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On Some Properties of the Max Algebra System Over Tensors

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Abstract. Recently we generalized the max algebra system to the class of nonnegative tensors. In this paper we give some basic properties for the left (right) inverse, under the new system. The existence of order 2 left (right) inverse of tensors is characterized. Also we generalize the direct product of matrices to the direct product of tensors (of the same order, but may be different dimensions) and investigate its properties relevant to the spectral theory.

1. Introduction

The algebraic system max algebra provides an attractive way of describing a class of non-linear problems appearing for instance in manufacturing and transportation scheduling, information technology, discrete event-dynamic systems, combinatorial optimization, mathematical physics, DNA analysis and etcetera. Max algebras usefulness arises from a fact that these non-linear problems become linear when described in the max algebra language. The max eigenproblem is well studied and there are important explicit applications of it in solving the problems mentioned above. We refer to [2, 3, 7] for a description of such systems and their applications.

Tensors are increasingly ubiquitous in various areas of applied, computational, and industrial mathematics and have wide applications in data analysis and mining, information science, signal/image processing, computational biology, and so on, see the workshop report [8] and references therein. A tensor can be regarded as a higher order generalization

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of a matrix, which takes the form

$$\mathbb{A} = \left(a_{i_1 \ldots i_m}\right), \qquad , a_{i_1 \ldots i_m} \in \Re, \qquad 1 \leq \ i_1, \ldots, i_m \leq n,$$

where $\Re$ is the real field. Such a multi-array $\mathbb{A}$ is said to be an mth order n-dimensional square real tensor with n^m entries $a_{i_1 \ldots i_m}$. In this regard, a vector is a first order tensor and a matrix is a second order tensor. Tensors of order more than two are called higher order tensors. Many important ideas, notions, and results have been successfully extended from matrices to higher order tensors.

Recently we generalized the concept of max algebra system from matrices to nonnegative tensors and some of their properties were investigated [1]. This paper is a sequel to our previous work. Here, we establish more properties of max algebra system over the nonnegative tensor.

The paper is organized as follows. In Section 2, the fundamental concept of max algebra system and tensors are given briefly. In Section 3, we give further results on the max algebra system on $\Re_+^{[m,n]}$, and generalize an other theorem from matrices to nonnegative tensors.
Also, some important characterization of left (right) inverse tensor in max algebra sense are analyzed.

We first add a comment on the notation that is used. Vectors are written as $(x, y, \ldots)$, matrices correspond to $(A, B, \ldots)$ and tensors are written as $(\mathbb{A}, \mathbb{B}, \ldots)$. The entry with row index i and column index j in a matrix A, i.e. $(A)_{ij}$ is symbolized by a_{ij} (also $(\mathbb{A})_{i_1 i_2 \ldots i_m} = a_{i_1 i_2 \ldots i_m}$). $\Re$ and $\mathcal{C}$ represents the real and complex field, respectively. For each nonnegative integer n, denote $[n] = \{1, 2, \ldots, n\}$. $\Re_+^n$ $\left(\Re_{++}^n\right)$ denotes the cone $\{x \in \Re^n : x_i \geq (>)\ 0,\ i = 1, \ldots, n\,\}$.

2. Preliminaries

2.1. **Max Algebra System.** We consider the set of nonnegative numbers equipped with two binary operations, defined as follows. If $a, b \in \Re_+$, then their sum, denoted $a \oplus b$, is defined as $\max\{a, b\}$ whereas their product is the usual product, ab. Addition and multiplication of vectors and matrices are defined in a natural way. If A, B are matrices compatible for matrix multiplication, then we denote their product by $A \odot B$, when $\oplus$ is used as a sum, to distinguish it from AB. For example, if

$$A = \begin{bmatrix} 1 & 0 & 4 \\ 2 & 3 & 5 \\ 7 & 1 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 3 & 2 & 0 \\ 1 & 4 & 2 \\ 2 & 1 & 1 \end{bmatrix},$$

then

$$A \oplus B = \begin{bmatrix} 3 & 2 & 4 \\ 2 & 4 & 5 \\ 7 & 1 & 1 \end{bmatrix}, \qquad A \odot B = \begin{bmatrix} 8 & 4 & 4 \\ 10 & 12 & 6 \\ 21 & 14 & 2 \end{bmatrix}.$$

It can be easily proved that the product $\odot$ is associative and that it distributes over the sum $\oplus$.

2.2. **Basic Definition of Tensor.** In this subsection, we will cover some fundamental notions and properties on tensors.

An mth order n-dimensional tensor $\mathbb{A}$ is called nonnegative if $a_{i_1 i_2 \dots i_m} \geq 0$. We denote the set of all nonnegative mth order n-dimensional tensors by $\Re_+^{[m,n]}$. For a vector $x = (x_1, \dots, x_n)^T$, let $\mathbb{A}x^{m-1}$ be a vector in $\Re^n$ whose ith component is defined as the following [10]:

$$\left(\mathbb{A}x^{m-1}\right)_i = \sum_{i_2,\dots,i_m=1}^{n} a_{ii_2\dots i_m} x_{i_2} \dots x_{i_m},$$

and let $x^{[m]} = (x_1^m, \dots, x_n^m)^T$.

