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Surjective Real-Linear Uniform Isometries Between Complex Function Algebras

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Abstract. In this paper, we first give a description of a surjective unit-preserving real-linear uniform isometry $T : A \longrightarrow B$, where A and B are complex function spaces on compact Hausdorff spaces X and Y, respectively, whenever $\mathrm{ER}(A, X) = \mathrm{Ch}(A, X)$ and $\mathrm{ER}(B, Y) = \mathrm{Ch}(B, Y)$. Next, we give a description of T whenever A and B are complex function algebras and T does not assume to be unit-preserving.

1. Introduction and Preliminaries

Let $\mathbb{F}$ denote either $\mathbb{R}$ or $\mathbb{C}$. Assume that X is a compact Hausdorff space. We denote by $C_{\mathbb{F}}(X)$ the Banach algebra of all continuous functions form X into $\mathbb{F}$, with the uniform norm

$$\|f\|_X = \sup\{|f(x)| : x \in X\}, \quad (f \in C_{\mathbb{F}}(X)).$$

However, we always write $C(X)$ instead of $C_{\mathbb{C}}(X)$ and we denote the uniform closure of A by $\bar{A}$, whenever A is a subset of $C(X)$.

Let A be a real or complex linear subspace of $C(X)$. A nonempty subset S of X is called a boundary for A with respect to X if for each $f \in A$ the function $|f|$ assumes its maximum on X at some $x \in S$. We denote by $\Gamma(A, X)$ the interesection of all closed boundaries for A. If $\Gamma(A, X)$ is a boundary for A, it is called the Shilov boundary for A with respect to X.

Let A be a linear subspace of $C(X)$ over $\mathbb{F}$ containing 1_X, the constant function on X with value 1. We denote by $K_{\mathbb{F}}(A)$ the set of all $\varphi \in A^\star$

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for which $\|\varphi\|_{op} = \varphi(1_X) = 1$, where $A^\star$ is the dual space of the normed space $(A, \|\cdot\|_X)$ over $\mathbb{F}$ and $\|\cdot\|_{op}$ is the operator norm. In fact, the elements of $A^\star$ are $\mathbb{F}$-valued continuous linear functionals on A over $\mathbb{F}$. $K_{\mathbb{F}}(A)$ is called the state space of A over $\mathbb{F}$. For each $x \in X$ the evaluation map on A at x is the linear mapping $e_{A,x} : A \longrightarrow \mathbb{C}$ over $\mathbb{F}$ defined by

$$e_{A,x}(f) = f(x), \quad (f \in A).$$

Clearly, $e_{A,x} \in K_{\mathbb{C}}(A)$ whenever $\mathbb{F} = \mathbb{C}$ and $\operatorname{Re} e_{A,x} \in K_{\mathbb{R}}(A)$ whenever $\mathbb{F} = \mathbb{R}$. It is well-known that $K_{\mathbb{F}}(A)$ is a convex set in $A^\star$ and a compact set in $A^\star$ with the weak* topology. Hence, $\operatorname{Ext}(K_{\mathbb{F}}(A))$ is nonempty by the Krein-Milman theorem, where $\operatorname{Ext}(K_{\mathbb{F}}(A))$ denotes the set of all extreme points of $K_{\mathbb{F}}(A)$.

Let A be a complex linear subspace of $C(X)$ containing 1_X. A representing measure for $\varphi \in A^\star$ is a complex regular Borel measure μ on X with $|\mu|(X) = \|\varphi\|_{op}$ such that $\varphi(f) = \int_X f d\mu$ for all $f \in A$. It is known that every $\varphi \in K_{\mathbb{C}}(A)$ has a representing measure and every representing measure for such φ is a probability measure [2, Section 2.1]. If $x \in X$, then δ_x, the point mass measure at x, is a representing measure for $e_{A,x}$. We denote by $\operatorname{Ch}(A, X)$ the set of all $x \in X$ for which δ_x is the only representing measure for $e_{A,x}$. If $\operatorname{Ch}(A, X)$ is a boundary for A, it is called the Choquet boundary for A with respect to X.

Let A be a real linear subspace of $C(X)$ containing 1_X. A real part representing measure for $\psi \in A^\star$ is a regular Borel measure μ on X with $|\mu|(X) = \|\psi\|_{op}$ such that $\psi(f) = \int_X \operatorname{Re} f d\mu$ for all $f \in A$. It is known that every $\psi \in K_{\mathbb{R}}(A)$ has a real part representing measure and every real part representing measure for such ψ is a probability measure [6, Theorem 1.5]. If $x \in X$, then δ_x is a real part representing measure for $\operatorname{Re} e_{A,x}$. We denote by $\operatorname{Ch}(A, X)$ the set of all $x \in X$ for which δ_x is the only real part representing measure for $\operatorname{Re} e_{A,x}$. If $\operatorname{Ch}(A, X)$ is a boundary for A with respect to X, it is called the Choquet boundary for A with respect to X.

