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Rational Geraghty Contractive Mappings and Fixed Point Theorems in Ordered b_2-metric Spaces

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Abstract. In 2014, Zead Mustafa introduced b_2-metric spaces, as a generalization of both 2-metric and b-metric spaces. Then new fixed point results for the classes of rational Geraghty contractive mappings of type I,II and III in the setup of b_2-metric spaces are investigated. Then, we prove some fixed point theorems under various contractive conditions in partially ordered b_2-metric spaces. These include Geraghty-type conditions, conditions that use comparison functions and almost generalized weakly contractive conditions. Berinde in [17–20] initiated the concept of almost contractions and obtained many interesting fixed point theorems. Results with similar conditions were obtained, *e.g.*, in [21] and [22]. In the last section of the paper, we define the notion of almost generalized $(\psi, \varphi)_{s,a}$-contractive mappings and prove some new results. In particular, we extend Theorems 2.1, 2.2 and 2.3 of Ćirić et.al. in [23] to the setting of b_2-metric spaces. Also, some examples are provided to illustrate the results presented herein and several interesting consequences of our theorems are also provided. The findings of the paper are based on generalization and modification of some recently reported theorems in the literature.

1. Introduction and Preliminaries

The concept of metric spaces has been generalized in many directions. The notion of a b-metric space was studied by S.G. Matthews. Many fixed point results were obtained for single and multivalued mappings by Czerwik and many other authors in [1, 2].

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On the other hand, the notion of a 2-metric was introduced by Gähler in [3], having the area of a triangle in $\mathbb{R}^2$ as the inspirative example. Similarly, several fixed point results were obtained for mappings in such spaces. Note that, unlike to many other generalizations of metric spaces introduced recently, 2-metric spaces are not topologically equivalent to metric spaces and there is no easy relationship between the results obtained in 2-metric and metric spaces. In this paper, we introduce a new type of generalized metric spaces, which we call b_2-metric spaces, as a generalization of both 2-metric and b-metric spaces. Then, we prove some fixed point theorems under various contractive conditions in partially ordered b_2-metric spaces. These include Geraghty-type conditions, conditions that use comparison functions and almost generalized weakly contractive conditions.

We illustrate these results by appropriate examples.

2. Mathematical Preliminaries

Definition 2.1 ([1])**.** Let X be a (nonempty) set and $s \geq 1$ be a given real number. A function $d : X \times X \to \mathbb{R}^+$ is a b-metric on X if, for all $x, y, z \in X$, the following conditions hold:

(b_1) $d(x,y) = 0$ if and only if $x = y$,
(b_2) $d(x,y) = d(y,x)$,
(b_3) $d(x,z) \leq s[d(x,y) + d(y,z)]$.

In this case, the pair (X,d) is called a b-metric space.

On the other hand, the notion of a 2-metric was introduced by Gähler in [3].

Definition 2.2 ([3])**.** Let X be a non-empty set and let $d : X^3 \to \mathbb{R}$ be a map satisfying the following conditions:

(1) For every pair of distinct points $x, y \in X$, there exists a point $z \in X$ such that
$$d(x,y,z) \neq 0.$$
(2) If at least two of three points x, y, z are the same, then
$$d(x,y,z) = 0.$$
(3) The symmetry:
$$d(x,y,z) = d(x,z,y) = d(y,x,z) = d(y,z,x) = d(z,x,y) = d(z,y,x),$$
for all $x, y, z \in X$.
(4) The rectangle inequality:
$$d(x,y,z) \leq d(x,y,t) + d(y,z,t) + d(z,x,t),$$
for all $x, y, z, t \in X$.

Then d is called a 2-metric on X and (X, d) is called a 2-metric space.

For some fixed point results on 2-metric spaces, the readers may refer to [5–14].

Very recently, Mustafa et.al. [26] introduced a new structure of generalized metric spaces, called b_2-metric spaces, as a generalization of 2-metric spaces.

Definition 2.3 ([26])**.** Let X be a non-empty set, $s \geq 1$ be a real number and let $d : X^3 \to \mathbb{R}$ be a map satisfying the following conditions:

(1) For every pair of distinct points $x, y \in X$, there exists a point $z \in X$ such that

$$d(x, y, z) \neq 0.$$

(2) If at least two of three points x, y, z are the same, then

$$d(x, y, z) = 0.$$

(3) The symmetry:

$$d(x, y, z) = d(x, z, y) = d(y, x, z) = d(y, z, x) = d(z, x, y) = d(z, y, x),$$

for all $x, y, z \in X$.

(4) The rectangle inequality:

$$d(x, y, z) \leq s[d(x, y, t) + d(y, z, t) + d(z, x, t)],$$

for all $x, y, z, t \in X$.

Then d is called a b_2-metric on X and (X, d) is called a b_2-metric space with parameter s.

Definition 2.4 ([26])**.** Let $\{x_n\}$ be a sequence in a b_2-metric space (X, d).

(1) $\{x_n\}$ is said to be b_2-convergent to $x \in X$, written as $\lim_n x_n = x$, if for all $a \in X$, $\lim_n d(x_n, x, a) = 0$.

(2) $\{x_n\}$ is said to be a b_2-Cauchy sequence in X if for all $a \in X$, $\lim_n d(x_n, x_m, a) = 0$.

(3) (X, d) is said to be b_2-complete if every b_2-Cauchy sequence in (X, d) is a b_2-convergent sequence in it.

The following are some easy examples of b_2-metric spaces.

Example 2.5 ([26])**.** Let $X = [0, +\infty)$ and $d(x, y, z) = [xy + yz + zx]^p$ if $x \neq y \neq z \neq x$, and otherwise $d(x, y, z) = 0$, where $p \geq 1$ is a real number. One can see that (X, d) is a b_2-metric space with $s = 3^{p-1}$.

Example 2.6 ([26]). Let a mapping $d : \mathbb{R}^3 \to [0, +\infty)$ be defined by

$$d(x, y, z) = \min\{|x - y|, |y - z|, |z - x|\}.$$

Then d is a 2-metric on $\mathbb{R}$ and

$$d_p(x, y, z) = [\min\{|x - y|, |y - z|, |z - x|\}]^p,$$

is a b_2-metric on $\mathbb{R}$ with $s = 3^{p-1}$.

Definition 2.7 ([26]). Let (X, d) and (X', d') be two b_2-metric spaces and let $f : X \to X'$ be a mapping. Then f is said to be b_2-continuous at a point $z \in X$ if for a given $\varepsilon > 0$, there exists $\delta > 0$ such that $x \in X$ and $d(z, x, a) < \delta$ for all $a \in X$ imply that $d'(fz, fx, a) < \varepsilon$. The mapping f is b_2-continuous on X if it is b_2-continuous at all $z \in X$.

Proposition 2.8 ([26]). *Let (X, d) and (X', d') be two b_2-metric spaces. Then a mapping $f : X \to X'$ is b_2-continuous at a point $x \in X$ if and only if it is b_2-sequentially continuous at x; that is, whenever $\{x_n\}$ is b_2-convergent to x, $\{fx_n\}$ is b_2-convergent to $f(x)$.*

We will need the following simple lemma about the b_2-convergent sequences in the proof of our main results.

Lemma 2.9 ([26]). *Let (X, d) be a b_2-metric space and suppose that $\{x_n\}$ and $\{y_n\}$ are b_2-convergent to x and y, respectively. Then we have*

$$\frac{1}{s^2} d(x, y, a) \le \liminf_{n\to\infty} d(x_n, y_n, a) \le \limsup_{n\to\infty} d(x_n, y_n, a) \le s^2 d(x, y, a),$$

for all $a \in X$. In particular, if $y_n = y$ is constant, then

$$\frac{1}{s} d(x, y, a) \le \liminf_{n\to\infty} d(x_n, y, a) \le \limsup_{n\to\infty} d(x_n, y, a) \le s d(x, y, a),$$

for all $a \in X$.

3. Main Results

3.1. **Results Under Rational Geraghty Condition of Type I.** In 1973, Geraghty [15] proved a fixed point result, generalizing the Banach contraction principle. Later several authors proved various results using the Geraghty-type conditions (See, e.g., [16]). Following [16], for a real number $s \ge 1$, let $\mathcal{F}_s$ denote, the class of all functions $\beta : [0, \infty) \to [0, \frac{1}{s})$ satisfying the following condition:

$$\beta(t_n) \to \frac{1}{s} \text{ as } n \to \infty \text{ implies } \ t_n \to 0 \text{ as } n \to \infty.$$

Definition 3.1. Let (X, d) be a complete 2-metric space. Assume that $f : X \to X$ be a self mapping and $\alpha : X \times X \times X \to [0, \infty)$ be a function. We say that T is a $2-\alpha$- admissible mapping if for all $a \in X$

$$x, y \in X, \qquad \alpha(x, y, a) \ge 1 \quad \Rightarrow \quad \alpha(Tx, Ty, a) \ge 1.$$

The following theorem extend Theorem 1 of [26] and also is the extension of Theorem 5 of [27].

Theorem 3.2. *Let (X,d) be a b_2 complete b_2-metric space (with parameter $s>1$), $f:X\to X$ be a self mapping and $\alpha: X\times X\times X\to[0,\infty)$ be a function such that f is an $2-\alpha$- admissible mapping. Suppose that*
(3.1)
$$s\alpha(x,fx,a)\alpha(y,fy,a)d(fx,fy,a)\le\beta(d(x,y,a))M(x,y,a)+LN(x,y,a),$$

for all elements $x,y,a\in X$, where

$$M(x,y,a)=\max\left\{d(x,y,a),\frac{d(x,fx,a)d(y,fy,a)}{1+d(x,y,a)},\frac{d(x,fx,a)d(y,fy,a)}{1+d(fx,fy,a)}\right\},$$

and

$$N(x,y,a)=\min\left\{d(x,fx,a),d(x,fy,a),d(y,fx,a),d(y,fy,a)\right\}.$$

Assume that f is b_2-continues and if there exist $x_0\in X$ such that $\alpha(x_0,fx_0,a)\ge1$, then f has a fixed point.