Definition 2.1 ([11])**.** Let $\mathbb{A}$ (and $\mathbb{B}$) be an order $m \geq 2$ (and order $k \geq 1$), dimension n tensor, respectively. The product $\mathbb{AB}$ is defined to be the following tensor $\mathbb{C}$ of order $(m-1)(k-1)+1$ and dimension n:

$$c_{i\alpha_1\dots\alpha_{m-1}} = \sum_{i_2,\dots,i_m=1}^{n} a_{ii_2\dots i_m} b_{i_2\alpha_1} \dots b_{i_m\alpha_{m-1}},$$

where $(i \in [n],\ \alpha_1, \dots, \alpha_{m-1} \in [n]^{k-1})$.

It is easy to check from the definition that $I_n\mathbb{A} = \mathbb{A} = \mathbb{A}I_n$, where I_n is the identity matrix of order n. When $k = 1$ and $\mathbb{B} = x \in \mathcal{C}^n$ is a vector of dimension n, then $(m-1)(k-1)+1 = 1$. Thus $\mathbb{AB} = \mathbb{A}x$ is still a vector of dimension n, and we have

$$(\mathbb{A}x)_i = (\mathbb{AB})_i = c_i = \sum_{i_2\dots i_m=1}^{n} a_{ii_2\dots i_m} x_{i_2} \dots x_{i_m} = \left(\mathbb{A}x^{m-1}\right)_i,$$

thus we have $\mathbb{A}x^{m-1} = \mathbb{A}x$. So the first application of the tensor product defined above is that now $\mathbb{A}x^{m-1}$ can be simply written as $\mathbb{A}x$.

Definition 2.2 ([6])**.** A tensor $\mathbb{A} \in \Re^{[m,n]}$ is called reducible, if there exists a nonempty proper index subset $I \subset \{1, \dots, n\}$ such that

$$a_{i_1\dots i_m} = 0, \quad \forall i_1 \in I, \quad \forall i_2, \dots, i_m \notin I.$$

If $\mathbb{A}$ is not reducible, then we call $\mathbb{A}$ irreducible.

Definition 2.3. For a given $\mathbb{A} = (a_{i_1\dots i_m}) \in \Re_+^{[m,n]}$, it is associated to a directed graph $G(\mathbb{A}) = (V, E(\mathbb{A}))$, where $V = \{1, 2, \dots, n\}$ and a directed edge $(i, j) \in E(\mathbb{A})$ if there exists indices $\{i_2, \dots, i_m\}$ such that $j \in \{i_2, \dots, i_m\}$ and $a_{ii_2\dots i_m} > 0$. In particular, we have

$$\sum_{j \in \{i_2,\dots,i_m\}} a_{ii_2\dots i_m} > 0.$$

A graph is strongly connected if it contains a directed path from i to j and a directed path from j to i for every pair of vertices i, j.

3. Further Results of the Max Algebra on Tensors

3.1. Basic Definitions and Properties. Recently, we generalized the max algebra system to the class of nonnegative tensors as the following [1]:

Definition 3.1. The max algebraic addition ($\oplus$) and multiplication ($\odot$) are defined as follows:

(i) Suppose that $\mathbb{A}, \mathbb{B} \in \Re_+^{[m,n]}$ then we have $\mathbb{A} \oplus \mathbb{B} \in \Re_+^{[m,n]}$ and

$$(\mathbb{A} \oplus \mathbb{B})_{i_1 \dots i_m} = a_{i_1 \dots i_m} \oplus b_{i_1 \dots i_m} = \max\left(a_{i_1 \dots i_m}, b_{i_1 \dots i_m}\right).$$

(ii) Suppose that $\mathbb{A} \in \Re_+^{[m,n]}$ and $\mathbb{B} \in \Re_+^{[k,n]}$ where $m \geq 2, k \geq 1$ then we have

$$\mathbb{A} \odot \mathbb{B} \in \Re_+^{[(m-1)(k-1)+1,n]},$$

and

$$(\mathbb{A} \odot \mathbb{B})_{i\alpha_1 \dots \alpha_{m-1}} = \bigoplus_{i_2 \dots i_m = 1}^{n} a_{ii_2 \dots i_m} b_{i_2 \alpha_1} \dots b_{i_m \alpha_{m-1}}$$

$$= \max_{1 \leq i_2 \dots i_m \leq n} \left\{ a_{ii_2 \dots i_m} b_{i_2 \alpha_1} \dots b_{i_m \alpha_{m-1}} \right\},$$

where $i \in \{1, \dots, n\}$, $\alpha_1, \dots, \alpha_{m-1} \in [n]^{k-1}$. In particular for $x \in \Re_+^n$ we have

$$(\mathbb{A} \odot x)_i = \max_{1 \leq i_2 \dots i_m \leq n} \left\{ a_{ii_2 \dots i_m} x_{i_2} \dots x_{i_m} \right\}.$$

Example 3.2. Let $\mathbb{A}$ and $\mathbb{B}$ be third-order two-dimensional tensors of the following form:

$$\begin{array}{llll} a_{111} = 1 & a_{121} = 2 & a_{112} = 1 & a_{122} = 2 \\ a_{211} = 2 & a_{221} = 1 & a_{212} = 2 & a_{222} = 1, \end{array}$$

$$\begin{array}{llll} b_{111} = 2 & b_{121} = 0 & b_{112} = 4 & b_{122} = 1 \\ b_{211} = 0 & b_{221} = 3 & b_{212} = 1 & b_{222} = 0. \end{array}$$

If $\mathbb{C} = \mathbb{A} \odot \mathbb{B}$, then for example $c_{12112} = 24$.
If $x = \binom{1}{3}$ then $(\mathbb{A} \odot x) = \binom{18}{9}$.

