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Observational Modeling of the Kolmogorov-Sinai Entropy

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Abstract. In this paper, Kolmogorov-Sinai entropy is studied using mathematical modeling of an observer Θ. The relative entropy of a sub-σ_Θ-algebra having finite atoms is defined and then the ergodic properties of relative semi-dynamical systems are investigated. Also, a relative version of Kolmogorov-Sinai theorem is given. Finally, it is proved that the relative entropy of a relative Θ-measure preserving transformations with respect to a relative sub-σ_Θ-algebra having finite atoms is affine.

1. Introduction

In 1948, Shannon introduced the concept of entropy to information theory. The Shannon entropy is taken to indicate the degree of uncertainty ascribed to a random variable. Examining a random phenomenon as a member of a σ-algebra, Kolmogorov introduced the concept of entropy to ergodic theory in 1958. Kolmogorov's entropy was improved by Sinai in [11]. Kolmogorov-Sinai entropy measures the rate of the loss of information for the iteration of finite partitions in a measure preserving transformation. Entropy as a mathematical device plays an important role in physical systems. On the other hand, one of the main objects in physical phenomena is the "observer". So, a method is needed to measure the entropy of a system from the point of view of an observer. A modeling for an observer of a set X is a fuzzy set $\Theta : X \to [0,1]$ [6]. In fact these kinds of fuzzy sets are called one dimensional observers. In this paper, the Kolmogorov-Sinai entropy is studied using mathematical modeling of an observer. Any mathematical model according to the

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view point of an observer Θ is called a relative model [6, 7]. The notion of a relative semi-dynamical system as a generalization of a fuzzy dynamical system has been defined in [6]. Also, the concept of the entropy of a relative semi-dynamical system has been introduced in [5, 10, 9]. This article is an attempt to present a new relative approach to the Kolmogorov-Sinai entropy. The first step in the evolution of relative entropy theory is to define the relative entropy of a sub-σ_Θ-algebra having finite atoms. Moreover, the definition of the relative entropy of a Θ-measure preserving transformations is based on the relative entropy of a sub-σ_Θ-algebra with finite atoms. Finally, the relative entropy of a relative semi-dynamical system is introduced and its ergodic properties are investigated.

2. Basic Notions

This section is devoted to provide the prerequisites that are necessary for the next section. Let (X, β) denotes a $\sigma-$finite measure space, i.e. a set equiped with a $\sigma-$algebra β of subsets of X. Further, let p denotes a probability measure on (X, β). Then (X, β, p) is called a probability space. Let $\varphi : X \to X$ be a measure preserving the invertible transformation of the probability space (X, β, p). In particular $\varphi(\beta) = \beta$ and $p(\varphi^{-1}(A)) = p(A)$ for all $A \in \beta$. Then (X, β, p, φ) is called a dynamical system. The entropy of the partition $\xi = \{A_1, \ldots, A_n\}$ of the probability space (X, β, p) is defined by

$$H(\xi, p) = -\sum_{i=1}^{n} p(A_i) \log p(A_i),$$

and the entropy of the dynamical system (X, β, p, φ) with respect to the finite partition ξ is given by

$$h(\varphi, \xi, p) = \lim_{n\to\infty} \frac{1}{n} H\left(\xi \vee \varphi^{-1}(\xi) \vee \cdots \vee \varphi^{-n}(\xi), p\right),$$

where $\varphi^{-1}(\xi) = \left\{\varphi^{-1}A_1, \ldots, \varphi^{-1}A_n\right\}$. Then the Kolmogrov- Sinai entropy of the automorphism φ is defined by

$$h(\varphi, p) = \sup_{\xi} h(\varphi, \xi, p),$$

where the supremum is taken over all finite partitions. In the following, we recall some known concepts of the relative structures.
Let Θ be an observer on X. Then we say $\lambda \subseteq \Theta$ if $\lambda(x) \leq \Theta(x)$ for all $x \in X$. Moreover, if $\lambda_1, \lambda_2 \subseteq \Theta$ then $\lambda_1 \vee \lambda_2$ and $\lambda_1 \wedge \lambda_2$ are subsets of Θ, and defined by

$$(\lambda_1 \vee \lambda_2)(x) = \sup\left\{\lambda_1(x), \lambda_2(x)\right\},$$

and

$$(\lambda_1 \wedge \lambda_2)(x) = \inf\{\lambda_1(x), \lambda_2(x)\},$$

where $x \in X$.

Definition 2.1. A collection F_Θ of subsets of Θ is said to be a σ_Θ-algebra in Θ if F_Θ satisfies the following conditions [10],

(i) $\Theta \in F_\Theta$,
(ii) if $\lambda \in F_\Theta$ then $\lambda^{'} = \Theta - \lambda \in F_\Theta$. $\lambda^{'}$ is called the complement of λ with respect to Θ,
(iii) if $\{\lambda_i\}_{i=1}^{\infty}$ is a sequence in F_Θ then $\vee_{i=1}^{\infty}\lambda_i = \sup_i \lambda_i \in F_\Theta$,
(iv) $\dfrac{\Theta}{2}$ doesn't belong to F_Θ.

If P_1 and P_2 are two σ_Θ-algebras on X then $P_1 \vee P_2$ is the smallest σ_Θ-algebra that contains $P_1 \cup P_2$, denoted by $[P_1 \cup P_2]$.