Proof. Let $x_0\in X$ such that $\alpha(x_0,fx_0,a)\ge1$. Define a sequence $\{x_n\}$ in X by

$$x_n=f^nx_0=fx_{n-1},$$

for all $n\in\mathbb{N}$. Since f is a $2-\alpha$-admissible mapping and $\alpha(x_0,fx_0,a)\ge1$, we deduce that $\alpha(x_1,fx_1,a)=\alpha\left(fx_0,f^2x_0,a\right)\ge1$. By continuing this process, we get that $\alpha\left(x_n,fx_n,a\right)\ge1$ for all $n\in\mathbb{N}\cup\{0\}$. Then,

$$\alpha(x_n,fx_n,a)\alpha(x_{n-1},fx_{n-1},a)\ge1,$$

for all $n\in\mathbb{N}\cup\{0\}$.
Step I: We will show that $\lim_n d(x_n,x_{n+1},a)=0$. By (3.1), we have

(3.2)
$$\begin{aligned}sd(x_n,x_{n+1},a)&=sd(fx_{n-1},fx_n,a)\\&\le s\alpha(x_{n-1},fx_{n-1},a)\alpha(x_n,fx_n,a)d(fx_{n-1},fx_n,a)\\&\le\beta(d(x_{n-1},x_n,a))M(x_{n-1},x_n,a)+LN(x_{n-1},x_n,a)\\&\le\frac{1}{s}d(x_{n-1},x_n,a)\\&\le d(x_{n-1},x_n,a),\end{aligned}$$

because

$$M(x_{n-1},x_n,a)=\max\left\{d(x_{n-1},x_n,a),\frac{d(x_{n-1},fx_{n-1},a)d(x_n,fx_n,a)}{1+d(fx_{n-1},fx_n,a)},\right.$$
$$\left.\frac{d(x_{n-1},fx_{n-1},a)d(x_n,fx_n,a)}{1+d(x_{n-1},x_n,a)}\right\}$$
$$=\max\left\{d(x_{n-1},x_n,a),\frac{d(x_{n-1},x_n,a)d(x_n,x_{n+1},a)}{1+d(x_n,x_{n+1},a)},\right.$$
$$\left.\frac{d(x_{n-1},fx_{n-1},a)d(x_n,fx_n,a)}{1+d(x_{n-1},x_n,a)}\right\}$$
$$\leq\max\left\{d(x_{n-1},x_n,a),d(x_n,x_{n+1},a)\right\},$$

and

$$N(x_{n-1},x_n,a)=\min\{d(x_{n-1},fx_n,a),d(x_n,fx_n,a),$$
$$d(x_{n-1},fx_{n-1},a),d(x_n,fx_{n-1},a)\}$$
$$=\min\{d(x_{n-1},x_{n+1},a),d(x_n,x_{n+1},a),$$
$$d(x_{n-1},x_n,a),d(x_n,x_n),a)\}$$
$$=0.$$

If

$$\max\{d(x_{n-1},x_n,a),d(x_n,x_{n+1},a)\}=d(x_n,x_{n+1},a),$$

then, from(3.2), we have

$$\begin{aligned}d(x_n,x_{n+1},a)&\leq\beta(M(x_n,x_{n+1},a))d(x_n,x_{n+1},a)\\&<\frac{1}{s}d(x_n,x_{n+1},a)\\&<d(x_n,x_{n+1},a),\end{aligned}\tag{3.3}$$

which is a contradiction. Hence,

$$\max\{d(x_{n-1},x_n),d(x_n,x_{n+1},a)\}=d(x_{n-1},x_n,a),$$

and so from (3.2),

$$d(x_n,x_{n+1},a)\leq\beta(M(x_{n-1},x_n,a))d(x_{n-1},x_n,a).\tag{3.4}$$

Therefore, the sequence $\{d(x_n,x_{n+1},a)\}$ is decreasing. Then there exists $r\geq 0$ such that $\lim_n d(x_n,x_{n+1},a)=r$. Suppose that $r>0$. Then, letting $n\to\infty$, from (3.2) we have

$$\frac{1}{s}r\leq sr\leq\lim_n\beta(d(x_{n-1},x_n,a))r\leq r.$$

So, we have $\lim_n\beta(d(x_{n-1},x_n,a))\geq\frac{1}{s}$ and since $\beta\in\mathcal{F}_s$ we deduce that $\lim_n d(x_{n-1},x_n,a)=0$ which is a contradiction. Hence, $r=0$, that is,

$$\lim_n d(x_n,x_{n+1},a)=0.\tag{3.5}$$

Step II: Now, we prove that the sequence $\{x_n\}$ is a b_2-Cauchy sequence. Using the rectangle inequality and by (3.1), we have

$$\begin{aligned}
d(x_n, x_m, a) &\leq sd(x_n, x_m, x_{n+1}) + sd(x_m, a, x_{n+1}) + sd(a, x_n, x_{n+1}) \\
&\leq sd(x_n, x_{n+1}, x_m) + s^2[d(x_m, x_{m+1}, a) + d(x_{n+1}, x_{m+1}, a) \\
&\quad + d(x_m, x_{m+1}, x_{n+1})] + sd(x_n, x_{n+1}, a) \\
&\leq sd(x_n, x_{n+1}, x_m) + s^2 d(x_m, x_{m+1}, a) \\
&\quad + s^2 \alpha(x_n, fx_n, a)\alpha(x_m, fx_m, a) d(x_{n+1}, x_{m+1}, a) \\
&\quad + s^2 d(x_m, x_{m+1}, x_{n+1}) + sd(x_n, x_{n+1}, a) \\
&\leq sd(x_n, x_{n+1}, x_m) + s^2 d(x_m, x_{m+1}, a) \\
&\quad + s\beta(d(x_n, x_m, a))M(x_n, x_m, a) + LN(x_n, x_m, a) \\
&\quad + s^2 d(x_m, x_{m+1}, x_{n+1}) + sd(x_n, x_{n+1}, a).
\end{aligned}$$

Letting $m, n \to \infty$ in the above inequality and applying (3.5), we have

$$\begin{aligned}
\lim_{m,n\to\infty} d(x_n, x_m, a) &\leq s \lim_{m,n\to\infty} \beta(d(x_n, x_m, a)) \lim_{m,n\to\infty} M(x_n, x_m, a) \\
&\quad + \lim_{m,n\to\infty} LN(x_n, x_m, a).
\end{aligned} \tag{3.6}$$

Here,

$$\begin{aligned}
d(x_n, x_m, a) &\leq M(x_n, x_m, a) \\
&= \max\left\{ d(x_n, x_m, a), \frac{d(x_n, fx_n, a)d(x_m, fx_m, a)}{1 + d(fx_n, fx_m, a)}, \right. \\
&\quad \left. \frac{d(x_n, fx_n, a)d(x_m, fx_m, a)}{1 + d(x_n, x_m, a)} \right\} \\
&= \max\left\{ d(x_n, x_m, a), \frac{d(x_n, x_{n+1}, a)d(x_m, x_{m+1}, a)}{1 + d(x_{n+1}, x_{m+1}, a)}, \right. \\
&\quad \left. \frac{d(x_n, x_{n+1}, a)d(x_m, x_{m+1}, a)}{1 + d(x_n, x_m, a)} \right\}.
\end{aligned}$$

Letting $m, n \to \infty$ in the above inequality, we get

$$\lim_{m,n\to\infty} M(x_n, x_m, a) = \lim_{m,n\to\infty} d(x_n, x_m, a), \tag{3.7}$$

and

$$
\begin{aligned}
\lim_{m,n\to\infty} N(x_n, x_m, a) &= \min\{d(x_n, fx_n, a), d(x_n, fx_m, a),\\
&\quad (x_m, fx_n, a), d(x_m, fx_m, a)\}\\
&= \lim_{m,n\to\infty} \min\{d(x_n, x_{n+1}, a), d(x_n, x_{m+1}, a),\\
&\quad d(x_m, x_{n+1}, a), d(x_m, x_{m+1}, a)\}\\
&= 0.
\end{aligned}
$$

Hence, from(3.6) and(3.7), we obtain

(3.8)
$$\lim_{m,n\to\infty} d(x_n, x_m, a) \leq s \lim_{m,n\to\infty} \beta(d(x_n, x_m, a)) \lim_{m,n\to\infty} d(x_n, x_m, a).$$

Now we claim that, $\lim_{m,n\to\infty} d(x_n, x_m, a) = 0$. On the contrary, if $\lim_{m,n\to\infty} d(x_n, x_m, a) \neq 0$, then we get

$$\frac{1}{s} \leq \lim_{m,n\to\infty} \beta(d(x_n, x_m, a)).$$

Since $\beta \in \mathcal{F}_s$ we deduce that

$$\lim_{m,n\to\infty} d(x_n, x_m, a) = 0, \tag{3.9}$$

Which is a contradiction. Consequently, $\{x_n\}$ is a b_2-Cauchy sequence in X. Since (X, d) is b_2-complete, the sequence $\{x_n\}$ b_2-converges to some $z \in X$, that is, $\lim_n d(x_n, z, a) = 0$.

Step III: Now, we show that z is a fixed point of f.

Using the rectangle inequality, we get

$$d(fz, z, a) \leq sd(fz, fx_n, z) + sd(z, a, fx_n) + sd(a, fz, fx_n).$$

Letting $n \to \infty$ and using the continuity of f, we have $fz = z$. Thus, z is a fixed point of f. □

Note that the continuity of f in Theorem 3.2 is not necessary and can be dropped.

Theorem 3.3. *Under the hypotheses of Theorem 3.2, without the b_2-continuity assumption on f, assume that if a sequence $\{x_n\}$ is such that $x_n \to x$ as $n \to \infty$, one has $x_n \preceq x$ for all $n \in \mathbb{N}$, and $\alpha(x_n, fx_n, a) \geq 1$ for all n, then $\alpha(x, fx, a) \geq 1$. If there exists $x_0 \in X$ such that $\alpha(x_0, fx_0, a) \geq 1$, then f has a fixed point*

Proof. Repeating the proof of Theorem 3.2, we construct an increasing sequence $\{x_n\}$ in X such that $x_n \to z \in X$. Using the assumptions on

X we have $x_n \preceq z$. Now, we show that $z = fz$. By (3.1) and Lemma 2.9,

$$\begin{aligned} s\left[\frac{1}{s}d(z,fz,a)\right] &\le s\limsup_{n\to\infty} d(x_{n+1},fz,a) \\ &\le \limsup_{n\to\infty} \alpha(x_n,fx_n,a)\alpha(z,fz,a)d(x_{n+1},fz,a) \\ &\le \limsup_{n\to\infty} \beta(d(x_n,z,a)) \limsup_{n\to\infty} M(x_n,z,a) \\ &\quad + \limsup_{n\to\infty} LN(x_n,z,a), \end{aligned}$$

where,

$$\begin{aligned} \lim_{n\to\infty} M(x_n,z,a) &= \lim_n \max\left\{d(x_n,z,a), \frac{d(x_n,fx_n,a)d(z,fz,a)}{1+d(fx_n,fz,a)},\right. \\ &\quad \left.\frac{d(x_n,fx_n,a)d(z,fz,a)}{1+d(x_n,z,a)}\right\} \\ &= \lim_n \max\left\{d(x_n,z,a), \frac{d(x_n,x_{n+1},a)d(z,fz,a)}{1+d(fx_n,fz,a)},\right. \\ &\quad \left.\frac{d(x_n,x_{n+1},a)d(z,fz,a)}{1+d(x_n,z,a)}\right\} \\ &= 0 \end{aligned}$$

(see(3.5)) and

$$\begin{aligned} \lim_{n\to\infty} N(x_n,z,a) &= \lim_{n\to\infty} \min\{d(x_n,fz,a), d(z,fx_n,a), \\ &\quad d(x_n,fx_n,a), d(z,fz,a)\} \\ &= \lim_{n\to\infty} \min\{d(x_n,fz,a), d(z,x_{n+1},a), \\ &\quad d(x_n,x_{n+1},a), d(z,fz,a)\} \\ &= 0. \end{aligned}$$

Therefore, we deduce that $d(z,fz,a) \le 0$. Since a is arbitrary, we have $z = fz$. □

Definition 3.4. Let (X,d) be a b_2-metric space. A mapping $f : X \to X$ is called a rational Geraghty contraction of type II if, there exists $\beta \in \mathcal{F}$ such that,