Theorem 3.3. *The max algebraic addition ($\oplus$) and multiplication ($\odot$) have the following properties:*

(i) *Let $\mathbb{A} \in \Re_+^{[m,n]}$, then $I_n \odot \mathbb{A} = \mathbb{A} = \mathbb{A} \odot I_n$ where I_n is a identity matrix.*
(ii) *$(\lambda \mathbb{A}) \odot \mathbb{B} = \lambda (\mathbb{A} \odot \mathbb{B})$ where λ is a nonnegative number.*

(iii) $\mathbb{A} \odot (\lambda\mathbb{B}) = \lambda^{m-1} (\mathbb{A} \odot \mathbb{B})$ *where* λ *be a nonnegative number.*

(iv) *Let* $\mathbb{A}_1$, $\mathbb{A}_2 \in \Re_+^{[m,n]}$ *and* $\mathbb{B} \in \Re_+^{[k,n]}$ *then*

$$(\mathbb{A}_1 \oplus \mathbb{A}_2) \odot \mathbb{B} = (\mathbb{A}_1 \odot \mathbb{B}) \oplus (\mathbb{A}_2 \odot \mathbb{B}).$$

(v) *Let* A *be an* $n \times n$ *matrix and* $\mathbb{B}_1, \mathbb{B}_2 \in \Re_+^{[k,n]}$ *then*

$$A \odot (\mathbb{B}_1 \oplus \mathbb{B}_2) = (A \odot \mathbb{B}_1) \oplus (A \odot \mathbb{B}_2).$$

(Note that in general when A *is not a matrix, then the right distributivity doesn't hold.)*

Proof. See [1]. □

Theorem 3.4. *Let* $\mathbb{A}$ *(*$\mathbb{B}$ *and* $\mathbb{C}$*) be an order* $m+1$ *(order* $k+1$ *and order* $r+1$*), dimension* n *tensor, respectively. Then we have*

$$\mathbb{A} \odot (\mathbb{B} \odot \mathbb{C}) = (\mathbb{A} \odot \mathbb{B}) \odot \mathbb{C}.$$

Proof. See [1]. □

Definition 3.5. Two mth order n dimensional tensor $\mathbb{A} = (a_{i_1 \dots i_m})$ and $\mathbb{B} = (b_{i_1 \dots i_m})$ are said to have the same zero pattern if $a_{i_1 \dots i_m} = 0$ whenever $b_{i_1 \dots i_m} = 0$, and vice versa.

Definition 3.6. A matrix that has the same zero pattern as a permutation matrix is called a generalized permutation matrix. Or a matrix obtained from a diagonal matrix by permuting the rows or columns is called a generalized matrix permutation.

Definition 3.7. A diagonal tensor is a tensor that only the entries of which all the indices are equal can be different from zero. A diagonal tensor with all diagonal entries equal to 1 is called the unit tensor and denoted by $\mathbb{I}$.

Theorem 3.8. *Let* $\mathbb{A}, \mathbb{I} \in \Re_+^{[m,n]}$*, then*

(i) *If* $\mathbb{A} \odot \mathbb{I} = 0$*, then* $\mathbb{A} = 0$.

(ii) *If* $\mathbb{I} \odot \mathbb{A} = 0$*, then* $\mathbb{A} = 0$.

(iii) *If* $\mathbb{A}$ *is irreducible, then*

$$\max_{1 \le i_2, \dots, i_m \le n} \{a_{i i_2 \dots i_m}\} > 0, \quad \forall\, 1 \le i \le n.$$

Proof.

(i) By Definition 3.1, we have

$$(\mathbb{A} \odot \mathbb{I})_{i\alpha_1 \dots \alpha_{m-1}} = \begin{cases} a_{ii_2 \dots i_m}, & \alpha_j = i_{j+1},\ j = 1, 2, \dots, m-1, \\ 0, & \text{otherwise} \end{cases}$$

(ii) Suppose that $\mathbb{A}$ has a nonzero entry $a_{i_1 i_2 \ldots i_m}$, by Definition 3.1, we have $\mathbb{I} \odot \mathbb{A}$ has a nonzero entry $(\mathbb{I} \odot \mathbb{A})_{i_1 \alpha} = a_{i_1 \alpha}^{(m-1)}$, for $\alpha = i_2 \ldots i_m$, which is a contradiction.

(iii) Suppose the contrary, then there exists i_0 so that

$$\max_{1 \leq i_2, \ldots, i_m \leq n} \{a_{i_0 i_2 \ldots i_m}\} = 0.$$

Thus $a_{i_0 i_2 \ldots i_m} = 0$, for all $i_2, \ldots, i_m$. In particular, if we assume $I = \{i_0\}$, then $a_{i_1 i_2 \ldots i_m} = 0$ for all $i_1 \in I$ and $i_2, \ldots, i_m \notin I$. This contradicts the irreducibility.