Definition 2.2. A positive $\Theta-$measure m_Θ over F_Θ is a function $m_\Theta : F_\Theta \to I$ which is countably additive. This means that if $\{\lambda_i\}$ is a disjoint countable collection of members of F_Θ, (i.e. $\lambda_i \subseteq \lambda_j^{'} = \Theta - \lambda_j$ whenever $i \neq j$) then

$$m_\Theta\left(\vee_{i=1}^{\infty}\lambda_i\right) = \sum_{i=1}^{\infty} m_\Theta(\lambda_i).$$

The $\Theta-$measure m_Θ has the following properties [10],

(i) $m_\Theta(\chi_\emptyset) = 0$,
(ii) $m_\Theta\left(\lambda^{'} \vee \lambda\right) = m_\Theta(\Theta)$ and $m_\Theta(\lambda^{'}) = m_\Theta(\Theta) - m_\Theta(\lambda)$ for all $\lambda \in F_\Theta$,
(iii) $m_\Theta(\lambda \vee \mu) + m_\Theta(\lambda \wedge \mu) = m_\Theta(\lambda) + m_\Theta(\mu)$ for each $\lambda, \mu \in F_\Theta$,
(iv) m_Θ is a nondecreasing function i.e. if $\lambda, \eta \in F_\Theta$ and $\lambda \subseteq \Theta$, then $m_\Theta(\lambda) \leq m_\Theta(\eta)$.

The triple (X, F_Θ, m_Θ) is called a $\Theta-$ measure space and the elements of F_Θ are called relative measurable sets. The $\Theta-$ measure space (X, F_Θ, m_Θ) is called a relative probability $\Theta-$measure space if $m_\Theta(\Theta) = 1$ [10].

3. Θ-relations and Atoms

Definition 3.1. Let (X, F_Θ, m) be a $\Theta-$measure space. The elements μ,λ of F_Θ are called m_Θ-disjoint if $m_\Theta(\lambda \wedge \mu) = 0$.

A $\Theta-$relation '=(mod m_Θ)' on F_Θ is defined as bellow

$$\lambda = \mu (\operatorname{mod} m_\Theta) \quad \text{iff} \quad m_\Theta(\lambda) = m_\Theta(\mu) = m_\Theta(\lambda \wedge \mu), \qquad \lambda, \mu \in F_\Theta.$$

The $\Theta-$relation '=(mod m_Θ)' is an equivalence relation. $\tilde{F_\Theta}$ denotes the set of all equivalence classes induced by this relation, and $\tilde{\mu}$ is the equivalence class determined by μ. For $\lambda, \mu \in F_\Theta$, $\lambda \wedge \mu = 0 \pmod{m_\Theta}$ iff λ, μ are m_Θ-disjoint. We shall identify $\tilde{\mu}$ with μ.

Definition 3.2. Let (X, F_Θ, m_Θ) be a $\Theta-$measure space, and P be a sub-σ_Θ-algebra of F_Θ. Then an element $\tilde{\lambda} \in \tilde{P}$ is an atom of P if

(i) $m_\Theta(\lambda) > 0$,
(ii) for each $\tilde{\mu} \in \tilde{P}$ such that $m_\Theta(\lambda \wedge \mu) = m_\Theta(\mu) \neq m_\Theta(\lambda)$ then $m_\Theta(\mu) = 0$.

Theorem 3.3. *Let (X, F_Θ, m) be a $\Theta-$measure space, and P be a sub-σ_Θ-algebra of F_Θ. If $\tilde{\lambda_1}, \tilde{\lambda_2}$ are disjoint atoms of P then they are m_Θ-disjoint.*

Proof. Since $\lambda_1 \wedge \lambda_2 \subseteq \lambda_1, \lambda_1 \wedge \lambda_2 \subseteq \lambda_2$, and $\lambda_1 \neq \lambda_2 (\mathrm{mod}\, m_\Theta)$, we get $\lambda_1 \wedge \lambda_2 \neq \lambda_i (\mathrm{mod}\, m_\Theta)$ for at least one $i = 1, 2$. Suppose $\lambda_1 \wedge \lambda_2 \neq \lambda_2 (\mathrm{mod}\, m_\Theta)$. Beacuse λ_2 is an atom, $\lambda_1 \wedge \lambda_2 = 0 (\mathrm{mod}\, m_\Theta)$. □

Now, we introduce $R_*(F_\Theta)$ as bellow,

$$R_*(F_\Theta) = \{P : P \text{ is a sub} - \sigma_\Theta - \text{algebra of } F_\Theta \text{with finite atoms}\}.$$

Assume that F_Θ is a σ_Θ-algebra, $P_1, P_2 \in R_*(F_\Theta)$, and $\{\lambda_i; i = 1, 2, \ldots, n\}$ and $\{\mu_j; j = 1, \ldots, m\}$ denote the atoms of P_1 and P_2, respectively, then the atoms of $P_1 \vee P_2$ are $\lambda_i \wedge \mu_j$ which $m_\Theta(\lambda_i \wedge \mu_j) > 0$ for each $1 \leq i \leq n, 1 \leq j \leq m$.