Let A be a real or complex linear subspace of $C(X)$ and S be a nonempty subset of X. Then S is called a peak set for A with respect to X, if there exists $f \in A$ such that $S = \{x \in X : f(x) = 1\}$ and $|f(y)| < 1$ for all $y \in X \setminus S$. We say that S is a weak peak set for A with respect to X, if S is the intersection of some collections of peak sets for A. If the peak set (weak peak set, respectively) S for A is a singleton $\{x\}$, then we call x as a peak point (weak peak point, respectively) for A with respect to X. We denote by $S(A, X)$ ($S_0(A, X)$, respectively) the set of all weak peak points (peak points, respectively) for A with respect to X. We say that A is extremely regular at S if for every open neighborhood U of S and for each $\epsilon > 0$ there is a function $f \in A$ with

$\|f\|_X = 1$ such that $f(x) = 1$ for all $x \in S$ and $|f(y)| < \epsilon$ for all $y \in X \setminus U$. If $x \in X$ and A is extremely regular at $\{x\}$, we say that A is extremely regular at x. We denote by $\mathrm{ER}(A, X)$ the set of all $x \in X$ for which A is extremely regular at x.

Let A be a subset of $C(X)$. We say that A separates the points of X if for each $x, y \in X$ with $x \neq y$ there exists a function $f \in A$ such that $f(x) \neq f(y)$.

Note that if A is a complex linear subspace of $C(X)$, then A separates the points of X if and only if $\mathrm{Re}\, A$ separates the points of X, where $\mathrm{Re}\, A = \{\mathrm{Re}\, f : f \in A\}$.

Definition 1.1. Let X be a compact Hausdorff space.

(a) A complex function space on X is a complex linear subspace of $C(X)$ containing 1_X and separating the points of X.
(b) A complex function algebra on X is a complex subalgebra of $C(X)$ containing 1_X and separating the points of X.
(c) A complex uniform function algebra on X is a complex function algebra on X which is uniformly closed, i.e., a closed set in $(C(X), \|\cdot\|_X)$.

Let X be a compact Hausdorff space and A be a complex function space on X. It is easy to see that for any two points $x, y \in X$ with $x \neq y$ there exists $f \in A$ such that $f(x) = 0$ and $f(y) = 1$. It is known that $S_0(A, X)$ is the intersection of all boundaries for A with respect to X. Applying [2, Theorem 2.2.1], we can easily show that $\mathrm{ER}(A, X)$ is a subset of $\mathrm{Ch}(A, X)$.

The following results are well-known and will be used in the sequel.

Theorem 1.2 (see [7], Theorem 4.3.3 and Corollary 4.3.4)**.** *Let A be a complex function space on X. Then the following statements hold.*

(i) *The element $\varphi \in K_{\mathbb{C}}(A)$ is an extreme point of $K_{\mathbb{C}}(A)$ if and only if there exists $x \in \mathrm{Ch}(A, X)$ such that $\varphi = e_{A,x}$. In particular,* $\mathrm{Ch}(A, X)$ *is nonempty.*
(ii) $\mathrm{Ch}(A, X)$ *is the set of all $x \in X$ such that $e_{A,x}$ is an extreme point of $K_{\mathbb{C}}(A)$.*
(iii) $\mathrm{Ch}(A, X)$ *is a boundary for A with respect to X.*
(iv) *The closure of* $\mathrm{Ch}(A, X)$ *is equal to* $\Gamma(A, X)$.

Theorem 1.3 (see [6], Lemma 2.1 and Theorem 2.2)**.** *Let A be a real linear subspace of $C(X)$ containing 1_X and $\mathrm{Re}\, A$ separates the points of X. Then the following statements hold.*

(i) *The element $\varphi \in K_{\mathbb{R}}(A)$ is an extreme point of $K_{\mathbb{R}}(A)$ if and only if $\varphi = \mathrm{Re}\, e_{A,x}$ for some $x \in \mathrm{Ch}(A, X)$. In particular,* $\mathrm{Ch}(A, X)$ *is nonempty.*

(ii) $\mathrm{Ch}\,(A,X)$ *is the set of all* $x \in X$ *such that* $\mathrm{Re}\, e_{A,x}$ *is an extreme point of* $K_{\mathbb{R}}\,(A)$.

Theorem 1.4 (see [1], Theorem 9 (iii)). *Let* A *be a complex function space on* X. *Then* $\mathrm{Ch}\,(A,X) = \mathrm{Ch}\,(\bar{A},X)$ *and* $\Gamma\,(A,X) = \Gamma\,(\bar{A},X)$.