□

Lemma 3.9. *Let $\mathbb{A} \in \Re_{+}^{[m,n]}$ and P, Q be both matrices, then*

$$(P \odot \mathbb{A} \odot Q)_{i_1 \ldots i_m} = \max_{1 \leq j_1, \ldots, j_m \leq n} \{a_{j_1 \ldots j_m} p_{i_1 j_1} q_{j_2 i_2} \ldots q_{j_m i_m}\}.$$

Proof. By Definition 3.1 we have

$$(P \odot \mathbb{A} \odot Q)_{i_1 \ldots i_m} = \max_{1 \leq j_2, \ldots, j_m \leq n} \left\{ \max_{1 \leq j_1 \leq n} (a_{j_1 \ldots j_m} p_{i_1 j_1}) q_{j_2 i_2} \ldots q_{j_m i_m} \right\}$$

$$= \max_{1 \leq j_1, \ldots, j_m \leq n} \{a_{j_1 \ldots j_m} p_{i_1 j_1} q_{j_2 i_2} \ldots q_{j_m i_m}\}.$$

□

Theorem 3.10. *Let $\sigma \in S_n$ be a permutation on the set $\{1, \ldots, n\}$, $P = P_\sigma = (p_{ij})$ be the corresponding permutation matrix of σ (where $p_{ij} = 0 \Leftrightarrow j = \sigma(i)$). Let $\mathbb{A}, \mathbb{B} \in \Re_{max}^{[m,n]}$ such that $\mathbb{B} = P \odot \mathbb{A} \odot P^T$. Then we have:*

(i) $b_{i_1 \ldots i_m} = a_{\sigma(i_1) \ldots \sigma(i_m)}$.
(ii) $P \odot \mathbb{I} \odot P^T = \mathbb{I}$.
(iii) *Let $D = diag(d_{11}, \ldots, d_{nn})$ be an invertible diagonal matrix. Then*

$$D^{-(m-1)} \odot \mathbb{I} \odot D = \mathbb{I}.$$

Proof. By using Lemma 3.9, we have

$$b_{i_1 \ldots i_m} = \left(P \odot \mathbb{A} \odot P^T\right)_{i_1 \ldots i_m}$$

$$= \max_{1 \leq j_1, \ldots, j_m \leq n} \left\{a_{j_1 \ldots j_m} p_{i_1 j_1} \left(P^T\right)_{j_2 i_2} \ldots \left(P^T\right)_{j_m i_m}\right\}$$

$$= \max_{1 \leq j_1, \ldots, j_m \leq n} \{a_{j_1 \ldots j_m} p_{i_1 j_1} p_{i_2 j_2} \ldots p_{i_m j_m}\} = a_{\sigma(i_1) \ldots \sigma(i_m)}.$$

The proofs of (ii) and (iii) are trivial by using part (i). □

3.2. **The Direct Product $\mathbb{A} \otimes \mathbb{B}$ in Max Algebra System.** It is well known that the direct product of matrices (denoted by $\otimes$) is a useful concept and tool in matrix theory. It has many applications in various fields. This section extends the concept of tropical tensor product defined by Butkovic and Fiedler to the max algebra system on $\Re_+^{[m,n]}$.

Definition 3.11. For two matrices Y and Z of dimensions $m \times n$ and $r \times s$, respectively, the tensor product of Y and Z is the following $mr \times ns$ matrix:

$$Y \otimes Z = \begin{bmatrix} Y \odot z_{11} & \cdots & Y \odot z_{1s} \\ \vdots & \ddots & \vdots \\ Y \odot z_{r1} & \cdots & Y \odot z_{rs} \end{bmatrix}.$$

This definition was first introduced by Butkovic and Fiedler [4].

Definition 3.12. Let $\mathbb{A} \in \Re_+^{[k,n]}$ and $\mathbb{B} \in \Re_+^{[k,m]}$. Define the direct product $\mathbb{A} \otimes \mathbb{B}$ to be the following tensor in $\Re_+^{[k,nm]}$ (the set of subscripts is taken as $[n] \times [m]$ in the lexicographic order):

$$(\mathbb{A} \otimes \mathbb{B})_{(i_1,j_1),(i_2,j_2),\ldots,(i_k,j_k)} = a_{i_1 i_2 \ldots i_k} \odot b_{j_1 j_2 \ldots j_k}.$$

Theorem 3.13. *Definition 3.12 has the following properties:*

(i) $(\lambda \mathbb{A}) \otimes \mathbb{B} = \mathbb{A} \otimes (\lambda \mathbb{B}) = \lambda (\mathbb{A} \otimes \mathbb{B})$.
(ii) $(\mathbb{A}_1 \oplus \mathbb{A}_2) \otimes \mathbb{B} = (\mathbb{A}_1 \otimes \mathbb{B}) \oplus (\mathbb{A}_2 \otimes \mathbb{B})$.
(iii) $\mathbb{A} \otimes (\mathbb{B}_1 \oplus \mathbb{B}_2) = (\mathbb{A} \otimes \mathbb{B}_1) \oplus (\mathbb{A} \otimes \mathbb{B}_2)$.

Proof. The theorem follows directly from Definition 3.12. □

Theorem 3.14. *Suppose that $\mathbb{A} \in \Re_+^{[k+1,n]}$, $\mathbb{B} \in \Re_+^{[k+1,m]}$, $\mathbb{C} \in \Re_+^{[r+1,n]}$ and $\mathbb{D} \in \Re_+^{[r+1,m]}$. Then we have:*

$$(\mathbb{A} \otimes \mathbb{B}) \odot (\mathbb{C} \otimes \mathbb{D}) = (\mathbb{A} \odot \mathbb{C}) \otimes (\mathbb{B} \odot \mathbb{D}).$$

Proof. Let

$$(\alpha, \beta) = ((a_1, b_1), \ldots, (a_r, b_r)) \in ([n] \times [m])^r,$$

such that

$$\alpha = (a_1, \ldots, a_r) \in [n]^r, \qquad \beta = (b_1, \ldots, b_r) \in [m]^r.$$