Definition 3.4. Let (X, F_Θ, m_Θ) be a relative probability $\Theta-$measure space and $P_1, P_2 \in R_*(F_\Theta)$. We say that P_2 is an m_Θ-refinement of P_1, denoted by $P_1 \leq_{m_\Theta} P_2$, if for each $\mu \in \bar{P}_2$ there exists $\lambda \in \bar{P}_1$ such that,

$$m_\Theta(\lambda \wedge \mu) = m_\Theta(\mu).$$

Theorem 3.5. *Let (X, F_Θ, m_Θ) be a relative probability $\Theta-$measure space and $P_1, P_2, P_3 \in R_*(F_\Theta)$. If $P_1 \leq_{m_\Theta} P_2$ then,*

$$P_1 \vee P_3 \leq_{m_\Theta} P_2 \vee P_3.$$

Proof. Let $\mu \in \overline{P_2 \vee P_3}$. Then $\mu = \lambda \wedge \gamma$ for some $\lambda \in \bar{P}_2$ and $\gamma \in \bar{P}_3$. Since $P_1 \leq_{m_\Theta} P_2$, there exists $\eta \in \bar{P}_1$ such that $m_\Theta(\eta \wedge \lambda) = m_\Theta(\lambda)$. Now,

$$\begin{aligned}
m_\Theta(\mu) &\geq m_\Theta(\eta \wedge \gamma \wedge \mu) \\
&= m_\Theta(\eta \wedge \gamma \wedge \lambda) \\
&= m_\Theta(\eta \wedge \lambda) + m_\Theta(\gamma) - m_\Theta((\eta \wedge \lambda) \vee \gamma) \\
&= m_\Theta(\lambda) + m_\Theta(\gamma) - m_\Theta((\eta \vee \lambda) \wedge (\lambda \vee \gamma)) \\
&= m_\Theta(\lambda) + m_\Theta(\gamma) - m_\Theta(\eta \vee \lambda) - m_\Theta(\lambda \vee \gamma) + m_\Theta(\eta \vee \gamma \vee \lambda) \\
&\geq m_\Theta(\lambda \wedge \gamma) \\
&= m_\Theta(\mu).
\end{aligned}$$

Hence,

$$m_\Theta(\eta \wedge \gamma \wedge \mu) = m_\Theta(\mu),$$

and the result follows. □

4. Relative Entropy of a sub-σ_Θ-algebra with Finite Atoms

Definition 4.1. Let (X, F_Θ, m_Θ) be a relative probability $\Theta-$measure space, and P be a sub σ_Θ-algebra of F_Θ which $P \in R_*(F_\Theta)$. The relative entropy of P is defined as

$$H_\Theta\left(P, m_\Theta\right) = -\sum_{\mu \in \bar{P}} m_\Theta(\mu) \log m_\Theta(\mu).$$

Example 4.2. Let (X, β, p) be a classical probability measure space and $\Theta = \chi_X$. Then $F_\Theta = \{\chi_A : A \in \beta\}$ is a σ_Θ-algebra on X. Define $m_\Theta(\chi_A) = p(A)$, $A \in \beta$. Then (X, F_Θ, m_Θ) is a relative probability $\Theta-$measure space. Let α be a finite sub-$\sigma-$algebra of β and $G = \{\chi_A : A \in \alpha\}$. So, $G \in R_*(F_\Theta)$ and the relative entropy of G is given by

$$\begin{aligned} H_\Theta(G, m_\Theta) &= -\sum_{A \in \bar{G}} m_\Theta(\chi_A) \log m_\Theta(\chi_A) \\ &= -\sum_{A \in \bar{G}} p(A) \log p(A), \end{aligned}$$

which is the Kolmogorov-Sinai entropy of the finite classical measurable sub-σ-algebra α of the space (X, β, p).

Thus, the concept of the relative entropy of a σ_Θ-algebra with finite atoms is a generalization of the Kolmogorov-Sinai entropy of a finite measurable σ-algebra.

Theorem 4.3. *Let (X, F_Θ, m_Θ) be a relative probability $\Theta-$measure space, and $P_1, P_2 \in R_*(F_\Theta)$ which $\bar{P}_1 = \{\lambda_i; 1 \le i \le n\}$ and $\bar{P}_2 = \{\mu_j; 1 \le j \le m\}$. If $P_1 \approx_{m_\Theta} P_2$ and $P_1 \le_{m_\Theta} P_2$ then,*

$$H_\Theta(P_1, m_\Theta) \le H_\Theta(P_2, m_\Theta).$$

Proof. Suppose that $P_1 \le_{m_\Theta} P_2$. Then for each $\mu_j \in \bar{P}_2$ there exists $\lambda_k \in \bar{P}_1$ such that, $m_\Theta(\mu_j \wedge \lambda_k) = m_\Theta(\mu_j)$. Since $P_1 \approx_{m_\Theta} P_2$ and λ_i's are pairwise m_Θ-disjoint, then

$$\begin{aligned} m_\Theta(\mu_j) &= m_\Theta(\mu_j \wedge (\vee \lambda_i)) \\ &= \sum_i m_\Theta(\mu_j \wedge \lambda_i). \end{aligned}$$

Therefore, $m_\Theta(\mu_j \wedge \lambda_i) = 0$ for each $i \ne k$. Hence,

$$m_\Theta(\mu_j) \log m_\Theta(\mu_j) = \sum_i m_\Theta(\mu_j \wedge \lambda_i) \log m_\Theta(\mu_j \wedge \lambda_i).$$

Now, set $\alpha = \{(i,j) : m_\Theta(\lambda_i \wedge \mu_j) > 0\}$, $\beta = \{i : m_\Theta(\lambda_i) > 0\}$. Then

$$\begin{aligned} H_\Theta(P_2, m_\Theta) &= -\sum_j \sum_i m_\Theta\,(\lambda_i \wedge \mu_j) \log m_\Theta(\lambda_i \wedge \mu_j) \\ &= -\sum_{(i,j)\in\alpha} m_\Theta(\lambda_i \wedge \mu_j) \log m_\Theta(\lambda_i \wedge \mu_j) \\ &\geq -\sum_{(i,j)\in\alpha} m_\Theta(\lambda_i \wedge \mu_j) \log m_\Theta(\lambda_i) \\ &\geq -\sum_{i\in\beta} \log m_\Theta(\lambda_i) \sum_j m_\Theta(\lambda_i \wedge \mu_j). \end{aligned}$$