(3.10)
$$s\alpha(x,fx,a)\alpha(y,fy,a)d(fx,fy,a) \le \beta(M(x,y,a))M(x,y,a)+LN(x,y,a),$$

for all elements $x, y \in X$, where

$$M(x,y,a) = \max\left\{d(x,y,a), \frac{d(x,fx,a)d(x,fy,a)+d(y,fy,a)d(y,fx,a)}{1+s[d(x,fx,a)+d(y,fy,a)+d(x,y,fy)]}, \frac{d(x,fx,a)d(x,fy,a)+d(y,fy,a)d(y,fx,a)}{1+d(x,fy,a)+d(y,fx,a)}\right\},$$

and

$$N(x,y,a) = \min\left\{d(x,fx,a), d(x,fy,a), d(y,fx,a), d(y,fy,a)\right\}.$$

Theorem 3.5. *Let (X,d) be a b_2 complete b_2-metric space (with parameter $s>1$), $f : X \to X$ be a self mapping and $\alpha : X \times X \times X \to [0,\infty)$ be a function such that f is a 2-α-admissible mapping. Suppose that f be a rational Geraghty contractive mapping of type II. If*
(I) f is continuous, or,
(II) assume that if a sequence $\{x_n\}$ is such that $x_n \to x$ as $n \to \infty$ and $\alpha(x_n, fx_n, a) \geq 1$ for all n, then $\alpha(x, fx, a) \geq 1$. If there exist $x_0 \in X$ such that $\alpha(x_0, fx_0, a) \geq 1$, then f has a fixed point,

Proof. Let $x_0 \in X$ such that $\alpha(x_0, fx_0, a) \geq 1$. Define a sequence $\{x_n\}$ in X by

$$x_n = f^n x_0 = fx_{n-1},$$

for all $n \in \mathbb{N}$. Since f is a 2-α-admissible mapping and $\alpha(x_0, fx_0, a) \geq 1$, we deduce that $\alpha(x_1, fx_1, a) = \alpha(fx_0, f^2x_0, a) \geq 1$. By continuing this process, we get that $\alpha(x_n, fx_n, a) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$. Then,

$$\alpha(x_n, fx_n, a)\alpha(x_{n-1}, fx_{n-1}, a) \geq 1,$$

for all $n \in \mathbb{N} \cup \{0\}$. We will do the proof in the following steps.
Step I: We will show that $\lim\limits_{n\to\infty} d(x_n, x_{n+1}, a) = 0$. Since $x_n \preceq x_{n+1}$, then for each $n \in \mathbb{N}$, by (3.10), we have

(3.11)

$$\begin{aligned} sd(x_n, x_{n+1}, a) &= sd(fx_{n-1}, fx_n, a) \\ &\leq s\alpha(x_{n-1}, fx_{n-1}, a)\alpha(x_n, fx_n, a)d(fx_{n-1}, fx_n, a) \\ &\leq \beta(d(x_{n-1}, x_n, a))M(x_{n-1}, x_n, a) + LN(x_{n-1}, x_n, a) \\ &\leq \frac{1}{s}d(x_{n-1}, x_n, a) \\ &\leq d(x_{n-1}, x_n, a), \end{aligned}$$

because

$$\begin{aligned}M(x_{n-1},x_n,a) = \max\Big\{ & d(x_{n-1},x_n,a),\\ & \frac{d(x_{n-1},fx_{n-1},a)d(x_{n-1},fx_n,a)+d(x_n,fx_n,a)d(x_n,fx_{n-1},a)}{1+s[d(x_{n-1},fx_{n-1},a)+d(x_n,fx_n,a)+d(x_{n-1},x_n,x_{n+1})]},\\ & \frac{d(x_{n-1},fx_{n-1},a)d(x_{n-1},fx_n,a)+d(x_n,fx_n,a)d(x_n,fx_{n-1},a)}{1+d(x_{n-1},fx_n,a)+d(x_n,fx_{n-1},a)}\Big\}\\ = \max\Big\{ & d(x_{n-1},x_n,a),\\ & \frac{d(x_{n-1},x_n,a)d(x_{n-1},x_{n+1},a)+d(x_n,x_{n+1},a)d(x_n,x_n,a)}{1+s[d(x_{n-1},x_n,a)+d(x_n,x_{n+1},a)+d(x_{n-1},x_n,x_{n+1})]},\\ & \frac{d(x_{n-1},x_n,a)d(x_{n-1},x_{n+1},a)+d(x_n,x_{n+1})d(x_n,x_n,a)}{1+d(x_{n-1},x_{n+1},a)+d(x_n,x_n,a)}\Big\}\\ = {} & d(x_{n-1},x_n,a).\end{aligned}$$

Since

(3.12)

$$d(x_{n-1},x_{n+1},a) < s[d(x_{n-1},x_{n+1},t)+d(x_{n+1},a,t)+d(a,x_{n-1},t)],$$

and

$$\begin{aligned}N(x_{n-1},x_n,a) &= \min\{d(x_{n-1},fx_n,a),d(x_n,fx_n,a),\\ &\quad d(x_{n-1},fx_{n-1},a),d(x_n,fx_{n-1},a)\}\\ &= \min\{d(x_{n-1},x_{n+1},a),d(x_n,x_{n+1},a),\\ &\quad d(x_{n-1},x_n,a),d(x_n,x_n),a\}\\ &= 0,\end{aligned}$$

from (3.12) and taking $t = x_n$, we have:

(3.13)

$$d(x_{n-1},x_{n+1},a) < s[d(x_{n-1},x_{n+1},x_n)+d(x_{n+1},a,x_n)+d(a,x_{n-1},x_n)],$$

Therefore, $\{d(x_n,x_{n+1},a)\}$ is decreasing. Then there exists $r \geq 0$ such that $\lim_{n\to\infty} d(x_n,x_{n+1},a) = r$. We will prove that $r = 0$. Suppose on contrary that $r > 0$. Then, letting $n \to \infty$, from (3.2), we have

$$\frac{1}{s}r \leq \lim_{n\to\infty} \beta\,(d(x_{n-1},x_n,a))r,$$

which implies that $d(x_{n-1},x_n,a) \to 0$. Hence, $r = 0$ which is a contradiction. So,

$$\lim_{n\to\infty} d(x_{n-1},x_n,a) = 0, \tag{3.14}$$

holds true.

Step II: Now, we prove that the sequence $\{x_n\}$ is a b_2-Cauchy sequence. Suppose the contrary, i.e., that $\{x_n\}$ is not a b_2-Cauchy sequence. Then there exists $\varepsilon > 0$ for which we can find two subsequences $\{x_{m_i}\}$ and $\{x_{n_i}\}$ of $\{x_n\}$ such that n_i is the smallest index for which

$$n_i > m_i > i, \qquad d(x_{m_i}, x_{n_i}, a) \geq \varepsilon. \tag{3.15}$$

This means that

$$d(x_{m_i}, x_{n_i-1}, a) < \varepsilon. \tag{3.16}$$

As in the proof of Theorem (3.2), we have,

$$\frac{\varepsilon}{s} \leq \limsup_{i\to\infty} d(x_{m_i+1}, x_{n_i}, a). \tag{3.17}$$

From the definition of $M(x, y, a)$ and the above limits, we obtain

$$\begin{aligned}
&\limsup_{i\to\infty} M(x_{m_i}, x_{n_i-1}, a)\\
&\quad= \limsup_{i\to\infty} \max\Big\{ d(x_{m_i}, x_{n_i-1}, a),\\
&\quad\frac{d(x_{m_i}, fx_{m_i}, a)d(x_{m_i}, fx_{n_i-1}, a) + d(x_{n_i-1}, fx_{n_i-1}, a)d(x_{n_i-1}, fx_{m_i}, a)}{1 + s[d(x_{m_i}, fx_{m_i}, a) + d(x_{n_i-1}, fx_{n_i-1}, a) + d(x_{m_i}, x_{n_i-1}, fx_{n_i-1})]},\\
&\quad\frac{d(x_{m_i}, fx_{m_i}, a)d(x_{m_i}, fx_{n_i-1}, a) + d(x_{n_i-1}, fx_{n_i-1}, a)d(x_{n_i-1}, fx_{m_i}, a)}{1 + d(x_{m_i}, fx_{n_i-1}, a) + d(x_{n_i-1}, fx_{m_i}, a)}\Big\}\\
&\quad= \limsup_{i\to\infty} \max\Big\{ d(x_{m_i}, x_{n_i-1}, a),\\
&\quad\frac{d(x_{m_i}, x_{m_i+1}, a)d(x_{m_i}, x_{n_i}, a) + d(x_{n_i-1}, x_{n_i}, a)d(x_{n_i-1}, x_{m_i+1}, a)}{1 + s[d(x_{m_i}, x_{m_i+1}, a) + d(x_{n_i-1}, x_{n_i}, a), d(x_{m_i}, x_{n_i-1}, x_{n_i})]},\\
&\quad\frac{d(x_{m_i}, x_{m_i+1}, a)d(x_{m_i}, x_{n_i}, a) + d(x_{n_i-1}, x_{n_i}, a)d(x_{n_i-1}, x_{m_i+1}, a)}{1 + d(x_{m_i}, x_{n_i}, a) + d(x_{n_i-1}, x_{m_i+1}, a)}\Big\}\\
&\quad\leq \varepsilon.
\end{aligned}$$

Now, from (3.10)and the above inequalities, we have

$$\begin{aligned}
\frac{\varepsilon}{s} &\leq \limsup_{i\to\infty} d(x_{m_i+1}, x_{n_i,a})\\
&\leq s \limsup_{i\to\infty} \alpha(x_{m_i}, fx_{m_i}, a)\alpha(x_{n_i}, fx_{n_i}, a)d(x_{m_i+1}, x_{n_i,a})\\
&\leq \limsup_{i\to\infty} \beta(M(x_{m_i}, x_{n_i-1}, a)) \limsup_{i\to\infty} M(x_{m_i}, x_{n_i-1}, a)\\
&\quad + LN(x_{m_i}, x_{n_i-1}, a)\\
&\leq \varepsilon \limsup_{i\to\infty} \beta(M(x_{m_i}, x_{n_i-1}, a)),
\end{aligned}$$

which implies that $\frac{1}{s} \leq \limsup_{i\to\infty} \beta(M(x_{m_i}, x_{n_i-1}, a))$. Now, as $\beta \in \mathcal{F}$ we conclude that $\{x_n\}$ is a b-Cauchy sequence. The b_2-Completeness of X yields that $\{x_n\}$ b_2-converges to a point $u \in X$.

Step III : u is a fixed point of f.