We assume that

$$\alpha_i \in [n]^r, \beta_j \in [m]^r, \quad i, j = 1, \ldots, k,$$

then

$$((\mathbb{A}\otimes\mathbb{B})\odot(\mathbb{C}\otimes\mathbb{D}))_{(i,j)(\alpha_1,\beta_1)\dots(\alpha_k,\beta_k)}$$

$$= \max_{(i_1,j_1),\dots,(i_k,j_k)\in[n]\times[m]} (\mathbb{A}\otimes\mathbb{B})_{(i,j)(i_1,j_1)\dots(i_k,j_k)}$$

$$\odot\,(\mathbb{C}\otimes\mathbb{D})_{(i_1,j_1)(\alpha_1,\beta_1)}\odot\dots\odot(\mathbb{C}\otimes\mathbb{D})_{(i_k,j_k)(\alpha_k,\beta_k)}$$

$$= \max_{1\le i_1,\dots,i_k\le n}\ \max_{1\le j_1,\dots,j_k\le m} a_{ii_1\dots i_k}\odot b_{jj_1\dots j_k}$$

$$\odot\,(c_{i_1\alpha_1}\odot d_{j_1\beta_1})\odot\dots\odot(c_{i_k\alpha_k}\odot d_{j_k\beta_k})$$

$$= \left(\max_{1\le i_1,\dots,i_k\le n} a_{ii_1\dots i_k}\odot c_{i_1\alpha_1}\odot\dots\odot c_{i_k\alpha_k}\right)$$

$$\left(\max_{1\le j_1,\dots,j_k\le m} b_{jj_1\dots j_k}\odot d_{j_1\beta_1}\odot\dots\odot d_{j_k\beta_k}\right)$$

$$= (\mathbb{A}\odot\mathbb{C})_{i\alpha_1\dots\alpha_k}\odot(\mathbb{B}\odot\mathbb{D})_{j\beta_1\dots\beta_k}$$

$$= ((\mathbb{A}\odot\mathbb{C})\otimes(\mathbb{B}\odot\mathbb{D}))_{(i,j)(\alpha_1,\beta_1)\dots(\alpha_k,\beta_k)}.$$

□

Definition 3.15 ([1])**.** Let $\mathbb{A}\in\Re_+^{[m,n]}$. We say that λ is a max eigenvalue of $\mathbb{A}$ if there exists a nonzero, nonnegative vector x such that $\mathbb{A}\odot x = \lambda x^{[m-1]}$. We refer to x as a corresponding max eigenvector.

Theorem 3.16. *Let $\mathbb{A}$ and $\mathbb{B}$ be two order k tensors with dimensions n and m, respectively. Suppose that we have $\mathbb{A}\odot x = \lambda x^{[k-1]}$, and $\mathbb{B}\odot y = \mu y^{[k-1]}$, and we also write $w = x\otimes y$. Then we have:*

(i) $(\mathbb{A}\otimes\mathbb{B})\odot w = (\lambda\mu)\, w^{[k-1]}$.
(ii) $\alpha\lambda\oplus\beta\mu$ *is an eigenvalue of* $\alpha\odot(\mathbb{A}\otimes\mathbb{I})\oplus\beta\odot(\mathbb{I}\otimes\mathbb{B})$, *for all nonnegative numbers* α,β.

Proof.

(i) By Theorem 3.14 we have

$$\begin{aligned}(\mathbb{A} \otimes \mathbb{B}) \odot w &= (\mathbb{A} \otimes \mathbb{B}) \odot (x \otimes y) \\ &= (\mathbb{A} \odot x) \otimes (\mathbb{B} \odot y) \\ &= \left(\lambda x^{[k-1]}\right) \otimes \left(\mu v^{[k-1]}\right) \\ &= (\lambda\mu) \left(x^{[k-1]} \otimes y^{[k-1]}\right) \\ &= (\lambda\mu)\, w^{[k-1]}.\end{aligned}$$

(ii) By Theorems 3.14 and 3.13 we have

$$\begin{aligned}&\{[\alpha \odot (\mathbb{A} \otimes \mathbb{I})] \oplus [\beta \odot (\mathbb{I} \otimes \mathbb{B})]\} \odot (x \otimes y) \\ &= \{[\alpha \odot (\mathbb{A} \otimes \mathbb{I})] \odot (x \otimes y)\} \oplus \{[\beta \odot (\mathbb{I} \otimes \mathbb{B})] \odot (x \otimes y)\} \\ &= \{\alpha \odot (\mathbb{A} \odot x) \otimes (\mathbb{I} \odot y)\} \oplus \{\beta \odot (\mathbb{I} \odot x) \otimes (\mathbb{B} \odot y)\} \\ &= \left\{\alpha \odot \left(\lambda x^{[k-1]}\right) \otimes y^{[k-1]}\right\} \oplus \left\{\beta \odot x^{[k-1]} \otimes \left(\mu y^{[k-1]}\right)\right\} \\ &= \alpha \odot \lambda (x \otimes y)^{[k-1]} \oplus \beta \odot \mu (x \otimes y)^{[k-1]} \\ &= (\alpha\lambda \oplus \beta\mu)\, (x \otimes y)^{[k-1]}.\end{aligned}$$

□

Definition 3.17 ([9])**.** Suppose that $\mathbb{A}$ is a nonnegative tensor of order m and dimension n. $\mathbb{A}$ is called essentially positive if $\mathbb{A}x \in \Re^n_{++}$ for any nonzero $x \in \Re^n_+$.