Since μ_j's are pairwise m_Θ-disjoint, we have

$$\begin{aligned} H_\Theta(P_2, m_\Theta) &\geq -\sum_{i\in\beta} \log m_\Theta(\lambda_i) m_\Theta(\vee_j(\lambda_i \wedge \mu_j)) \\ &= -\sum_i m_\Theta(\lambda_i) \log m_\Theta(\lambda_i) \\ &= H_\Theta(P_1, m_\Theta). \end{aligned}$$

□

Definition 4.4. Let (X, F_Θ, m_Θ) be a relative probability $\Theta-$measure space and $P_1, P_2 \in R_*(F_\Theta)$. We say that P_1 and P_2 are m_Θ-equivalent, denoted by $P_1 \approx_{m_\Theta} P_2$, if the following axioms are satisfied:

(i) If $\lambda \in \bar{P}_1$ then $m_\Theta(\lambda \wedge (\vee\{\mu; \mu \in \bar{P}_2\})) = m_\Theta(\lambda)$.

(ii) If $\mu \in \bar{P}_2$ then $m_\Theta(\mu \wedge (\vee\{\lambda; \lambda \in \bar{P}_1\})) = m_\Theta(\mu)$.

Theorem 4.5. *Let (X, F_Θ, m_Θ) be a relative probability $\Theta-$measure space, and $P_1, P_2 \in R_*(F_\Theta)$. If $P_1 \approx_{m_\Theta} P_2$ then,*

$$P_1 \approx_{m_\Theta} P_1 \vee P_2.$$

Proof. Assume that $\bar{P}_1 = \{\lambda_i; 1 \leq i \leq n\}$ and $\bar{P}_2 = \{\mu_j; 1 \leq j \leq m\}$. We know that,

$$\overline{P_1 \vee P_2} = \big\{\lambda_i \wedge \mu_j; \lambda_i \in \bar{P}_1, \mu_j \in \bar{P}_2, m_\Theta(\lambda_i \wedge \mu_j) > 0\big\}.$$

If $\alpha = \{(i,j) : V_{ij} = \lambda_i \wedge \mu_j \in \overline{P_1 \vee P_2}\}$ then $\alpha = \bigcup_i \{(i,j); j \in \beta_i\}$ where $\beta_i = \{j; m_\Theta(V_{ij}) > 0\}$ and $1 \leq i \leq n$. Note that if $j \notin \beta_i$ then $m_\Theta(V_{ij}) = 0$ and we have

$$\begin{aligned} \vee_{i,j\in\mathbb{N}} V_{ij} &= \vee_{i\in\mathbb{N}} \left(\vee_{j\in\beta_i} V_{ij}\right) \\ &= \vee_{1\leq i\leq n} \left(\lambda_i \wedge (\vee_{j\in\beta_i} \mu_j)\right). \end{aligned}$$

Since the collections $\{\lambda_i; 1 \leq i \leq n\}$ and $\{\mu_j; 1 \leq j \leq m\}$ are m_Θ-disjoint, then we have

$$m_\Theta(\lambda_k \wedge (\vee_{i,j} V_{ij})) = m_\Theta(\lambda_k \wedge (\vee_i \lambda_i \wedge (\vee_{j\in\beta_i} \mu_j)))$$

$$
\begin{aligned}
&= m_\Theta(\lambda_k \wedge (\vee_{j\in\beta_i}\mu_j)) \\
&= m_\Theta(\lambda_k \wedge (\vee_{j\in\beta_k}\mu_j)) \\
&= m_\Theta(\vee_{j\in\beta_k}(\lambda_k \wedge \mu_j)) \\
&= \sum_{j\in\beta_k} m_\Theta(\vee V_{kj}) \\
&= \sum_{j} m_\Theta(\vee V_{kj}) \\
&= m_\Theta(\lambda_k \wedge (\vee_j \mu_j)) \\
&= m_\Theta(\lambda_k).
\end{aligned}
$$

□

Theorem 4.6. *Let (X, F_Θ, m_Θ) be a relative probability $\Theta-$measure space, and $P_1, P_2 \in R_*(F_\Theta)$. If $P_1 \approx_{m_\Theta} P_2$ then,*

$$H_\Theta(P_1, m_\Theta) \leq H_\Theta(P_1 \vee P_2, m_\Theta).$$

Proof. Suppose that $P_1 \approx_{m_\Theta} P_2$. By Theorem 3.8 we have $P_1 \approx_{m_\Theta} P_1 \vee P_2$. Now suppose that $\delta \in \overline{P_1 \vee P_2}$. Then $\delta = \lambda_i \wedge \mu_j$ which $\lambda_i \in \bar{P_1}$ and $\mu_j \in \bar{P_2}$. So for $\lambda_i \in \bar{P_1}$, $m_\Theta(\delta) = m_\Theta(\delta \wedge \lambda_i)$ and therefore we have $P_1 \leq_{m_\Theta} P_1 \vee P_2$. Now use Theorem 4.3. □

Theorem 4.7. *Let (X, F_Θ, m_Θ) be a relative probability $\Theta-$measure space, and $P_1, P_2 \in R_*(F_\Theta)$. If $P_1 \approx_{m_\Theta} P_2$ then,*

$$H_\Theta(P_1 \vee P_2, m_\Theta) \leq H_\Theta(P_1, m_\Theta) + H_\Theta(P_2, m_\Theta).$$