First, let f be continuous. So, we have

$$u = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} fx_n = fu.$$

Now, let (II) holds. Using the assumptions on X we have $x_n \preceq u$. Now, we show that $u = fu$. By Lemma (2.9)

$$\begin{aligned}
\frac{1}{s}d(u, fu, a) &\leq \limsup_{n\to\infty} d(x_{n+1}, fu, a) \\
&\leq s \limsup_{n\to\infty} d(x_{n+1}, fu, a) \\
&\leq \limsup_{n\to\infty} s\alpha(x_n, fx_n, a)\alpha(u, fu, a)d(x_{n+1}, fu, a) \\
&\leq \limsup_{n\to\infty} \beta(M(x_n, u, a)) \limsup_{n\to\infty} M(x_n, u, a) \\
&= 0,
\end{aligned}$$

because,

$$\begin{aligned}
&\lim_{n\to\infty} M(x_n, u, a) \\
&\quad = \lim_{n\to\infty} \max\Bigg\{d(x_n, u, a), \\
&\quad \frac{d(x_n, fx_n, a)d(x_n, fu, a) + d(u, fu, a)d(u, fx_n, a)}{1 + s[d(x_n, fx_n, a) + d(u, fu, a) + d(x_n, u, fu)]}, \\
&\quad \frac{d(x_n, fx_n, a)d(x_n, fu, a) + d(u, fu, a)d(u, fx_n, a)}{1 + d(x_n, fu, a) + d(x_n, fu, a)}\Bigg\} \\
&\quad = \max\{0, 0, 0\} \\
&\quad = 0.
\end{aligned}$$

Therefore, $d(u, fu) = 0$, so, $u = fu$. □

Definition 3.6. Let (X, d) be a b_2-metric space. A mapping $f : X \to X$ is called a rational Geraghty contraction of type III if there exists $\beta \in \mathcal{F}$ such that,

(3.18)

$$s\alpha(x, fx, a)\alpha(y, fy, a)d(fx, fy, a) \leq \beta(M(x, y, a))M(x, y, a)+LN(x, y, a),$$

for all elements $x, y, a \in X$, where

$$\begin{aligned}
M(x, y, a) = \max\Bigg\{&d(x, y, a), \frac{d(x, fx, a)d(y, fy, a)}{1 + s[d(x, y, a) + d(x, fy, a) + d(y, fx, a) + d(x, y, fy)]}, \\
&\frac{d(x, fx, a)d(x, y, a)}{1 + sd(x, fx, a) + s^3[d(y, fx, a) + d(y, fy, a) + d(x, y, fy)]}\Bigg\}.
\end{aligned}$$

and

$$N(x, y, a) = \min\{d(x, fx, a), d(x, fy, a), d(y, fx, a), d(y, fy, a)\}.$$

Theorem 3.7. *Let (X, d) be a b_2 complete b_2-metric space (with parameter $s > 1$), $f : X \to X$ be a self mapping and $\alpha : X \times X \times X \to [0, \infty)$ be a function such that f is a 2-α- admissible mapping. Suppose that f be a rational Geraghty contractive mapping of type III. Assume that either*

(I) *f is continuous, or,*
(II) *if a sequence $\{x_n\}$ is such that $x_n \to x$ as $n \to \infty$ and $\alpha(x_n, fx_n, a) \geq 1$ for all n, then $\alpha(x, fx, a) \geq 1$.*

If there exist $x_0 \in X$ such that $\alpha(x_0, fx_0, a) \geq 1$, then f has a fixed point.

Proof. Let $x_0 \in X$ such that $\alpha(x_0, fx_0, a) \geq 1$. Define a sequence $\{x_n\}$ in X by

$$x_n = f^n x_0 = f x_{n-1},$$

for all $n \in \mathbb{N}$. Since f is a $2-\alpha$-admissible mapping and

$$\alpha(x_0, fx_0, a) \geq 1,$$

we deduce that $\alpha(x_1, fx_1, a) = \alpha(fx_0, f^2 x_0, a) \geq 1$. By continuing this process, we get that $\alpha(x_n, fx_n, a) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$. Then,

$$\alpha(x_n, fx_n, a)\alpha(x_{n-1}, fx_{n-1}, a) \geq 1,$$

for all $n \in \mathbb{N} \cup \{0\}$.

Step I: We will show that $\lim_{n\to\infty} d(x_n, x_{n+1}, a) = 0$. By (3.18) we have

$$\begin{aligned} d(x_n, x_{n+1}, a) &= d(fx_{n-1}, fx_n, a) \\ &\leq \beta(M(x_{n-1}, x_n, a))M(x_{n-1}, x_n, a) \\ &\leq \beta(d(x_{n-1}, x_n, a))d(x_{n-1}, x_n, a) \\ &< \frac{1}{s} d(x_{n-1}, x_n, a), \\ &\leq d(x_{n-1}, x_n, a), \end{aligned} \tag{3.19}$$

with suppose

$A_1 = d(x_{n-1}, x_n, a) + d(x_{n-1}, fx_n, a) + d(x_n, fx_{n-1}, a) + d(x_{n-1}, x_n, fx_n)$,
$A_2 = d(x_n, fx_{n-1}, a) + d(x_n, fx_n a) + d(x_{n-1}, x_n, fx_n)$,
$A_3 = d(x_{n-1}, x_n, a) + d(x_{n-1}, x_{n+1}, a) + d(x_n, x_n, a) + d(x_{n-1}, x_n, x_{n+1})$,
$A_4 = d(x_n, x_n, a) + d(x_n, x_{n+1}, a) + d(x_{n-1}, x_n, x_{n+1})$,

because

$$\begin{aligned}M(x_{n-1},x_n,a) &= \max\Big\{d(x_{n-1},x_n,a), \frac{d(x_{n-1},fx_{n-1},a)d(x_n,fx_n,a)}{1+sA_1},\\ &\qquad \frac{d(x_{n-1},fx_{n-1},a)d(x_{n-1},x_n,a)}{1+sd(x_{n-1},fx_{n-1},a)+s^3A_2}\Big\}\\ &= \max\Big\{d(x_{n-1},x_n,a), \frac{d(x_{n-1},x_n,a)d(x_n,x_{n+1},a)}{1+sA_3},\\ &\qquad \frac{d(x_{n-1},x_n,a)d(x_{n-1},x_n,a)}{1+sd(x_{n-1},x_n,a)+s^3A_4}\Big\},\end{aligned}$$

Therefore, $\{d(x_n,x_{n+1},a)\}$ is decreasing. By similar arguments as done in Theorems 3.2 and 3.5, we have,

$$\lim_{n\to\infty} d(x_{n-1},x_n,a)=0. \tag{3.20}$$

Step II: Now, we prove that the sequence $\{x_n\}$ is a b_2-Cauchy sequence. Suppose the contrary, i.e., that $\{x_n\}$ is not a b_2-Cauchy sequence. Then there exists $\varepsilon>0$ for which we can find two subsequences $\{x_{m_i}\}$ and $\{x_{n_i}\}$ of $\{x_n\}$ such that n_i is the smallest index for which

$$n_i>m_i>i \text{ and } d(x_{m_i},x_{n_i},a)\ge\varepsilon. \tag{3.21}$$

This means that

$$d(x_{m_i},x_{n_i-1},a)<\varepsilon. \tag{3.22}$$

As in the proof of Theorem 3.2, we have

$$\frac{\varepsilon}{s}\le\limsup_{i\to\infty} d(x_{m_i+1},x_{n_i},a). \tag{3.23}$$

From the definition of $M(x,y,a)$ and the above limits,we have

$$\begin{aligned}B_1 &= d(x_{m_i},x_{n_i-1},a)+d(x_{m_i},fx_{n_i-1},a)+d(x_{n_i-1},fx_{m_i},a),\\ &\quad +d(x_{m_i},x_{n_i-1},fx_{n_i-1}),\\ B_2 &= d(x_{n_i-1},fx_{m_i},a)+d(x_{n_i-1},fx_{n_i-1},a)+d(x_{m_i},x_{n_i-1},fx_{n_i-1}),\\ B_3 &= d(x_{m_i},x_{n_i-1},a)+d(x_{m_i},x_{n_i},a)+d(x_{n_i-1},x_{m_i+1},a),\\ &\quad +d(x_{m_i},x_{n_i-1},x_{n_i}),\\ B_4 &= d(x_{n_i-1},x_{m_i+1},a)+d(x_{n_i-1},x_{n_i},a)+d(x_{m_i},x_{n_i-1},x_{n_i}),\end{aligned}$$

we have

$$\begin{aligned}\limsup_{i\to\infty} M(x_{m_i}, x_{n_i-1}, a) &= \limsup_{i\to\infty} \max\Bigg\{ d(x_{m_i}, x_{n_i-1}, a), \\ &\quad \frac{d(x_{m_i}, fx_{m_i}, a)d(x_{n_i-1}, fx_{n_i-1}, a)}{1+sB_1}, \\ &\quad \frac{d(x_{m_i}, fx_{m_i}, a)d(x_{m_i}, x_{n_i-1}, a)}{1+sd(x_{m_i}, fx_{m_i}, a)+s^3B_2}\Bigg\} \\ &= \limsup_{i\to\infty} M(x_{m_i}, x_{n_i-1}, a) \\ &= \limsup_{i\to\infty} \max\Bigg\{ d(x_{m_i}, x_{n_i-1}, a), \\ &\quad \frac{d(x_{m_i}, x_{m_i+1}, a)d(x_{n_i-1}, x_{n_i}, a)}{1+sB_3}, \\ &\quad \frac{d(x_{m_i}, x_{m_i+1}, a)d(x_{m_i}, x_{n_i-1}, a)}{1+sd(x_{m_i}, x_{m_i+1}, a)+s^3B_4}\Bigg\}.\end{aligned}$$

Now, from (3.10) and the above inequalities, we have

$$\begin{aligned}\frac{\varepsilon}{s} &\leq \limsup_{i\to\infty} d(x_{m_i+1}, x_{n_i,a}) \\ &\leq s\limsup_{i\to\infty} \alpha(x_{m_i}, fx_{m_i}, a)\alpha(x_{n_i}, fx_{n_i}, a)d(x_{m_i+1}, x_{n_i,a}) \\ &\leq \limsup_{i\to\infty} \beta(M(x_{m_i}, x_{n_i-1}, a)) \limsup_{i\to\infty} M(x_{m_i}, x_{n_i-1}, a) \\ &\quad + LN(x_{m_i}, x_{n_i-1}, a) \\ &\leq \varepsilon \limsup_{i\to\infty} \beta(M(x_{m_i}, x_{n_i-1}, a)),\end{aligned}$$

which implies that $\frac{1}{s} \leq \limsup_{i\to\infty} \beta(M(x_{m_i}, x_{n_i-1}, a))$. Now, as $\beta \in \mathcal{F}$ we conclude that $\{x_n\}$ is a b-Cauchy sequence. The b_2-Completeness of X yields that $\{x_n\}$ b_2-converges to a point $u \in X$.

Step III : u is a fixed point of f.

First, let f is continuous. So, we have

$$u = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} fx_n = fu.$$

Now, let (II) holds. We show that $u = fu$. By Lemma 2.9, we have

$$\begin{aligned}\frac{1}{s}d(u, fu, a) &\leq \limsup_{n\to\infty} d(x_{n+1}, fu, a) \leq s \limsup_{n\to\infty} d(x_{n+1}, fu, a)\\ &\leq \limsup_{n\to\infty} s\alpha(x_n, fx_n, a)\alpha(u, fu, a)d(x_{n+1}, fu, a)\\ &\leq \limsup_{n\to\infty} \beta(M(x_n, u, a)) \limsup_{n\to\infty} M(x_n, u, a)\\ &= 0,\end{aligned}$$

and

$$\begin{aligned}&\lim_{n\to\infty} M(x_n, u, a)\\ &\quad = \lim_{n\to\infty} \max \Bigg\{ d(x_n, u, a),\\ &\quad \frac{d(x_n, fx_n, a)d(u, fu, a)}{1 + s[d(x_n, u, a) + d(x_n, fu, a) + d(u, fx_n, a) + d(x_n, u, fu)]},\\ &\quad \frac{d(x_n, fx_n, a)d(x_n, u, a)}{1 + sd(x_n, fx_n, a) + s^3 d(u, fx_n, a) + d(u, fu, a) + d(x_n, u, fu)} \Bigg\}\\ &\quad = \max\{0, 0, 0\}\\ &\quad = 0.\end{aligned}$$

Therefore, $d(u, fu) = 0$, so, $u = fu$. □

3.2. **Results Using Comparison Functions.** Let Ψ denotes the family of all nondecreasing and continuous functions $\psi : [0, \infty) \to [0, \infty)$ such that $\lim_n \psi^n(t) = 0$ for all $t > 0$, where ψ^n denotes the n-th iterate of ψ. It is easy to show that, for each $\psi \in \Psi$, the followings are satisfied:

(a) $\psi(t) < t$ for all $t > 0$;
(b) $\psi(0) = 0$.