Definition 3.18 ([1])**.** We define NC to be the set of all $\mathbb{A} \in \Re^{[m,n]}_+$ such that, there exist $x \neq 0$, $x \in \Re^n_+$ and $\lambda > 0$ such that $\mathbb{A} \odot x = \lambda x^{[m-1]}$, and

$$\left\{(i,j) :\ a_{ij\ldots j} x_j^{m-1} = \lambda x_i^{m-1},\ 1 \leq i,j \leq n\right\},$$

has at least a circuit.

Definition 3.19 ([1])**.** Let $\mathbb{A} \in \Re^{[m,n]}_+$ be an essentially positive tensor. Consider the directed graph $G(\mathbb{A}) = (V, E(\mathbb{A}))$. In this directed graph, k is a simple cycle of length q described by a sequence of distinct integers $i_1, \ldots, i_q \in \{1, \ldots, n\}$. Then with $|k| = q$,

$$\mu(\mathbb{A}) = \max_k \left\{ \left(a_{i_1 i_2 \ldots i_2}\, a_{i_2 i_3 \ldots i_3} \ldots\, a_{i_q i_1 \ldots i_1}\right)^{\frac{1}{|k|}} \right\}.$$

Theorem 3.20 ([1])**.** *Let $\mathbb{A}$ be an essentially positive tensor which belongs to NC, Then $\lambda = \mu(\mathbb{A})$.*

Theorem 3.21 ([1])**.** *Let $\mathbb{A}$ be an essentially positive tensor which belongs to NC. Then there exists a positive vector x such that*

$$\mathbb{A} \odot x = \mu(\mathbb{A})\, x^{[m-1]}.$$

Now we state and prove the main theorem in this subsection.

Theorem 3.22. *If $\mathbb{A}$ and $\mathbb{B}$ have positive eigenvectors, in particular if they are essentially positive and belong to NC, then*

$$\mu(\mathbb{A}\otimes\mathbb{B}) = \mu(\mathbb{A}) \odot \mu(\mathbb{B}).$$

Proof. By Theorem 3.21, $\mu(\mathbb{A})$ and $\mu(\mathbb{B})$ are the eigenvalues of $\mathbb{A}$ and $\mathbb{B}$ respectively with associated positive eigenvectors u and v. By Theorem 3.16, $u\otimes v$ is a positive eigenvector of $\mathbb{A}\otimes\mathbb{B}$ with the associated eigenvalue $\mu(\mathbb{A})\otimes\mu(\mathbb{B})$. But by Theorems 3.20 and 3.21, we have the only eigenvalue of $\mathbb{A}\otimes\mathbb{B}$ with positive eigenvector is $\mu(\mathbb{A}\otimes\mathbb{B})$. □

Example 3.23. Suppose that

$$\mathbb{A}(:,:,1) = \begin{bmatrix} a_{111}=2 & a_{121}=0 & a_{131}=1 \\ a_{211}=3 & a_{221}=1 & a_{231}=5 \\ a_{311}=1 & a_{321}=5 & a_{331}=1 \end{bmatrix},$$

$$\mathbb{A}(:,:,2) = \begin{bmatrix} a_{112}=2 & a_{122}=5 & a_{132}=4 \\ a_{212}=1 & a_{222}=4 & a_{232}=0 \\ a_{312}=3 & a_{322}=1 & a_{332}=7 \end{bmatrix},$$

$$\mathbb{A}(:,:,3) = \begin{bmatrix} a_{113}=2 & a_{123}=0 & a_{133}=1 \\ a_{213}=0 & a_{223}=6 & a_{233}=2 \\ a_{313}=0 & a_{323}=3 & a_{333}=7 \end{bmatrix}.$$

Then, there are 8 cycles in $G(\mathbb{A}) = (V, E(\mathbb{A}))$ as follows (see figure 1):

$$\begin{array}{ll}
\Gamma_1 = (1,1), & w(\Gamma_1) = a_{111} = 2, \\
\Gamma_2 = (1,2), & w(\Gamma_2) = a_{122}a_{211} = \sqrt{15}, \\
\Gamma_3 = (1,3), & w(\Gamma_3) = a_{133}a_{311} = 1, \\
\Gamma_4 = (2,2), & w(\Gamma_4) = a_{222} = 4, \\
\Gamma_5 = (2,3), & w(\Gamma_5) = a_{233}a_{322} = \sqrt{2}, \\
\Gamma_6 = (3,3), & w(\Gamma_6) = a_{333} = 7, \\
\Gamma_7 = (1,2,3), & w(\Gamma_7) = a_{122}a_{233}a_{311} = \sqrt[3]{10}, \\
\Gamma_8 = (1,3,2), & w(\Gamma_8) = a_{133}a_{322}a_{211} = \sqrt[3]{3}.
\end{array}$$

Then $\mu(\mathbb{A}) = \max\left\{1,2,4,7,\sqrt{2},\sqrt{15},\sqrt[3]{3},\sqrt[4]{10}\right\} = 7$, $\mu(\mathbb{A}\otimes\mathbb{A}) = 49$.

Example 3.24. Consider the positive order 3, dimension 2 tensors given by $a_{122} = a_{211} = 2$, $a_{ijk} = 1$ otherwise, and $b_{122} = b_{211} = 1$, $b_{ijk} = 0.5$ otherwise. Then $\mu(\mathbb{A}) = 2$, $\mu(\mathbb{B}) = 1$ and $\mu(\mathbb{A}\otimes\mathbb{B}) = 2$.