Proof. Suppose that $g : [0, 1] \to \mathbb{R}$ be the convex function $g(x) = x \log x$. Assume that $\bar{P_1} = \{\lambda_i; 1 \leq i \leq n\}$ and $\bar{P_2} = \{\mu_j; 1 \leq j \leq m\}$. Take $\alpha_j = m_\Theta(\mu_j), 1 \leq j \leq m$ and for a fixed i $(1 \leq i \leq n)$ put

$$x_j = \frac{m_\Theta(\lambda_i \wedge \mu_j)}{m_\Theta(\mu_j)}.$$

We have,

$$
\begin{aligned}
\sum_{j=1}^{m} \alpha_j x_j &= \sum_{j=1}^{m} m_\Theta(\lambda_i \wedge \mu_j) \\
&= m_\Theta(\lambda_i \wedge (\vee_{j=1}^{m} \mu_j)) \\
&= m_\Theta(\lambda_i).
\end{aligned}
$$

Put

$$\eta = \{(i, j) : 1 \leq i \leq n, 1 \leq j \leq m, m_\Theta(\lambda_i \wedge \mu_j) > 0\},$$

and $\beta_i = \{j; m_\Theta(\lambda_i \wedge \mu_j) > 0\}$. Let $\alpha = \sum_j \alpha_j$, then we get

$$m_\Theta(\lambda_i) g\left(\frac{m_\Theta(\lambda_i)}{\alpha}\right) \leq \sum_{j=1}^{m} m_\Theta(\mu_j) g\left(\frac{m_\Theta(\lambda_i \wedge \mu_j)}{m_\Theta(\mu_j)}\right)$$
$$= \sum_{j\in\beta_i} m_\Theta(\lambda_i \wedge \mu_j) \log \frac{m_\Theta(\lambda_i \wedge \mu_j)}{m_\Theta(\mu_j)},$$

or

$$m_\Theta(\lambda_i) \log\left(\frac{m_\Theta(\lambda_i)}{\alpha}\right) \leq \sum_{j\in\beta_i} m_\Theta(\lambda_i \wedge \mu_j) \log\left(\frac{m_\Theta(\lambda_i \wedge \mu_j)}{m_\Theta(\mu_j)}\right).$$

Now,

$$\begin{aligned} H_\Theta(P_1, m_\Theta) &= -\sum_{i=1}^{n} m_\Theta(\lambda_i) \log m_\Theta(\lambda_i) \\ &\geq -\sum_i m_\Theta(\lambda_i)\log\alpha - \sum_i \sum_{j\in\beta_i} m_\Theta(\lambda_i\wedge\mu_j)\log m_\Theta(\lambda_i\wedge\mu_j) \\ &\quad + \sum_i \sum_{j\in\beta_i} m_\Theta(\lambda_i\wedge\mu_j)\log m_\Theta(\mu_j) \\ &= -\sum_{(i,j)\in\eta} m_\Theta(\lambda_i\wedge\mu_j)\log m_\Theta(\lambda_i\wedge\mu_j) \\ &\quad + \sum_j \log m_\Theta(\mu_j) \sum_i m_\Theta(\lambda_i\wedge\mu_j) - \log\alpha \sum_i m_\Theta(\lambda_i) \\ &\geq H_\Theta(P_1 \vee P_2, m_\Theta) + \sum_j m_\Theta(\mu_j \wedge (\vee\lambda_i)) \log m_\Theta(\mu_j) \\ &= H_\Theta(P_1 \vee P_2, m_\Theta) - H_\Theta(P_2, m_\Theta). \end{aligned}$$

Thus, $H_\Theta(P_1, m_\Theta) \leq H_\Theta(P_1 \vee P_2, m_\Theta)$. □

5. Relative Entropy of a $\Theta-$measure Preserving Transformations

Definition 5.1. Suppose (X, F_Θ, m_Θ) be an $\Theta-$measure space and Θ be a constant observer on X. A transformation $\varphi : (X, F_\Theta, m_\Theta) \to (X, F_\Theta, n_\Theta)$, is said to be a $\Theta-$measure preserving if $m_\Theta(\varphi^{-1}(\mu)) = n_\Theta(\mu)$ for all $\mu \in \bar{F}_\Theta$.

Theorem 5.2. *Suppose that*

$$\varphi : (X, F_\Theta, m_\Theta) \to (X, F_\Theta, n_\Theta)$$

be a $\Theta-$measure preserving transformations. Then for each $P \in R_*(F_\Theta)$ we have,

$$H_\Theta(P, m_\Theta) = H_\Theta\left(\varphi^{-1}(P), m_\Theta\right).$$

Proof. Since φ is a $\Theta-$measure preserving, we have

$$m_\Theta(\varphi^{-1}(\mu)) = n_\Theta(\mu),$$

then,

$$\begin{aligned} H_\Theta\left(\varphi^{-1}(P), m_\Theta\right) &= -\sum_{\mu \in \bar{P}} m_\Theta\left(\varphi^{-1}(\mu)\right) \log m_\Theta\left(\varphi^{-1}(\mu)\right) \\ &= -\sum_{\mu \in \bar{P}} n_\Theta(\mu) \log n_\Theta(\mu) \\ &= H_\Theta(P, m_\Theta). \end{aligned}$$