Theorem 3.8. *Let (X, d) be a b_2-complete b_2-metric space, $f : X \to X$ be a self mapping and $\alpha : X \times X \times X \to [0, \infty)$ be a function such that f is 2-α- admissible and suppose that*

$$s\alpha(x, y, a)d(fx, fy, a) \leq \psi(M(x, y, a)), \tag{3.24}$$

for all elements $x, y, a \in X$ and

$$M(x, y, a) = \max\left\{d(x, y, a), \frac{d(x, fx, a)d(y, fy, a)}{1 + d(x, y, a)}, \frac{d(x, fx, a)d(y, fy, a)}{1 + d(fx, fy, a)}\right\}.$$

Assume that either

(i) *f is continuous, or*
(ii) *if a sequence $\{x_n\}$ is such that $x_n \to x$ as $n \to \infty$ and $\alpha(x_n, x_{n+1}, a) \geq 1$, then $\alpha(x_n, x, a) \geq 1$.*

If there exists $x_0 \in X$ such that $\alpha(x_0, fx_0, a) \geq 1$, then f has a fixed point.

Proof. Let $x_0 \in X$ be such that $\alpha(x_0, fx_0, a) \geq 1$. Define a sequence $\{x_n\}$ by $x_n = f^n x_0$ for all $n \in \mathbb{N}$. Since f is a $2-\alpha$-admissible mapping and

$$\alpha(x_0, x_1, a) = \alpha(x_0, fx_0, a) \geq 1,$$

we deduce that

$$\alpha(x_1, x_2, a) = \alpha(fx_0, fx_1, a) \geq 1.$$

Continuing this process, we get that $\alpha(x_n, x_{n+1}, a) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$. Now, we do the proof in three steps.

Step I. We will prove that

$$\lim_{n\to\infty} d(x_n, x_{n+1}, a) = 0.$$

Using condition (3.24) and since $\alpha(x_n, x_{n+1}, a) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$, we obtain

$$\begin{aligned} d(x_{n+1}, x_n, a) &\leq s\alpha(x_{n-1}, x_n, a) d(x_{n+1}, x_n, a) \\ &= s\alpha(x_{n-1}, x_n, a) d(fx_n, fx_{n-1}, a) \\ &\leq \psi(M(x_n, x_{n-1}, a)). \end{aligned}$$

Here,

$$\begin{aligned} M(x_{n-1}, x_n, a) &= \max\left\{ d(x_{n-1}, x_n, a), \frac{d(x_{n-1}, fx_{n-1}, a) d(x_n, fx_n, a)}{1 + d(fx_{n-1}, fx_n, a)}, \right. \\ &\quad \left. \frac{d(x_{n-1}, fx_{n-1}, a) d(x_n, fx_n, a)}{1 + d(x_{n-1}, x_n, a)} \right\} \\ &= \max\left\{ d(x_{n-1}, x_n, a), \frac{d(x_{n-1}, x_n, a) d(x_n, x_{n+1}, a)}{1 + d(x_n, x_{n+1}, a)}, \right. \\ &\quad \left. \frac{d(x_{n-1}, fx_{n-1}, a) d(x_n, fx_n, a)}{1 + d(x_{n-1}, x_n, a)} \right\} \\ &\leq \max\left\{ d(x_{n-1}, x_n, a), d(x_n, x_{n+1}, a) \right\}. \end{aligned}$$

Hence,

$$d(x_n, x_{n+1}, a) \leq s\alpha(x_{n-1}, x_n, a) d(x_{n+1}, x_n, a) \leq \psi(d(x_{n-1}, x_n, a)).$$

If $\max\{d(x_{n-1}, x_n, a), d(x_n, x_{n+1}, a)\} = d(x_n, x_{n+1}, a)$, then from (3.11) we have,

$$d(x_n, x_{n+1}, a) \leq sd(x_n, x_{n+1}, a) \leq \psi d(x_n, x_{n+1}, a) < (d(x_n, x_{n+1}, a).$$

and this is a contraction. If $\max\{d(x_{n-1}, x_n, a), d(x_n, x_{n+1}, a)\} = d(x_{n-1}, x_n, a)$, then we have:

$$d(x_n, x_{n+1}, a) \leq sd(x_n, x_{n+1}, a) \leq \psi d(x_{n-1}, x_n, a).$$

By induction, we get that

$$\begin{aligned} d(a, x_{n+1}, x_n) &\leq \psi(d(a, x_n, x_{n-1})) \\ &\leq \psi^2(d(a, x_{n-1}, x_{n-2})) \\ &\ \ \vdots \\ &\leq \psi^n(d(a, x_1, x_0)). \end{aligned}$$

As $\psi \in \Psi$, we conclude that

$$\lim_n d(x_n, x_{n+1}, a) = 0. \tag{3.25}$$

Step II. We will prove that $\{x_n\}$ is a b_2-Cauchy sequence. Suppose the contrary. Then there exists $a \in X$ and $\varepsilon > 0$ for which we can find two subsequences $\{x_{m_i}\}$ and $\{x_{n_i}\}$ of $\{x_n\}$ such that n_i is the smallest index for which

$$n_i > m_i > i, \qquad d(x_{m_i}, x_{n_i}, a) \geq \varepsilon. \tag{3.26}$$

This means that

$$d(x_{m_i}, x_{n_i-1}, a) < \varepsilon. \tag{3.27}$$

From (3.26) and using the rectangle inequality, we get

$$\varepsilon \leq d(x_{m_i}, x_{n_i}, a) \leq sd(x_{m_i}, x_{n_i}, x_{m_i+1}) + sd(x_{m_i+1}, x_{n_i}, a) + sd(x_{m_i+1}, x_{m_i}, a).$$

Taking the upper limit as $i \to \infty$, we get

$$\frac{\varepsilon}{s} \leq \limsup_{i\to\infty} d(x_{m_i+1}, x_{n_i}, a). \tag{3.28}$$

From the definition of $M(x, y, a)$ we have:

$$\begin{aligned} M(x_{m_i}, x_{n_i-1}, a) &= \max\left\{ d(x_{m_i}, x_{n_i-1}, a), \frac{d(x_{m_i}, fx_{m_i}, a)d(x_{n_i-1}, fx_{n_i-1}, a)}{1 + d(fx_{m_i}, fx_{n_i-1}, a)}, \right. \\ &\qquad \left. \frac{d(x_{m_i}, fx_{m_i}, a)d(x_{n_i-1}, fx_{n_i-1}, a)}{1 + d(x_{m_i}, x_{n_i-1}, a)} \right\} \\ &= \max\left\{ d(x_{m_i}, x_{n_i-1}, a), \frac{d(x_{m_i}, a, x_{m_i+1})d(x_{n_i-1}, a, x_{n_i})}{1 + d(x_{m_i+1}, x_{n_i}, a)}, \right. \\ &\qquad \left. \frac{d(x_{m_i}, x_{m_i+1}, a)d(x_{n_i-1}, x_{n_i}, a)}{1 + d(x_{m_i}, x_{n_i-1}, a)} \right\}. \end{aligned}$$

$$\limsup_{i\to\infty} M(x_{m_i}, x_{n_i-1}, a) \leq \varepsilon.$$

Now, from (3.24) we have

$$\begin{aligned} sd(x_{m_i+1}, x_{n_i}, a) &= sd(fx_{m_i}, fx_{n_i-1}, a) \\ &\leq s\alpha(d(x_{m_i}, x_{n_i-1}, a))d(fx_{m_i}, fx_{n_i-1}, a)\psi(M(x_{m_i}, x_{n_i-1}, a)). \end{aligned}$$

Again, if $i \to \infty$, by (3.28), we obtain

$$\begin{aligned}
\varepsilon &= s\left(\frac{\varepsilon}{s}\right) \\
&\leq s \limsup_{i\to\infty} d(x_{m_{i+1}}, x_{n_i}, a) \\
&\leq s \limsup_{i\to\infty} \alpha(x_{m_i}, x_{n_{i-1}}, a) d(fx_{m_i}, fx_{n_{i-1}}, a) \\
&\leq \psi(\varepsilon) \\
&< \varepsilon,
\end{aligned} \tag{3.29}$$

which is a contradiction. Consequently, $\{x_n\}$ is a b_2-Cauchy sequence in X. Therefore, the sequence $\{x_n\}$ b_2-converges to some $z \in X$, that is, $\lim_n d(x_n, z, a) = 0$ for all $a \in X$.

Step III. Now we show that z is a fixed point of f.

Using the rectangle inequality, we get

$$d(z, fz, a) \leq sd(z, fz, fx_n) + sd(fx_n, fz, a) + sd(fx_n, z, a).$$

Letting $n \to \infty$ and using the continuity of f, we get

$$d(z, fz, a) \leq 0.$$

Hence, we have $fz = z$. Thus, z is a fixed point of f. □

Theorem 3.9. *Under the hypotheses of Theorem 3.8, without the b_2-continuity assumption on f. Assume for any sequence $\{x_n\}$ in X with $\alpha(x_n, x_{n+1}, a) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$ and $x_n \to x$ as $n \to +\infty$, we have $\alpha(x_n, x, a) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$. Then f has a fixed point.*

Proof. Following the proof of Theorem 3.8, we construct an increasing sequence $\{x_n\}$ in X such that $x_n \to z \in X$. Now, we show that $z = fz$. By (3.24), we have

$$\begin{aligned}
sd(fz, x_n, a) &= sd(fz, fx_{n-1}, a) \\
&\leq s\alpha(z, x_{n-1}, a) d(fz, fx_{n-1}, a)\psi(M(z, x_{n-1}, a)),
\end{aligned} \tag{3.30}$$

where

$$M(z, x_{n-1}, a) = \max\left\{ d(z, x_{n-1}, a), \frac{d(z, fz, a)d(x_{n-1}, fx_{n-1}, a)}{1 + d(fz, fx_{n-1}, a)}, \right.$$
$$\left. \frac{d(z, fz, a)d(x_{n-1}, fx_{n-1}, a)}{1 + d(z, x_{n-1}, a)} \right\}.$$

Letting $n \to \infty$ in the above relation, we get

$$\limsup_{n\to\infty} M(z, x_{n-1}, a) = 0. \tag{3.31}$$

Again, taking the upper limit as $n \to \infty$ in (3.30) and using Lemma 2.9 and (3.31) we get

$$\begin{aligned} s\left[\frac{1}{s}d(z,fz,a)\right] &\leq s\limsup_{n\to\infty} d(x_n,fz,a) \\ &\leq s\limsup_{n\to\infty} \alpha(x_{n-1},z,a)d(x_n,fz,a) \\ &\leq \limsup_{n\to\infty} \psi(M(z,x_{n-1},a)) \\ &= 0. \end{aligned}$$

So we get $d(z,fz,a)=0$, i.e., $fz=z$. □

Corollary 3.10. *Let (X,d) be a b_2-complete b_2-metric space, $f : X \to X$ be a self mapping and $\alpha : X \times X \times X \to [0,\infty)$ be a function such that f is 2-α- admissible and suppose that*

$$s\alpha(x,y,a)d(fx,fy,a) \leq rM(x,y,a),$$

where $0 \leq r < 1$ and

$$M(x,y,a) = \max\left\{d(x,y,a), \frac{d(x,fx,a)d(y,fy,a)}{1+d(x,y,a)}, \frac{d(x,fx,a)d(y,fy,a)}{1+d(fx,fy,a)}\right\}.$$

Assume that for all elements $x,y,a \in X$. either

(i) *f is continuous, or*
(ii) *if a sequence $\{x_n\}$ is such that $x_n \to x$ as $n \to \infty$ and $\alpha(x_n,x_{n+1},a) \geq 1$, then $\alpha(x_n,x,a) \geq 1$.*

If there exists $x_0 \in X$ such that $\alpha(x_0,fx_0,a) \geq 1$, then f has a fixed point.

