_2, t)} - 1\right)\\
&\quad \left. + \cdots + \left(\frac{1}{M(T^{k-(p+1)}\gamma_{N-1}, x_k, t)} - 1\right)\right]\\
&\leq t\left[\frac{\varepsilon_{k-1}}{t} + \left(\frac{1}{M(t_{k-1}, T^{k-(p+1)-1}\gamma_1, t)} - 1\right)\right.\\
&\quad + \left(\frac{1}{M(T^{k-(p+1)-1}\gamma_1, T^{k-(p+1)-1}\gamma_2, t)} - 1\right)\\
&\quad \left. + \cdots + \left(\frac{1}{M(T^{k-(p+1)-1}\gamma_{N-1}, x_{k-1}, t)} - 1\right)\right]\\
&= \varepsilon_{k-1} + r(t_{k-1}, x_{k-1}).
\end{aligned}$$

Now by using induction, we can show that

$$r(t_k, x_k) \leq \sum_{i=p}^{k} \varepsilon_i - \varepsilon_k.$$

Choose $q > p$ such that

$$\frac{1}{M(x_k, F, t)} - 1 < \frac{\delta}{2},$$

for all $k \geq q$. Then, for each $k \geq q$ we have

$$\begin{aligned}\frac{1}{M(t_k, F, t)} - 1 &\leq \left(\frac{1}{M(t_k, x_k, t)} - 1\right) + \left(\frac{1}{M(x_k, F, t)} - 1\right)\\ &\leq \sum_{i=k}^{\infty} \varepsilon_i + \frac{\delta}{2} < \frac{\delta}{2} + \frac{\delta}{2} = \delta.\end{aligned}$$

This shows that

$$\frac{1}{M(t_i, F, t)} - 1 \to 0.$$

□

References

1. R.P. Agarwal, M.A. El-Gebeily, and D. O'Regan, *Generalized contractions in partially ordered metric spaces*, Appl. Analysis, 87 (2008), pp. 109-116.
2. C. Alaca, D. Turkoghlu, and C. Yildiz, *Fixed points in intuitionistic fuzzy metric spaces*, Chaos, Solitons and Fractals, 29 (2006), pp. 1073-1078.
3. S.M.A. Aleomraninejad, Sh. Rezapour, and N. Shahzad, *Some fixed point results on metric space witha graph*, Topology Appl., 159 (2012), pp. 659-663.
4. I. Altun and G. Durmaz, *Some fixed point results in cone metric spaces*, Rend Circ. Math. Palermo, 58 (2009), pp. 319-325.
5. K. Atanassov, *Intuitionistic fuzzy Sets*, Fuzzy Sets and Systems, 20 (1986), pp. 87-96.
6. S. Banach, *Sur les operations dans les ensembles abstraits et leur application aux equations integrales*, Fund. Math., 3 (1922), pp. 133-181.
7. I. Beg, A.R. Butt, and S. Radojević, *The contraction principle for set valued mappings on a metric space with a graph*, Comput. Math. Appl., 60 (2010), pp. 1214-1219.
8. V. Berinde, *Generalized coupled fixed point theorems for mixed monotone mappings in partially ordered metric spaces*, Nonlinear Analysis, 74 (2011), pp. 7347-7355.
9. J. Caristi, *Fixed point theorems for mapping satisfying inwardness conditions*, Trans. Amer. Math. Soc., 215 (1976), pp. 241-251.
10. D. Coker, *An introduction to intuitionistic fuzzy metric spaces*, Fuzzy Sets and Systems, 88 (1997), pp. 81-89.