□

Definition 5.3. Suppose $\varphi : (X, F_\Theta, m_\Theta) \to (X, F_\Theta, n_\Theta)$, be a $\Theta-$measure preserving transformation. If $P \in R_*(F_\Theta)$, we define the relative entropy of φ with respect to P as

$$h_\Theta(\varphi, P, m_\Theta) = \lim_{n\to\infty} \frac{1}{n} H_\Theta\left(\vee_{i=0}^{n-1} \varphi^{-i}(P), m_\Theta\right).$$

Theorem 5.4. $\lim\limits_{n\to\infty} \frac{1}{n} H_\Theta\left(\vee_{i=0}^{n-1} \varphi^{-i}(P), m_\Theta\right)$ *exsists.*

Proof. Let

$$a_n = H_\Theta\left(\vee_{i=0}^{n-1} \varphi^{-i}(P), m_\Theta\right) \geq 0.$$

Using Theorem 4.7 and Theorem 5.2, we have

$$\begin{aligned} a_{n+k} &= H_\Theta\left(\vee_{i=0}^{n+k-1} \varphi^{-i}(P), m_\Theta\right) \\ &\leq H_\Theta\left(\vee_{i=0}^{n-1} \varphi^{-i}(P), m_\Theta\right) + H_\Theta\left(\vee_{i=n}^{n+k-1} \varphi^{-i}(P), m_\Theta\right) \\ &= a_n + a_k. \end{aligned}$$

So, for each n, k we have $a_{n+k} \leq a_n + a_k$. Now, by Theorem 4.9 in [12] $\lim_{n\to\infty} \dfrac{a_n}{n}$ exists. □

Theorem 5.5. *Let* $\varphi : (X, F_\Theta, m_\Theta) \to (X, F_\Theta, n_\Theta)$*, be a* $\Theta-$*measure preserving transformations and* $P \in R_*(F_\Theta)$*. Then,*

(i) $h_\Theta\left(\varphi, \varphi^{-1}(P)\right) = h_\Theta(\varphi, P)$,
(ii) $h_\Theta\left(\varphi, \vee_{i=0}^{r-1} \varphi^{-i}(P)\right) = h_\Theta(\varphi, P)$ *for every* $r \geq 1$.

Proof. (i) It is obvious.
(ii) We have

$$\begin{aligned} h_\Theta\left(\varphi, \vee_{i=1}^{\infty} \varphi^{-i}(P), m_\Theta\right) &= \lim_{n\to\infty} \frac{1}{n} H_\Theta\left(\vee_{j=0}^{n-1} \varphi^{-j}\left(\vee_{i=0}^{r-1} \varphi^{-i}(P)\right), m_\Theta\right) \\ &= \lim_{n\to\infty} \frac{1}{n} H_\Theta\left(\vee_{i=0}^{r+n-2} \varphi^{-i}(P), m_\Theta\right) \end{aligned}$$

$$= \lim_{n\to\infty} \left(\frac{r+n-2}{n}\right)\left(\frac{1}{r+n-2}\right)$$
$$\times H_\Theta \left(\vee_{i=0}^{r+n-2}\varphi^{-i}(P), m_\Theta\right)$$
$$= h_\Theta\left(\varphi, \varphi(P), m_\Theta\right).$$

□

Theorem 5.6. *Let $\varphi : (X, F_\Theta, m_\Theta) \to (X, F_\Theta, n_\Theta)$, be a $\Theta-$measure preserving transformations and $P_1, P_2 \in R_*(F_\Theta)$. If $P_1 \approx_{m_\Theta} P_2$ and $P_1 \leq_{m_\Theta} P_2$ then $h_\Theta(\varphi, P_1, m_\Theta) \leq h_\Theta(\varphi, P_2, m_\Theta)$.*

Proof. The result follows from Theorem 4.3. □

6. Relative Entropy and (Θ_1, Θ_2)-isomorphic Dynamical Systems

Definition 6.1. A relative semi-dynamical system is denoted by $(X, F_\Theta, m_\Theta, \varphi)$ which (X, F_Θ, m_Θ) is a relative probability Θ-measure space and φ is a $\Theta-$measure preserving transformations.

Definition 6.2. Let $(X, F_\Theta, m_\Theta, \varphi)$ be a relative semi-dynamical system and $L \in R_*(F_\Theta)$. Suppose $[L]_\Theta$ denotes the m_Θ-equivalence class induced by L. Then the relative entropy $h_\Theta(\varphi, [L]_\Theta)$ of φ on L is defined as

$$h_\Theta\left(\varphi, [L]_\Theta, m_\Theta\right) = \sup_{P\in[L]_\Theta} h_\Theta\left(\varphi, P, m_\Theta\right).$$

Definition 6.3. Suppose $(X_1, F_{\Theta_1}, m_{\Theta_1})$ be a Θ_1-measure space and $(X_2, F_{\Theta_2}, n_{\Theta_2})$ be a Θ_2-measure space. A transformation

$$\varphi : (X_1, F_{\Theta_1}, m_{\Theta_1}) \to (X_2, F_{\Theta_2}, n_{\Theta_2}),$$

is said to be a $(\Theta_1, \Theta_2)-$measure preserving if

(i) $\varphi^{-1}(\mu) \in F_{\Theta_1}$ for every $\mu \in F_{\Theta_2}$, where $\varphi^{-1}(\mu)(x) = \mu(\varphi(x)), \forall x \in X$,
(ii) $m_{\Theta_1}(\varphi^{-1}(\mu)) = n_{\Theta_2}(\mu)$ for all $\mu \in \bar{F}_{\Theta_2}$.