Remark 3.11. In Theorems 3.8 and 3.9, we can replace $M(x,y,a)$ by the following:

$$\begin{aligned} M(x,y,a) = \max\Bigg\{ & d(x,y,a), \frac{d(x,fx,a)d(x,fy,a)+d(y,fy,a)d(y,fx,a)}{1+s[d(x,fx,a)+d(y,fy,a)+d(x,y,fy)]}, \\ & \frac{d(x,fx,a)d(x,fy,a)+d(y,fy,a)d(y,fx,a)}{1+d(x,fy,a)+d(y,fx,a)}\Bigg\}, \end{aligned}$$

or,

$$\begin{aligned} M(x,y,a) = \max\Bigg\{ & d(x,y,a), \\ & \frac{d(x,fx,a)d(y,fy,a)}{1+s[d(x,y,a)+d(x,fy,a)+d(y,fx,a)+d(x,y,fy)}, \\ & \frac{d(x,fy,a)d(x,y,a)}{1+sd(x,fx,a)+s^3[d(y,fx,a)+d(y,fy,a)+d(x,y,fy)]}\Bigg\}. \end{aligned}$$

Corollary 3.12 ([28]). *Let $(X, \preceq)$ be a partially ordered set and suppose that there exists a b_2-metric d on X such that (X, d) is a complete b_2-metric space. Let $f : X \to X$ be an increasing mapping with respect to $\preceq$ such that there exists an element $x_0 \in X$ with $x_0 \preceq f(x_0)$. Suppose that*

$$sd(fx, fy, a) \le \beta(d(x, y, a))M(x, y, a), \tag{3.32}$$

where

$$M(x, y, a) = \max\left\{d(x, y, a), \frac{d(x, fx, a)d(y, fy, a)}{1 + d(x, y, a)}, \frac{d(x, fx, a)d(y, fy, a)}{1 + d(fx, fy, a)}\right\}.$$

Assume that either

(i) *f is continuous, or*
(ii) *if a sequence $\{x_n\}$ is a nondecreasing sequence such that $x_n \to x$ as $n \to \infty$, then $x \preceq fx$.*

Then f has a fixed point.

Proof. Define the mapping $\alpha : X \times X \times X \to [0, \infty)$ as follows

$$\alpha(x, y, a) = \begin{cases} 1 & \text{if } x \preceq y \quad or \quad y \preceq x \\ 0 & \text{if otherwise.} \end{cases}$$

So, we have

$$s\alpha(x, fx, a)\alpha(y, fy, a)d(fx, fy, a) \le \beta(d(x, y, a))M(x, y, a), \tag{3.33}$$

for all $x, y \in X$. Since $x_0 \preceq f(x_0)$, by the definition of α, we have $\alpha(x_0, fx_0, a) \ge 1$. Now, we show that f is 2-α-admissible. If $\alpha(x, y, a) \ge 1$, then we conclude that $x \preceq y$ or $y \preceq x$. Since f is nondecreasing, so we deduce $fx \preceq fy$ or $fy \preceq fx$. By the definition of α, we get $\alpha(fx, fy, a) \ge 1$. So, f is 2-α admissible. Now, if condition (ii) hold, since $\{x_n\}$ is a nondecreasing sequence, we have $\alpha(x_n, fx_n, a) \ge 1$ as $n \to \infty$, then $x \preceq fx$, so $\alpha(x, fx, a) \ge 1$. So, all conditions of Theorem 3.2 are hold and f has a fixed point. $\square$

Now, we take $\beta(t) = r$, Where $0 \le r < \frac{1}{s}$, then we have the following corollary.

Corollary 3.13 ([28]). *Let $(X, \preceq)$ be a partially ordered set and suppose that there exists a b_2-metric d on X such that (X, d) is a b_2-complete b_2-metric space. Let $f : X \to X$ be an increasing mapping with respect to $\preceq$ such that there exists an element $x_0 \in X$ with $x_0 \preceq f(x_0)$. Suppose that*

$$sd(fx, fy, a) \le rM(x, y, a), \tag{3.34}$$

where

$$M(x, y, a) = \max\left\{d(x, y, a), \frac{d(x, fx, a)d(y, fy, a)}{1 + d(x, y, a)}, \frac{d(x, fx, a)d(y, fy, a)}{1 + d(fx, fy, a)}\right\}.$$

Assume that either

(i) *f is continuous, or*

(ii) *if a sequence $\{x_n\}$ is a nondecreasing sequence such that $x_n \to x$ as $n \to \infty$, then $x \preceq fx$.*

Then f has a fixed point.

Corollary 3.14 ([28])**.** *Let $(X, \preceq)$ be a partially ordered set and suppose that there exists a b_2-metric d on X such that (X, d) is a b_2-complete b_2-metric space. Let $f : X \to X$ be an increasing mapping with respect to $\preceq$ such that there exists an element $x_0 \in X$ with $x_0 \preceq f(x_0)$. Suppose that*

$$sd(fx, fy, a) \leq \alpha d(x, y, a) + \beta \frac{d(x, fx, a)d(y, fy, a)}{1 + d(x, y, a)} + \gamma \frac{d(x, fx, a)d(y, fy, a)}{1 + d(fx, fy, a)},$$

for all $x, y, a \in X$, where $\alpha, \beta, \gamma \geq 0$ and $0 \leq \alpha + \beta + \gamma < \frac{1}{s}$. Assume that either

(i) *f is continuous, or*

(ii) *if a sequence $\{x_n\}$ is a nondecreasing sequence such that $x_n \to x$ as $n \to \infty$, then $x \preceq fx$.*

Then f has a fixed point.

Corollary 3.15 ([28])**.** *Let $(X, \preceq)$ be a partially ordered set and suppose that there exists a b-metric d on X such that (X, d) is a b_2-complete b_2-metric space. Let $f : X \to X$ be an increasing mapping with respect to $\preceq$ such that there exists an element $x_0 \in X$ with $x_0 \preceq f(x_0)$. Suppose that*

$$sd(fx, fy, a) \leq \psi(M(x, y, a)), \tag{3.35}$$

where

$$M(x, y, a) = \max\left\{d(x, y, a), \frac{d(x, fx, a)d(y, fy, a)}{1 + d(x, y, a)}, \frac{d(x, fx, a)d(y, fy, a)}{1 + d(fx, fy, a)}\right\}.$$

Assume that either

(i) *f is continuous, or*

(ii) *if a sequence $\{x_n\}$ is a nondecreasing sequence such that $x_n \to x$ as $n \to \infty$, then $x \preceq fx$.*

Then f has a fixed point.

3.3. **Results for Almost Generalized Weakly Contractive Mappings.** Berinde in [17–20] initiated the concept of almost contractions and obtained many interesting fixed point theorems. Results with similar conditions were obtained, *e.g.*, in [21] and [22]. In this section, we define the notion of almost generalized $(\psi, \varphi)_{s,a}$-contractive mappings and prove some new results. In particular, we extend Theorems 2.1, 2.2 and 2.3 of Ćirić et.al. in [23] to the setting of b_2-metric spaces.

The concept of an altering distance function which is introduced in [24] by Khan et.al.

Definition 3.16 ([24])**.** A function $\varphi : [0, +\infty) \to [0, +\infty)$ is called an altering distance function, if the following properties hold:

(i) φ is continuous and non-decreasing.
(ii) $\varphi(t) = 0$ if and only if $t = 0$.

Let (X, d) be a b_2-metric space and let $f : X \to X$ be a mapping. For $x, y, a \in X$, set

$$M(x, y, a) = \max\left\{d(x, y, a), \frac{d(x, fx, a)d(y, fy, a)}{1 + d(fx, fy, a)}, \frac{d(x, fx, a)d(y, fy, a)}{1 + d(x, y, a)}\right\},$$

and

$$N_a(x, y) = \min\left\{d(x, fx, a), d(x, fy, a), d(y, fx, a), d(y, fy, a)\right\}.$$

Definition 3.17. Let (X, d) be a b_2-metric space. We say that a mapping $f : X \to X$ is an almost generalized $(\psi, \varphi)_{s,a}$-contractive mapping if there exist $L \geq 0$ and two altering distance functions ψ and φ such that

$$\psi(sd(fx, fy, a)) \leq \psi(M_a(x, y)) - \varphi(M_a(x, y)) + L\psi(N_a(x, y)), \tag{3.36}$$

for all $x, y, a \in X$.

Now, let us to prove our new results.