Theorem 1.5 (see [7], Theorem 4.3.6). *Let* A *be a uniformly closed complex subalgebra of* $C\,(X)$ *containing* 1_X *and* $x \in X$. *Then the following statements are equivalent.*

(i) $x \in \mathrm{Ch}\,(A,X)$.
(ii) *For every* γ *and* δ *with* $0 < \gamma < \delta < 1$ *and for every neighborhood* U *of* X, *there exists* $k \in A$ *with* $\|k\|_X \leq 1$, $k\,(x) > \delta$ *and* $|k\,(y)\,| < \gamma$ *for all* $y \in X \setminus U$.
(iii) *There exist* c *and* M *with* $0 < c < 1 \leq M$ *such that for every neighberhood* V *of* x, *there exists* $f \in A$ *with* $\|f\|_X \leq M$, $f\,(x) = 1$ *and* $|f\,(y)\,| < c$ *for all* $y \in X \setminus V$.
(iv) $x \in S\,(A,X)$.
(v) $x \in \mathrm{ER}\,(A,X)$.

Let X and Y be compact Hausdorff spaces and A and B be real subspaces of $C\,(X)$ and $C\,(Y)$, respectively. A map $T : A \longrightarrow B$ is called a uniform isometry if $\|T\,(f) - T\,(g)\,\|_Y = \|f - g\|_X$ for all $f, g \in A$.

Some descriptions of real-linear uniform isometries between complex function spaces have been given by many authors. For instance see [3, 4, 8]. In this paper we give a description of surjective real-linear uniform isometries between certain function spaces and function algebras on compact Hausdorff spaces using the extreme points of their state spaces of them.

2. Results

In this section, we first study unit-preserving surjective real-linear uniform isometries between complex uniform function spaces on compact Hausdorff spaces. For this purpose, we need the following lemmas.

Lemma 2.1 (see [7], Lemma 5.1.1). *Let* X *and* Y *be compact Hausdorff spaces and let* A *and* B *be real linear subspaces of* $C\,(X)$ *and* $C\,(Y)$ *containing* 1_X *and* 1_Y, *respectively. Suppose that* $x \in \mathrm{ER}\,(A,X)$ *and* $y \in \mathrm{ER}\,(B,Y)$. *Let* $T : A \longrightarrow B$ *be a surjective real-linear uniform isometry with* $T1_X = 1_Y$ *and* $\mathrm{Re}\,(Tf)\,(y) = \mathrm{Re}\, f\,(x)$ *for all* $f \in A$. *Then* $(Tf)\,(y) = f\,(x)$ *for all* $f \in A$ *or* $(Tf)\,(y) = \overline{f\,(x)}$ *for all* $f \in A$.

Lemma 2.2. *Let* X *be a compact Hausdorff space,* A *be a complex linear subspace of* $C\,(X)$ *containing* 1_X *and* $A_{\mathbb{R}}$ *denotes* A *regarded as a real linear space. Then the following statements hold.*

(i) $K_{\mathbb{R}}\,(A_{\mathbb{R}}) = \{\mathrm{Re}\,\varphi : \varphi \in K_{\mathbb{C}}\,(A)\}$.

(ii) *If* $\operatorname{Re} A$ *separates the points of* X*, then* $\operatorname{Ch}(A_{\mathbb{R}}, X) = \operatorname{Ch}(A, X)$.

(iii) *If* A *separates the points of* X*, then* $\operatorname{Ch}(A_{\mathbb{R}}, X)$ *is a boundary for* A *with respect to* X.

Proof. Define the map $\Theta : A^\star \longrightarrow (A_{\mathbb{R}})^\star$ by

$$\Theta(\Lambda) = \operatorname{Re}\Lambda, \quad (\Lambda \in A^\star). \tag{2.1}$$

It is known that Θ is a surjective real-linear isometry from $A^\star$ with the operator norm to $(A_{\mathbb{R}})^\star$ with the operator norm. To prove (i), we first assume that $\varphi \in K_{\mathbb{C}}(A)$. Then $\varphi \in A^\star$ and

$$\|\varphi\|_{op} = \varphi(1_X) = 1. \tag{2.2}$$

Hence, $\operatorname{Re}\varphi \in (A_{\mathbb{R}})^\star$ and $\|\operatorname{Re}\varphi\|_{op} = (\operatorname{Re}\varphi)(1_X) = 1$ by (2.2). This implies that $\operatorname{Re}\varphi \in K_{\mathbb{R}}(A_{\mathbb{R}})$. Therefore,

$$\{\operatorname{Re}\varphi : \varphi \in K_{\mathbb{C}}(A)\} \subseteq K_{\mathbb{R}}(A_{\mathbb{R}}). \tag{2.3}$$