Theorem 6.4. *Suppose $\varphi : (X_1, F_{\Theta_1}, m_{\Theta_1}) \to (X_2, F_{\Theta_2}, n_{\Theta_2})$, be a $(\Theta_1, \Theta_2)-$measure preserving transformations. Then for each $P \in R_*(F_{\Theta_2})$ we have,*

$$H_{\Theta_2}(P, m_{\Theta_2}) = H_{\Theta_1}(\varphi^{-1}(P), m_{\Theta_1}).$$

Proof. By Theorem 5.2, the proof is clear. □

Definition 6.5. A relative semi-dynamical system $\phi_1 = (X_1, F_{\Theta_1}, m_{\Theta_1})$ is a $(\Theta_1, \Theta_2)-$factor of the relative semi-dynamical system $\phi_2 = (X_2, F_{\Theta_2}, n_{\Theta_2})$ if there exists an onto $(\Theta_1, \Theta_2)-$measure preserving transformations (called homomorphism) $\psi : \phi_2 \to \phi_1$ such that,

$$\psi o \varphi_2 = \varphi_1 o \psi.$$

Theorem 6.6. *Let $\phi_1 = (X_1, F_{\Theta_1}, m_{\Theta_1})$ be a $(\Theta_1, \Theta_2)-$factor of the relative semi-dynamical system $\phi_2 = (X_2, F_{\Theta_2}, n_{\Theta_2})$. Then for each $L \in R_*(F_\Theta)$,*

$$h_{\Theta_1}\left(\varphi_1, \left[\psi^{-1}(L)\right]_{\Theta_1}, m_{\Theta_1}\right) \leq h_{\Theta_2}\left(\varphi_2, [L]_{\Theta_2}, m_{\Theta_2}\right),$$

Where $\psi : \phi_2 \to \phi_1$ is the corresponding homomorphism.

Proof. Suppose that $P \in [L]_{\Theta_2}$. Then by Theorem 6.4,

$$H_{\Theta_2}(P, m_{\Theta_2}) = H_{\Theta_1}(\psi^{-1}(P), m_{\Theta_1}).$$

Now,

$$\begin{aligned} h_{\Theta_2}\left(\varphi_2, P, m_{\Theta_2}\right) &= \lim_{n\to\infty} \frac{1}{n} H_{\Theta_2}\left(\vee_{i=0}^{n-1} \varphi_2^{-i}(P), m_{\Theta_2}\right) \\ &= \lim_{n\to\infty} \frac{1}{n} H_{\Theta_1}\left(\psi^{-1}(\vee_{i=0}^{n-1} \varphi_2^{-i}(P)), m_{\Theta_1}\right) \\ &= \lim_{n\to\infty} \frac{1}{n} H_{\Theta_1}\left(\vee_{i=0}^{n-1} \psi^{-1}\varphi_2^{-i}(P), m_{\Theta_1}\right) \\ &= \lim_{n\to\infty} \frac{1}{n} H_{\Theta_1}\left(\vee_{i=0}^{n-1} \varphi_1^{-i}\psi^{-1}(P), m_{\Theta_1}\right) \\ &= h_{\Theta_1}\left(\varphi_1, \psi^{-1}(P), m_{\Theta_1}\right). \end{aligned}$$

As P ranges over an m_{Θ_2}-equivalence class $[L]_{\Theta_2}$ in $R_*(F_{\Theta_2})$, $\psi^{-1}(P)$ ranges over a subset of the m_{Θ_1}-equivalence class $[\psi^{-1}(L)]_{\Theta_1}$ in $R_*(F_{\Theta_1})$. □

Definition 6.7. Two relative semi-dynamical systems $\phi_1 = (X_1, F_{\Theta_1}, m_{\Theta_1})$ and $\phi_2 = (X_2, F_{\Theta_2}, m_{\Theta_2})$ are said to be (Θ_1, Θ_2)-isomorphic if there exists an invertible relative $(\Theta_1, \Theta_2)-$measure preserving transformations $\psi : \phi_1 \to \phi_2$ (i.e both ψ and ψ^{-1} are relative measure preserving transformations) such that,

$$\psi o \varphi_1 = \varphi_2 o \psi.$$

The mapping ψ is called (Θ_1, Θ_2)-isomorphism.

Theorem 6.8. *Let ϕ_1 and ϕ_2 be (Θ_1, Θ_2)-isomorphic semi-dynamical systems. Then for each $L \in R_*(F_{\Theta_2})$,*

$$h_{\Theta_1}\left(\varphi_1, \left[\psi^{-1}(L)\right]_{\Theta_1}, m_{\Theta_1}\right) = h_{\Theta_2}\left(\varphi_2, [L]_{\Theta_2}, m_{\Theta_2}\right),$$

which $\psi : \phi_1 \to \phi_2$ is the corresponding (Θ_1, Θ_2)-isomorphism.

Proof. The result follows from Theorem 6.6. □

7. Relative Entropy and m_Θ-generators of Relative Semi-Dynamical Systems

Definition 7.1. The relative entropy of the relative semi-dynamical system $(X, F_\Theta, m_\Theta, \varphi)$ is the number $h_\Theta(\varphi, m_\Theta)$ defined by,

$$h_\Theta(\varphi, m_\Theta) = \sup_P h_\Theta(\varphi, P, m_\Theta),$$

where the supremum is taken over all sub-σ_Θ-algebras of F_Θ which $P \in R_*(F_\Theta)$.