Theorem 3.18. *Let $(X, \preceq)$ be a partially ordered set and suppose that there exists a b_2-metric d on X such that (X, d) is a b_2-complete b_2-metric space. Let $f : X \to X$ be a continuous mapping, non-decreasing with respect to $\preceq$. Suppose that f satisfies condition* (3.36)*, for all elements $x, y, a \in X$, where x, y are comparable. If there exists $x_0 \in X$ such that $x_0 \preceq fx_0$, then f has a fixed point.*

Proof. Starting with the given x_0, define a sequence $\{x_n\}$ in X such that $x_{n+1} = fx_n$, for all $n \geq 0$. Since $x_0 \preceq fx_0 = x_1$ and f is non-decreasing, we have $x_1 = fx_0 \preceq x_2 = fx_1$, and by induction

$$x_0 \preceq x_1 \preceq \cdots \preceq x_n \preceq x_{n+1} \preceq \cdots .$$

If $x_n = x_{n+1}$, for some $n \in \mathbb{N}$, then $x_n = fx_n$ and hence x_n is a fixed point of f. So, we may assume that $x_n \neq x_{n+1}$, for all $n \in \mathbb{N}$. By (3.36), we have

(3.37)
$$\begin{aligned}
\psi(d(x_n, x_{n+1}, a)) &\leq \psi(sd(x_n, x_{n+1}, a)) \\
&= \psi(sd(fx_{n-1}, fx_n, a)) \\
&\leq \psi(M_a(x_{n-1}, x_n)) - \varphi(M_a(x_{n-1}, x_n)) + L\psi(N_a(x_{n-1}, x_n)),
\end{aligned}$$

where

$$\begin{aligned}M(x_{n-1},x_n,a) &= \max\left\{d(x_{n-1},x_n,a), \frac{d(x_{n-1},fx_{n-1},a)d(x_n,fx_n,a)}{1+d(fx_{n-1},fx_n,a)},\right.\\ &\quad\left.\frac{d(x_{n-1},fx_{n-1},a)d(x_n,fx_n,a)}{1+d(x_{n-1},x_n,a)}\right\}\\ &= \max\left\{d(x_{n-1},x_n,a), \frac{d(x_{n-1},x_n,a)d(x_n,x_{n+1},a)}{1+d(x_n,x_{n+1},a)},\right.\\ &\quad\left.\frac{d(x_{n-1},fx_{n-1},a)d(x_n,fx_n,a)}{1+d(x_{n-1},x_n,a)}\right\}\\ &\le \max\left\{d(x_{n-1},x_n,a), d(x_n,x_{n+1},a)\right\}.\end{aligned}$$

If $\max\{d(x_{n-1},x_n,a),d(x_n,x_{n+1},a)\}=d(x_n,x_{n+1},a)$, and

(3.38)

$$\begin{aligned}N_a(x_{n-1},x_n) &= \min\{d(x_{n-1},fx_{n-1},a), d(x_{n-1},fx_n,a),\\ &\quad d(x_n,fx_{n-1},a), d(x_n,fx_n,a)\}\\ &= \min\{d(x_{n-1},x_n,a), d(x_{n-1},x_{n+1},a), 0, d(x_n,x_{n+1},a)\}\\ &= 0,\end{aligned}$$

then from (3.37)-(3.38) and the properties of ψ and φ, we get

(3.39)

$$\begin{aligned}\psi(d(x_n,x_{n+1},a)) &\le \psi\left(\max\left\{d(x_{n-1},x_n,a), \frac{d(x_{n-1},fx_{n-1},a)d(x_n,fx_n,a)}{1+d(x_{n-1},x_n,a)}\right\}\right)\\ &\quad -\varphi\Bigg(\max\left\{d(x_{n-1},x_n,a), \frac{d(x_{n-1},fx_{n-1},a)d(x_n,fx_n,a)}{1+d(fx_{n-1},fx_n,a)},\right.\\ &\quad\left.\frac{d(x_{n-1},fx_{n-1},a)d(x_n,fx_n,a)}{1+d(x_{n-1},x_n,a)}\right\}\Bigg),\end{aligned}$$

since

$$\begin{aligned}&\max\left\{d(x_{n-1},x_n,a), \frac{d(x_{n-1},fx_{n-1},a)d(x_n,fx_n,a)}{1+d(fx_{n-1},fx_n,a)}, \frac{d(x_{n-1},fx_{n-1},a)d(x_n,fx_n,a)}{1+d(x_{n-1},x_n,a)}\right\}\\ &\quad< \max\left\{d(x_{n-1},x_n,a), d(x_n,x_{n+1},a)\right\}.\end{aligned}$$

Then by (3.39) we have if

$$\max\{d(x_{n-1},x_n,a),d(x_n,x_{n+1},a)\}=d(x_n,x_{n+1},a),$$

then

$$\psi(d(x_n,x_{n+1},a)) < \psi(d(x_n,x_{n+1},a)),$$

and this is a contradiction.

if $\max\{d(x_{n-1},x_n,a),d(x_n,x_{n+1},a)\}=d(x_{n-1},x_n,a)$, then

$$\psi(d(x_n,x_{n+1},a)) < \psi(d(x_{n-1},x_n,a)),$$

Thus, $\{d(x_n, x_{n+1}, a) : n \in \mathbb{N} \cup \{0\}\}$ is a non-increasing sequence of positive numbers. Hence, there exists $r \geq 0$ such that

$$\lim_n d(x_n, x_{n+1}, a) = r.$$

Letting $n \to \infty$ in (3.39), we get

$$\psi(r) \leq \psi\left(\max\left(r, \frac{r.r}{1+r}, \frac{r.r}{1+r}\right)\right) - \varphi\left(\max\left\{r, \frac{r.r}{1+r}, \frac{r.r}{1+r}\right\}\right) \leq \psi(r).$$

Therefore,

$$\varphi\left(\max\left\{r, \frac{r.r}{1+r}, \frac{r.r}{1+r}\right\}\right) = 0,$$

and hence $r = 0$. Thus, we have

$$\lim_n d(x_n, x_{n+1}, a) = 0, \tag{3.40}$$

for each $a \in X$.

Next, we show that $\{x_n\}$ is a b_2-Cauchy sequence in X. For this purpose, we use the relation (2.12), page 5 of [5] which is as follows:

$$d(x_i, x_j, x_k) = 0, \tag{3.41}$$

for all $i, j, k \in N$ (note that this can be obtained as $\{d(x_n, x_{n+1}, a)$: $n \in \mathbb{N} \cup \{0\}\}$ be a non-increasing sequence of positive numbers).

Suppose the contrary, that is, $\{x_n\}$ is not a b_2-Cauchy sequence. Then there exist $a \in X$ and $\varepsilon > 0$ for which we can find two subsequences $\{x_{m_i}\}$ and $\{x_{n_i}\}$ of $\{x_n\}$ such that n_i is the smallest index for which

$$n_i > m_i > i, \ \ d(x_{m_i}, x_{n_i}, a) \geq \varepsilon. \tag{3.42}$$

This means that

$$d(x_{m_i}, x_{n_i - 1}, a) < \varepsilon. \tag{3.43}$$

Using (3.43) and taking the upper limit as $i \to \infty$, we get

$$\limsup_{n\to\infty} d(x_{m_i}, x_{n_i - 1}, a) \leq \varepsilon. \tag{3.44}$$

On the other hand, we have

$$\begin{aligned} &d(x_{m_i}, x_{n_i}, a) \\ &\qquad \leq sd(x_{m_i}, x_{n_i}, x_{m_i+1}) + sd(x_{n_i}, a, x_{m_i+1}) + sd(a, x_{m_i}, x_{m_i+1}). \end{aligned}$$

Using (3.40), (3.42) and taking the upper limit as $i \to \infty$, we get

$$\frac{\varepsilon}{s} \leq \limsup_{n\to\infty} d(x_{m_i+1}, x_{n_i}, a). \tag{3.45}$$

Again, using the rectangular inequality, we have

$$\begin{aligned} &d(x_{m_i+1}, x_{n_i-1}, a) \\ &\qquad \leq sd(x_{m_i+1}, x_{n_i-1}, x_{m_i}) + sd(x_{n_i-1}, a, x_{m_i}) + sd(a, x_{m_i+1}, x_{m_i}), \end{aligned}$$

and

$$d(x_{m_i}, x_{n_i}, a) \leq sd(x_{m_i}, x_{n_i}, x_{n_i-1}) + sd(x_{n_i}, a, x_{n_i-1}) + sd(a, x_{m_i}, x_{n_i-1}).$$

Taking the upper limit as $i \to \infty$ in the first inequality in the above, and using (3.40) and (3.44), we get

$$\limsup_{n\to\infty} d(x_{m_i+1}, x_{n_i-1}, a) \leq \varepsilon s. \tag{3.46}$$

Similarly, taking the upper limit as $i \to \infty$ in the second inequality in the above, and using (3.40) and (3.43), we get

$$\limsup_{n\to\infty} d(x_{m_i}, x_{n_i}, a) \leq \varepsilon s. \tag{3.47}$$

From (3.36), we have

$$\begin{aligned}\psi(sd(x_{m_i+1}, x_{n_i}, a)) &= \psi(sd(fx_{m_i}, fx_{n_i-1}, a)) \\ &\leq \psi(M_a(x_{m_i}, x_{n_i-1})) - \varphi(M_a(x_{m_i}, x_{n_i-1})) \\ &\quad + L\psi(N_a(x_{m_i}, x_{n_i-1})),\end{aligned} \tag{3.48}$$

where

(3.49)

$$\begin{aligned}M_a(x_{m_i}, x_{n_i-1}) &= \max\left\{d(x_{m_i}, x_{n_i-1}, a), \frac{d(x_{m_i}, fx_{m_i}, a)d(x_{n_i-1}, fx_{n_i-1}, a)}{1 + d(x_{m_i}, x_{n_i-1})},\right. \\ &\qquad \left.\frac{d(x_{m_i}, fx_{m_i}, a)d(x_{n_i-1}, fx_{n_i-1}, a)}{1 + d(fx_{m_i}, fx_{n_i-1})}\right\} \\ &= \max\left\{d(x_{m_i}, x_{n_i-1}, a), \frac{d(x_{m_i}, x_{m_i+1}, a)d(x_{n_i-1}, x_{n_i}, a)}{1 + d(x_{m_i}, x_{n_i-1})},\right. \\ &\qquad \left.\frac{d(x_{m_i}, x_{m_i+1}, a)d(x_{n_i-1}, x_{n_i}, a)}{1 + d(x_{m_i+1}, x_{n_i})}\right\},\end{aligned}$$

and

$$\begin{aligned}N_a(x_{m_i}, x_{n_i-1}) &= \min\{d(x_{m_i}, fx_{m_i}, a), d(x_{m_i}, fx_{n_i-1}, a)\,, \\ &\qquad d(x_{n_i-1}, fx_{m_i}, a), d(x_{n_i-1}, fx_{n_i-1}, a)\} \\ &= \min\{d(x_{m_i}, x_{m_i+1}, a), d(x_{m_i}, x_{n_i}, a), \\ &\qquad d(x_{n_i-1}, x_{m_i+1}, a), d(x_{n_i-1}, x_{n_i}, a)\} \\ &= 0.\end{aligned}$$

(3.50)

Taking the upper limit as $i \to \infty$ in (3.49) and (3.50) and using (3.40), (3.44), (3.46) and (3.47), we get

(3.51)

$$\limsup_{n\to\infty} M_a(x_{m_i-1}, x_{n_i-1}) = \max\left\{\limsup_{n\to\infty} d(x_{m_i}, x_{n_i-1}, a), 0, 0\right\} \leq \varepsilon.$$

So, we have So, we have

$$\limsup_{n\to\infty} M_a(x_{m_i-1}, x_{n_i-1}) \leq \varepsilon, \tag{3.52}$$

and

$$\limsup_{n\to\infty} N_a(x_{m_i}, x_{n_i-1}) = 0. \tag{3.53}$$

Now, taking the upper limit as $i \to \infty$ in (3.48) and using (3.45), (3.52) and (3.53) we have

$$\begin{aligned}
\psi\left(s \cdot \frac{\varepsilon}{s}\right) &\leq \psi\left(s \limsup_{n\to\infty} d(x_{m_i+1}, x_{n_i}, a)\right) \\
&\leq \psi\left(\limsup_{n\to\infty} M_a(x_{m_i}, x_{n_i-1})\right) - \liminf_{n\to\infty} \varphi(M_a(x_{m_i}, x_{n_i-1})) \\
&\leq \psi(\varepsilon) - \varphi\left(\liminf_{n\to\infty}) M_a(x_{m_i}, x_{n_i-1})\right),
\end{aligned}$$

which further implies that

$$\varphi\left(\liminf_{n\to\infty} M_a(x_{m_i}, x_{n_i-1})\right) = 0,$$

so $\liminf\limits_{n\to\infty} M_a(x_{m_i}, x_{n_i-1}) = 0$, which is a contradiction to (3.42). Thus, $\{x_{n+1} = fx_n\}$ is a b_2-Cauchy sequence in X.