11. C. Di Bari and, C. Vetro, *A fixed point theorem for a family of mappings in a fuzzy metric space*, Rend Circ. Math. Palermo, 52 (2003), pp. 315-321.
12. F. Echenique, *A short and constructive proof of Tarski's fixed point theorem*, Internat, J. Game Theory, 33 (2005), pp. 215-218.
13. M.S. El Naschie, *A review of E-infinity theory and the mass spectrum of high energy particle physics*, Chaos, solitons and Fractals, 19 (2004), pp. 209-236.
14. M.S. El Naschie, *On a fuzzy Kähler-like manifold which is consistent with two-slit experiment*, Int. J. Nonlinear Science and Numerical Simul., 6 (2005), pp. 95-98.
15. M.S. El Naschie, *On the uncertainty of Contorian geometry and two-slit experiment*, Chaos, Soliton and Fractals, 9 (1998), pp. 517-529.
16. M.S. El Naschie, *On the verification of heterotic strings theory and* $\epsilon^{(\infty)}$ *theory*, Chaos, Soliton and Fractals, 11 (2000), pp. 2397-2407.
17. M.S. El Naschie, *On two new fuzzy K"ahler manifolds, Klein modular space and Hooft holographic principles*, Chaos, Solitons and Fractals, 29 (2006), pp. 876-881.
18. M.S. El Naschie, *The idealized quantum two-slit Gedanken experiment revisited criticism and reinterpretation*, Chaos, Solitons and Fractals, 27 (2006), pp. 843-849.
19. R. Espinola and W.A. Kirk, *Fixed point theorems in R-trees with applications to graph theory*, Topology Appl., 153 (2006), pp. 1046-1055.
20. M. Grabiec, *Fixed points in fuzzy metric spaces*, Fuzzy Sets and Systems, 27 (1988), pp. 385-389.
21. A. George and P. Veeramani, *On some results in fuzzy metric spaces*, Fuzzy Sets and Systems, 64 (1994), pp. 395-399.
22. V. Gregori and A. Sapena, *On fixed point theorems in fuzzy metric spaces*, Fuzzy Sets and Systems, 125 (2002), pp. 245-252.
23. G. Gwozdz-Lukawska and J. Jachymski, *IFS on metric space with a graph structure and extentions of the Kelisky-Rivilin theorem*, J. Math. Anal. Appl., 356 (2009), pp. 453-463.
24. T.L. Hicks, *Fixed point theorems for quasi-metric spaces*, Math. Japon., 33 (1988), pp. 231-236.
25. J. Jachymski, *The contraction principle for mappings on metric space with a graph*, Proc, Amer. Math. Soc., 136 (2008), pp. 1359-1373.
26. S.G. Jeong and B.E. Rhoades, *More maps for which* $F(T) = F(T^n)$, Demonestraio Math., 40 (2007), pp. 671-680.
27. H. Karayilan and M. Telci, *Common fixed point theorem for contractive type mappings in fuzzy metric spaces*, Rend. Circ. Mat.

Palermo., 60 (2011), pp. 145-152.
28. O. Kramosil and J. Michalek, *Fuzzy metric and statistical metric spaces*, Kybernetica, 11 (1975), pp. 326-334.
29. K. Menger, *Statistical metrices*, Proc. Natl. Acad. Sci., 28 (1942) 535-537.
30. D. Mihet, *A banach contraction theorem in fuzzy metric spaces*, Fuzzy Sets and Systems, 144 (2004), pp. 431-439.
31. J.H. Park, *Intiutionistic fuzzy metric spaces*, Chaos, Solitons and Fractals, 22 (2004), pp. 1039-1046.
32. J.S. Park, Y.C. Kwun, and J.H. Park, *A fixed point theorem in the intiutionistic fuzzy metric spaces*, Far East J. Math. Sci., 16 (2005), pp. 137-149.
33. M. Rafi and M.S.M. Noorani, *Fixed point theorem on intuitionistic fuzzy metric spaces*, Iranian J. of Fuzzy Systems, 3 (2006), pp. 23-29.
34. Sh. Rezapour and P. Amiri, *Some fixed point results for multivalued operators in generalized metric spaces*, Computers and Mathematics with Applications, 61 (2011) 2661-2666.
35. Sh. Rezapour and R. Hamlbarani, *Cone metric spaces and fixed point theorems of contractive mappings*, J. Math. Analysis Appl., 345 (2008), pp. 719-724.
36. J. Rodriquez-Lopez and S. Romaguera, *The Hausdorff fuzzy metric on compact sets*, Fuzzy Sets and Systems, 147 (2004), pp. 173-176.
37. B. Samet, C. Vetro, and P. Vetro, *Fixed point theorem for α-ψ-contractive type mappings*, Nonlinear Analysis, 75 (2012), pp. 2154-2165.
38. B. Schweizer and A. Sklar, *Statistical metric spaces*, Pacific J. Math., 10 (1960), pp. 314-334.
39. L.D.J. Sigalotte and A. Mejias, *On El Naschie's conjugate complex time, fractal $E(\infty)$ space-time and faster-than-light particles*, Int. J. Nonlinear Sci. Number Simul., 7 (2006), pp. 467-472.
40. P. Veeramani, *Best approximation in fuzzy metricspaces*, Journal of fuzzy mathematics, 9 (2001), pp. 75-80.
41. L.A. Zadeh, *Fuzzy sets*, Information and Control, 8 (1965), pp. 338-353.

Department of Mathematics, Faculty of Science, University of Bu-Ali Sina, P.O.Box 6517838695, Hamedan, IRAN.

E-mail address: mesamei@basu.ac.ir