Definition 7.2. $P \in R_*(F_\Theta)$ is said to be an m_Θ-generator of the relative semi-dynamical system $(X, F_\Theta, m_\Theta, \varphi)$ if there exists an integer $r > 0$ such that,

$$Q \leq_{m_\Theta} \vee_{i=0}^{r} \varphi^{-i} P,$$

for each $Q \in R_*(F_\Theta)$.

Theorem 7.3. *If P is an m_Θ-generator of the relative semi-dynamical system $(X, F_\Theta, m_\Theta, \varphi)$ then,*

$$h_\Theta(\varphi, Q, m_\Theta) \leq h_\Theta(\varphi, P, m_\Theta),$$

for each $Q \in R_*(F_\Theta)$.

Proof. Let $Q \in R_*(F_\Theta)$ be any arbitrary sub-σ_Θ-algebra of F_Θ. Since P is an m_Θ-generator, $Q \leq_{m_\Theta} \vee_{i=0}^{r} \varphi^{-i} P$ follows from Theorem 5.6,

$$\begin{aligned} h_\Theta(\varphi, Q, m_\Theta) &\leq h_\Theta\left(\varphi, \vee_{i=0}^{r} \varphi^{-i} P, m_\Theta\right) \\ &= h_\Theta(\varphi, P, m_\Theta). \end{aligned}$$

□

Now we can deduce the following version of the Kolmogorov-Sinai theorem.

Theorem 7.4. *If P is an m_Θ-generator of the relative semi-dynamical system $(X, F_\Theta, m_\Theta, \varphi)$ then,*

$$h_\Theta(\varphi, m_\Theta) = h_\Theta(\varphi, P, m_\Theta).$$

Proof. It is obvious. □

Theorem 7.5. *Let $(X, F_\Theta, m_\Theta, \varphi)$ be a relative semi-dynamical system. Then, the map $m_\Theta \mapsto h_\Theta(\varphi, m_\Theta)$ is affine, i.e,*

$$h_\Theta(\varphi, \lambda m_\Theta + (1-\lambda) n_\Theta) = \lambda h_\Theta(\varphi, m_\Theta) + (1-\lambda) h_\Theta(\varphi, n_\Theta),$$

for each pair m_Θ and n_Θ of the relative probability Θ-measures and $\lambda \in [0, 1]$.

Proof. Suppose that $P \in R_*(F_\Theta)$. If m_Θ and n_Θ are two relative probability $\Theta-$measures and $\lambda \in [0,1]$ then,

$$H_\Theta\left(P, \lambda m_\Theta + (1-\lambda) n_\Theta\right) \geq \lambda H_\Theta\left(P, m_\Theta\right) + (1-\lambda) H_\Theta\left(P, n_\Theta\right). \tag{7.1}$$

The 'concavity' inequality (7.1) is a direct consequence of the definition of $H_\Theta(P, m_\Theta)$ and the 'concavity' of the function $x \mapsto -x\log x$. Conversely, one has inequalities

$$-\log\left(\lambda m_\Theta(\mu_i) + (1-\lambda) n_\Theta(\mu_i)\right) \leq -\log\lambda - \log m_\Theta(\mu_i),$$

and

$$-\log(\lambda m_\Theta(\mu_i) + (1-\lambda) n_\Theta(\mu_i)) \leq -\log(1-\lambda) - \log n_\Theta(\mu_i).$$

Because $x \mapsto -\log x$ is decreasing, therfore, one obtains the 'convexity' bound,

$$\begin{aligned} H_\Theta(P, \lambda m_\Theta + (1-\lambda) n_\Theta) \leq \lambda H(P, m_\Theta) + (1-\lambda) H(P, n_\Theta) \\ - \lambda\log\lambda - (1-\lambda)\log(1-\lambda). \end{aligned} \tag{7.2}$$

Now replacing P by $\vee_{i=0}^{n-1}\varphi^{-i}(P)$ in (7.1), dividing by n and taking the $\lim_{n\to\infty}$ gives

$$h_\Theta(\varphi, P, \lambda m_\Theta + (1-\lambda) n_\Theta) \geq \lambda h_\Theta(\varphi, P, m_\Theta) + (1-\lambda) h_\Theta(\varphi, P, n_\Theta).$$

Similarly from (7.2), since

$$\frac{-(\lambda\log\lambda + (1-\lambda)\log(1-\lambda))}{n} \to 0,$$

as $n \to \infty$, one deduces the converse inequality

$$h_\Theta\left(\varphi, P, \lambda m_\Theta + (1-\lambda) n_\Theta\right) \leq \lambda h_\Theta\left(\varphi, P, m_\Theta\right) + (1-\lambda) h_\Theta\left(\varphi, P, n_\Theta\right).$$

Hence one concludes that the map $m_\Theta \mapsto h_\Theta(\varphi, P, m_\Theta)$ is affine. Finally, it follows from Theorem 7.4 that the relative entropy is affine. □

This is somewhat surprising and is of great significance in the application of the relative entropy.

8. Concluding Remarks and Open Problems

In this paper, the notion of the relative entropy for a sub-σ_Θ-algebra with finite atoms is presented. The entropy of a relative semi-dynamical system is defined using the observer notion and its properties are investigated. Also, the notion of an m_Θ-generator for a relative semi-dynamical system is introduced and a relative version of Kolmogorov-Sinai theorem concerning the entropy of a relative semi-dynamical system is given. Finally, it is proved that the relative entropy of a relative Θ-measure preserving transformations with respect to a relative sub-σ_Θ-algebra having finite atoms is affine.

An interesting open problem is to establish a theorem on existence of m_Θ-generators for relative semi-dynamical systems.

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