As X is a b_2-complete space, there exists $u \in X$ such that $x_n \to u$ as $n \to \infty$, that is,

$$\lim_n x_{n+1} = \lim_n fx_n = u.$$

Now, suppose that f is continuous. Using the rectangle inequality, we get

$$d(u, fu, a) \leq sd(u, fu, fx_n) + sd(fu, a, fx_n) + sd(a, u, fx_n).$$

Letting $n \to \infty$, we get

$$d(u, fu, a) \leq s \lim_n d(u, fu, fx_n) + s \lim_n d(fu, a, fx_n) + s \lim_{n\to\infty} d(a, u, fx_n) = 0.$$

So, we have $fu = u$. Thus, u is a fixed point of f. Note that the continuity of f in Theorem 3.18 is not necessary and can be dropped. □

Theorem 3.19. *Under the hypotheses of Theorem 3.18, without the continuity assumption on f. Assume that whenever $\{x_n\}$ is a non-decreasing sequence in X such that $x_n \to x \in X$, one has $x_n \preceq x$, for all $n \in \mathbb{N}$. Then f has a fixed point in X.*

Proof. Following similar arguments to those given in the proof of Theorem 3.18, we construct an increasing sequence $\{x_n\}$ in X such that

$x_n \to u$, for some $u \in X$. Using the assumptions on X, we have that $x_n \preceq u$, for all $n \in \mathbb{N}$. Now, we show that $fu = u$. By (3.36), we have

(3.54)

$$\begin{aligned}\psi(sd(x_{n+1}, fu, a)) &= \psi(sd(fx_n, fu, a)) \\ &\leq \psi(M_a(x_n, u)) - \varphi(M_a(x_n, u)) + L\psi(N_a(x_n, u)),\end{aligned}$$

where

$$\begin{aligned}M_a(x_n, u) &= \max\left\{d(x_n, u, a), \frac{d(x_n, fx_n, a)d(u, fu, a)}{1 + d(x_n, u, a)}, \frac{d(x_n, fx_n, a)d(u, fu, a)}{1 + d(fx_n, fu, a)}\right\} \\ &= \max\left\{d(x_n, u, a), \frac{d(x_n, x_{n+1}, a), d(u, fu, a)}{1 + d(x_n, u, a)}, \frac{d(x_n, x_{n+1}, a)d(u, fu, a)}{1 + d(x_{n+1}, fu, a)}\right\},\end{aligned}$$

and

(3.55)

$$\begin{aligned}N_a(x_n, u) &= \min\left\{d(x_n, fx_n, a), d(x_n, fu, a), d(u, fx_n, a), d(u, fu, a)\right\} \\ &= \min\left\{d(x_n, x_{n+1}, a), d(x_n, fu, a), d(u, x_{n+1}, a), d(u, fu, a)\right\},\end{aligned}$$

and

$$N_a(x_n, u) \to 0, M_a(x_n, u) \to 0.$$

Again, taking the upper limit as $i \to \infty$ in (3.54) and using Lemma 2.9 and (3.55), we get

$$\begin{aligned}\psi(d(u, fu, a) &= \psi(s \cdot \frac{1}{s} d(u, fu, a)) \\ &\leq \psi\big(s \limsup_{n\to\infty} d(x_{n+1}, fu, a)\big) \\ &\leq \psi\big(\limsup_{n\to\infty} M_a(x_n, u)\big) - \liminf_{n\to\infty} \varphi(M_a(x_n, u)).\end{aligned}$$

Therefore, $\psi(d(u, fu, a) = 0$ then $(d(u, fu, a) = 0$. Thus, letting $n \to \infty$, we get

$$\begin{aligned}d(u, fu, a) &\leq s \lim_n d(u, fu, fx_n) + s \lim_n d(fu, a, fx_n) + s \lim_{n\to\infty} d(a, u, fx_n) \\ &= 0.\end{aligned}$$

So, we have $fu = u$. Thus, u is a fixed point of f. □

Corollary 3.20. *Let $(X, \preceq)$ be a partially ordered set and suppose that there exists a b_2-metric d on X such that (X, d) is a b_2-complete b_2-metric space. Let $f : X \to X$ be a non-decreasing continuous mapping with respect to $\preceq$. Suppose that there exist $k \in [0, 1)$ and $L \geq 0$ such that*

$$\begin{aligned}d(fx, fy, a) \leq \frac{k}{s} \max&\left\{d(x, y, a), \frac{d(x, fx, a)d(y, fy, a)}{1 + d(x, y, a)}, \frac{d(x, fx, a)d(y, fy, a)}{1 + d(fx, fy, a)}\right\} \\ &+ \frac{L}{s} \min\left\{d(x, fx, a), d(y, fx, a)\right\},\end{aligned}$$

for all elements $x, y, a \in X$ where x, y are comparable. If there exists $x_0 \in X$ such that $x_0 \preceq fx_0$, then f has a fixed point.

Proof. The result follows from Theorem 3.18 by taking $\psi(t) = t$ and $\varphi(t) = (1-k)t$, for all $t \in [0, +\infty)$. □

Corollary 3.21. *Under the hypotheses of Corollary 3.20, without the continuity assumption of f. Assume that for any non-decreasing sequence $\{x_n\}$ in X such that $x_n \to x \in X$ we have $x_n \preceq x$, for all $n \in \mathbb{N}$. Then, f has a fixed point in X.*

4. Example

Example 4.1. Let $X = [0, +\infty)$ and $d(x, y, z) = [xy + yz + zx]^2$ if $x \neq y \neq z \neq x$, and otherwise $d(x, y, z) = 0$. Where $s > 3$. d is a b_2-metric on X . Consider the mapping $f : X \to X$ defined by $f(x) = \frac{1}{4}\ln(x+1)$ and the function $\psi \in \Psi$ given by $\psi(t) = \frac{1}{4}t$, $t \geq 0$. It is easy to see that f is increasing and $0 \leq f(0) = 0$. For all comparable elements $x, y \in X$, using the mean value theorem, we have,

$$\begin{aligned} d(fx, fy, a) &= \left[\frac{1}{4}\ln(x+1)\frac{1}{4}\ln(y+1) + \frac{1}{4}\ln(y+1)a + \frac{1}{4}\ln(x+1)a\right]^2 \\ &\leq \frac{1}{4}[xy + ya + ax]^2 \\ &\leq \frac{1}{4}d(x, y, a) \\ &= \psi(d(x, y, a)) \\ &\leq \psi(M(x, y, a)), \end{aligned}$$

so, using Theorem 3.8, f has a fixed point.

Example 4.2. Let $X = [0, +\infty)$ and $d(x, y, z) = [xy + yz + zx]^2$ if $x \neq y \neq z \neq x$, and otherwise $d(x, y, z) = 0$. Where $s > 3$. d is a b_2-metric on X . Consider the mapping $f : X \to X$ defined by $f(x) = \frac{1}{16}xe^{-x^2}$ and the function β given by $\beta(t) = \frac{1}{4}$. Define the mapping $\alpha : X \times X \times X \to [0, \infty)$ as follow

$$\alpha(x, y, a) = \begin{cases} 1 & \text{if } x \preceq y \text{ } or \text{ } y \preceq x, \\ 0 & \text{if otherwise.} \end{cases}$$

Since $x_0 \preceq f(x_0)$, by the definition of α, we have $\alpha(x_0, fx_0, a) \geq 1$ for all $a \in X$. Now, we show that f is $2-\alpha$-admissible. If $\alpha(x, y, a) \geq 1$, then we conclude that $x \preceq y$ or $y \preceq x$. Since f is non-decreasing, so we deduce $fx \preceq fy$ or $fy \preceq fx$. By the definition of α, we get $\alpha(fx, fy, a) \geq 1$. So, f is $2-\alpha$ admissible. Now, since f is continues therefore, all conditions of Corollary 3.20 hold and f has a fixed point, it is easy to see that f

is increasing and $0 \leq f(0) = 0$. For all comparable elements $x, y \in X$, using the mean value theorem, we have,

$$\begin{aligned} d(fx, fy, a) &= \left[\frac{1}{16}xe^{-x^2}\frac{1}{16}ye^{-y^2} + \frac{1}{16}y^2e^{-y^2}a + \frac{1}{16}xe^{-x^2}a\right]^2 \\ &\leq \frac{1}{16}[xy + ya + ax]^2 \\ &\leq \frac{1}{8}d(x, y, a) \\ &\leq \frac{1}{4}d(x, y, a) \\ &= \beta(d(x, y, a))d(x, y, a) \\ &\leq \beta(d(x, y, a))M(x, y, a). \end{aligned}$$

So, from Theorem 3.2, f has a fixed point.

Example 4.3. Let $X = \{(\alpha, 0) : \alpha \in [0, +\infty)\} \cup \{(0, 2)\} \subset \mathbb{R}^2$ and let $d(x, y, z)$ denotes the square of the area of the triangle with vertices $x, y, z \in X$. E.g.,

$$d((\alpha, 0), (\beta, 0), (0, 2)) = (\alpha - \beta)^2.$$

It is easy to check that d is a b_2-metric with parameter $s = 2$. Introduce an order $\preceq$ in X by

$$(\alpha, 0) \preceq (\beta, 0) \quad \Leftrightarrow \quad \alpha \geq \beta,$$

with all other pairs of distinct points in X which are incomparable.

Consider the mapping $f : X \to X$ given by

$$f(\alpha, 0) = \left(\frac{\alpha}{3}, 0\right) \text{ for } \alpha \in [0, +\infty) \text{ and } f(0, 2) = (0, 2),$$

and the function $\beta \in \mathcal{F}_2$ given as

$$\beta(t) = \frac{1 + t}{2 + 4t} \text{ for } t \in [0, +\infty).$$

Then f is an increasing mapping with $(\alpha, 0) \preceq f(\alpha, 0)$ for each $\alpha \geq 0$. Finally, in order to check the contractive condition, only the case when $x = (\alpha, 0)$, $y = (\beta, 0)$, $a = (0, 2)$ is nontrivial. Let $d(x, y, a) = (\alpha - \beta)^2$

and

$$\begin{aligned} sd(fx,fy,a) &= 2d\left(\left(\tfrac{1}{3}\alpha,0\right),\left(\tfrac{1}{3}\beta,0\right),(0,2)\right) \\ &= 2\cdot\frac{1}{9}(\alpha-\beta)^2 \\ &\le \frac{1}{4}(\alpha-\beta)^2 \\ &\le \beta(d(x,y,a))d(x,y,a) \\ &\le \beta(d(x,y,a))M(x,y,a). \end{aligned}$$

All the conditions of above Theorem are satisfied and f has two fixed points, $(0,0)$ and $(0,2)$. But if we define: $d_1(x,y)=\sqrt{d(x,y,a)}$, then (X,d_1) is a b_2-metric space and f is not satisfied in contractive conditions. Only the case when $x=(\alpha,0)$, $y=(\beta,0)$, $a=(0,2)$ is nontrivial, since $d_1(fx,fy)=(\frac{1}{3})|\alpha-\beta|$ and $d_1(x,y)=|\alpha-\beta|$.

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