3.3. **Inverse Tensor Under the New System.** Since the operation $\oplus$ in max algebra is not invertible, inverse matrices are almost nonexistent and thus some tools used in linear algebra are unavailable. We show that in max algebra system, generalized permutation matrices are the only type of invertible matrices.

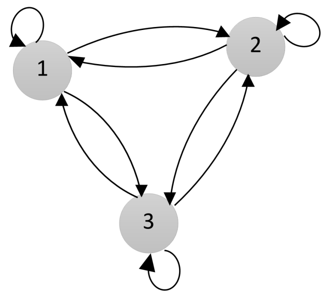

FIGURE 1. Directed graph $G(\mathbb{A}) = (V, E(\mathbb{A}))$

Theorem 3.25. *The inverse of a nonnegative matrix A in the max algebra sense is nonnegative if and only if A is a generalized permutation matrix.*

Proof. The sufficiency of the condition is obvious. Now, let $B = A^{-1} \geq 0$ and $A \odot B = I = B \odot A$. We show that A is a generalized permutation matrix. We have

$$\max_{1\leq k\leq n} a_{ik}b_{kj} = \max_{1\leq k\leq n} b_{ik}a_{kj} = \begin{cases} 1, & i = j, \\ 0, & i \neq j, \end{cases}$$

therefore for every $1 \leq i \leq n$ there is a $1 \leq t \leq n$ such that $a_{it} \odot b_{ti} = 1$, thus a_{it}, b_{ti} are positive. If there was an $a_{il} > 0$ for an $l \neq t$, then $b_{ti} \odot a_{il} > 0$ which would imply $\max\limits_{1\leq k\leq n} b_{tk} \odot a_{kl} > 0$, a contradiction. Therefore every row of A contains a unique positive entry. It is proved in a similar way that the same holds about every column of A. Hence A is a generalized permutation matrix. □

Recently in [5] the left and right inverse of tensors under the general product, are defined. Now we generalize this definition to max algebra system.

Definition 3.26. Let $\mathbb{A}$ be a tensor of order m and dimension n, and let $\mathbb{B}$ be a tensor of order k and dimension n. If $\mathbb{A} \odot \mathbb{B} = \mathbb{I}$, then $\mathbb{A}$ is called an order m left inverse of $\mathbb{B}$ in the max algebra sense, and $\mathbb{B}$ is called an order k right inverse of $\mathbb{A}$ in the max algebra sense.

Theorem 3.27. *Let $\mathbb{A} \in \Re_+^{[m,n]}$ be a diagonal tensor. Then*

(i) *$\mathbb{A}$ has an order k left inverse if and only if $\mathbb{A}$ has nonzero diagonal entries. Moreover, an order k diagonal tensor $\mathbb{L}$ with diagonal entry $l_{ii\ldots i} = {a_{ii\ldots i}}^{-(k-1)}$ is the unique order k left inverse of $\mathbb{A}$.*

(ii) *$\mathbb{A}$ has an order k right inverse if and only if $a_{ii\ldots i} \neq 0$, $i = 1, 2, \ldots, n$. In this case, an order k diagonal tensor $\mathbb{R}$ with* *diagonal entry*

$$r_{i\dots i} = \sqrt[(m-1)]{a^{-1}_{ii\dots i}}$$

is the unique order k right inverse of $\mathbb{A}$.

Proof. (i) By Definition 3.26, $\mathbb{A}$ has an order k left inverse if and only if there exists an order k dimension n tensor $\mathbb{L}$ such that $\mathbb{L} \odot \mathbb{A} = \mathbb{I}$. Since $\mathbb{A}$ is diagonal, by Definition 3.1, $\mathbb{A}$ has an order k left inverse if and only if $a_{ii\dots i} \neq 0$, $i = 1, 2, \dots, n$. By $\mathbb{L} \odot \mathbb{A} = \mathbb{I}$ and Definition 3.1 we have $l_{ii\dots i} = a_{ii\dots i}{}^{-(k-1)}$.

(ii) If $\mathbb{A}$ has an order k right inverse, then there exists an order k dimension n tensor $\mathbb{R}$ such that $\mathbb{A} \odot \mathbb{R} = \mathbb{I}$. Since $\mathbb{A}$ is diagonal, by Definition 3.1, $\mathbb{A}$ has an order k right inverse if and only if $a_{ii\dots i} \neq 0$, $i = 1, 2, \dots, n$. By $\mathbb{A} \odot \mathbb{R} = \mathbb{I}$ and Definition 3.1 the second assertion is clear.

□

We will now characterize the left (right) inverse of order 2 by using the Theorems 3.3 and 3.13 for a tensor $\mathbb{A} \in \Re_+^{[m,n]}$. For this purpose we require the following lemma.

Lemma 3.28. *If $\mathbb{A} \in \Re_+^{[m,n]}$ and it has an order 2 left (right) inverse, then*

(i) *This left (right) inverse does not have a row with all entries nonzero (zero).*
(ii) *This left (right) inverse does not have a column with all entries nonzero (zero).*
(iii) *$\mathbb{A}$ does not have a face with all entries nonzero (zero).*

Proof. This follows from Definition 3.1 and Definition 3.26. □

Theorem 3.29. *If $\mathbb{A} \in \Re_+^{[m,n]}$, and it has an order 2 left (right) inverse G, then G must be a generalized permutation matrix.*

Proof. Let $\mathbb{A}$ has an order 2 left (right) inverse G. We know that G does not have a zero row (a zero column), so there will be at least one positive entry in each row (column). Notice that if there exists one column of G such that has two positive entries then $\mathbb{A}$ has a zero face. Therefore there exists exactly one positive entry in each column and row. □

Theorem 3.30. *If $\mathbb{A} \in \Re_+^{[m,n]}$, then it has an order 2 left inverse if and only if there exists a generalized permutation matrix G such that $\mathbb{A} = G \odot \mathbb{I}$. Moreover, G^{-1} is the unique order 2 left inverse of $\mathbb{A}$.*

Proof. If $\mathbb{A} = G \odot \mathbb{I}$, for a generalized permutation matrix G, then $\mathbb{A}$ has an order 2 left inverse G^{-1}. Assume C is an order 2 left inverse of $\mathbb{A}$.

Then $C \odot \mathbb{A} = \mathbb{I}$, this equation conclude that C must be a generalized permutation matrix (by Theorem 3.29), thus $\mathbb{A} = C^{-1} \odot \mathbb{I}$. Suppose that B is also an order 2 left inverse of $\mathbb{A}$. We can also get $\mathbb{A} = B^{-1} \odot \mathbb{I}$. Hence $\left(C^{-1} - B^{-1}\right) \odot \mathbb{I} = 0$, By Theorem 3.8, we have $C^{-1} = B^{-1}$. By the fact that a nonsingular matrix has a unique inverse matrix, it follows that $B = C$ and the desired results hold. □

Theorem 3.31. *If $\mathbb{A} \in \Re_+^{[m,n]}$, then it has an order 2 right inverse if and only if there exists a generalized permutation matrix Q such that $\mathbb{A} = \mathbb{I} \odot Q$. In this case, Q^{-1} is the unique order 2 right inverse of $\mathbb{A}$.*

Proof. If $\mathbb{A} = \mathbb{I} \odot Q$ for a generalized permutation matrix Q, then $\mathbb{A}$ has an order 2 left inverse Q^{-1}. If T is an order 2 right inverse of $\mathbb{A}$, then $\mathbb{A} \odot T = \mathbb{I}$, implies that T is a generalized permutation matrix (by Theorem 3.29). So $\mathbb{A} = \mathbb{I} \odot T^{-1}$. Hence if $\mathbb{A}$ has an order 2 right inverse, then there exists a generalized permutation matrix T such that $\mathbb{A} = \mathbb{I} \odot T$.

If $\mathbb{R}$ is any order 2 right inverse of $\mathbb{A}$, then $\mathbb{A} \odot R = \mathbb{I} \odot Q \odot R = \mathbb{I}$. Set $D = Q \odot R$, then $\mathbb{I} = \mathbb{I} \odot D$. By Definition 3.1, D must be the identity matrix of dimension n. Hence the proof is complete. □

Notice that for $m = 2$ (when $\mathbb{A}$ is a matrix), we have, the right inverse is equal to left inverse, (refer to max algebra theory and Theorem 3.25).

Theorem 3.32. *Let $\mathbb{A}$ and $\mathbb{B}$ be tensors such that $\mathbb{A} \odot \mathbb{B} = 0$. Then the following hold:*

(i) *If the order 2 left inverse of a tensor $\mathbb{A}$ (resp. $\mathbb{B}$) exists, then $\mathbb{B} = 0$ (resp. $\mathbb{A} = 0$).*

(ii) *If the order 2 right inverse of a tensor $\mathbb{A}$ (resp. $\mathbb{B}$) exists, then $\mathbb{B} = 0$ (resp. $\mathbb{A} = 0$).*

Proof. If the order 2 left inverse of a tensor $\mathbb{A}$ exists, then by Theorem 3.30, there exists a generalized permutation matrix G such that $G \odot \mathbb{I} \odot \mathbb{B} = 0$. Thus $\mathbb{I} \odot \mathbb{B} = 0$. By Theorem 3.8 we get $\mathbb{B} = 0$. Similarly if the order 2 left inverse of a tensor $\mathbb{B}$ exists, then Theorems 3.30 and 3.8 imply that $\mathbb{A} = 0$. Hence part (i) holds. The proof of (ii) follows in a similar manner of the proof of (i), using Theorems 3.31 and 3.8. □

Definition 3.33. We define a new class for tensors as follows:

$$\mathrm{M} = \left\{ \begin{array}{l} \mathbb{A} \in \Re_+^{[m,n]} : \mathbb{A} = G \odot \mathbb{I} = \mathbb{I} \odot G, \\ \text{where } G \text{ is } a \text{ generalized permutation matrix} \end{array} \right\}.$$

For example the unit tensor belongs to this class.

Theorem 3.34. *Let $\mathbb{A} \in \Re_+^{[m,n]}$. Then a matrix B with*

$$\mathbb{A} \odot B = \mathbb{I} = B \odot \mathbb{A},$$

exists if and only if $\mathbb{A}$ *belongs to* M.

Proof. Let $\mathbb{A} \in \Gamma$, thus there exists a generalized permutation matrix G such that $\mathbb{A} = G \odot \mathbb{I} = \mathbb{I} \odot G$. By puting $B = G^T$, we will have $\mathbb{A} \odot B = \mathbb{I} = B \odot \mathbb{A}$. On the other hand, if $\mathbb{A} \odot B = \mathbb{I} = B \odot \mathbb{A}$, Theorems 3.30 and 3.31 conclude that $\mathbb{A}$ belongs to M. □

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