We now assume that $\psi \in K_{\mathbb{R}}(A_{\mathbb{R}})$. Then $\psi \in (A_{\mathbb{R}})^\star$ and $\|\psi\|_{op} = \psi(1_X) = 1$. On the other hand, there exists $\varphi \in A^\star$ such that $\operatorname{Re}\varphi = \psi$ and $\|\varphi\|_{op} = \|\psi\|_{op}$. Thus

$$\begin{aligned}
\|\varphi\|_{op} &= 1 \\
&= \|1_X\|_X \\
&\geq |\varphi(1_X)| \\
&\geq |\operatorname{Re}(\varphi(1_X))| \\
&= |(\operatorname{Re}\varphi)(1_X)| \\
&= |\psi(1_X)| \\
&= 1 \\
&= \|\psi\|_{op} \\
&= \|\varphi\|_{op},
\end{aligned}$$

and so $|\varphi(1_X)| = 1$. This implies $\varphi(1_X) = 1$ since

$$\operatorname{Re}\varphi(1_X) = (\operatorname{Re}\varphi)(1_X) = \psi(1_X) = 1.$$

Hence, $\varphi \in K_{\mathbb{C}}(A)$. Therefore,

$$K_{\mathbb{R}}(A_{\mathbb{R}}) \subseteq \{\operatorname{Re}\varphi : \varphi \in K_{\mathbb{C}}(A)\}. \tag{2.4}$$

From (2.3) and (2.4), we conclude that (i) holds.

(ii) Let $\operatorname{Re} A$ separates the points of X. To prove $\operatorname{Ch}(A_{\mathbb{R}}, X) = \operatorname{Ch}(A, X)$, we first assume that $x \in \operatorname{Ch}(A, X)$. By Theorem 1.2, we have $e_{A,x} \in \operatorname{Ext}(K_{\mathbb{C}}(A))$. Let

$$\operatorname{Re} e_{A,x} = (1-t)\psi_1 + t\psi_2, \tag{2.5}$$

where $\psi_1, \psi_2 \in K_{\mathbb{R}}(A)$ and $t \in (0,1)$. Suppose that $\varphi_1, \varphi_2 \in K_{\mathbb{C}}(A)$ such that $\operatorname{Re}\varphi_1 = \psi_1$ and $\operatorname{Re}\varphi_2 = \psi_2$. Then

$$\operatorname{Re} e_{A,x} = \operatorname{Re}\left((1-t)\varphi_1 + t\varphi_2\right),$$

by (2.5). This implies that $e_{A,x} = (1-t)\varphi_1 + t\varphi_2$, since Θ is injective. Therefore, $\varphi_1 = \varphi_2 = e_{A,x}$ and so $\psi_1 = \psi_2 = \operatorname{Re} e_{A,x}$. Hence, $\operatorname{Re} e_{A,x} \in \operatorname{Ext}(K_{\mathbb{R}}(A_{\mathbb{R}}))$. This implies that $x \in \operatorname{Ch}(A_{\mathbb{R}}, X)$ by Theorem 1.3. Therefore,

$$\operatorname{Ch}(A, X) \subseteq \operatorname{Ch}(A_{\mathbb{R}}, X). \tag{2.6}$$

We now assume that $x \in \operatorname{Ch}(A_{\mathbb{R}}, X)$. Then $\operatorname{Re} e_{A,x} \in \operatorname{Ext}(K_{\mathbb{R}}(A))$ by Theorem 1.3. Let $e_{A,x} = (1-t)\varphi_1 + t\varphi_2$, where, $\varphi_1, \varphi_2 \in K_{\mathbb{C}}(A)$ and $t \in (0,1)$. Then $\operatorname{Re} e_{A,x} = (1-t)\operatorname{Re}\varphi_1 + t\operatorname{Re}\varphi_2$, and $\operatorname{Re}\varphi_1, \operatorname{Re}\varphi_2 \in K_{\mathbb{R}}(A_{\mathbb{R}})$ by (i). Therefore, $\operatorname{Re} e_{A,x} = \operatorname{Re}\varphi_1 = \operatorname{Re}\varphi_2$. Hence, $e_{A,x} = \varphi_1 = \varphi_2$ since Θ^{-1} is injective. Therefore,

$$\operatorname{Ch}(A_{\mathbb{R}}, X) \subseteq \operatorname{Ch}(A, X). \tag{2.7}$$

From (2.6) and (2.7), we conclude that (ii) holds.

(iii) Assume that A separates the points of X. Then $\operatorname{Ch}(A, X)$ is a boundary for A with respect to X by Theorem 1.2. Hence, $\operatorname{Ch}(A_{\mathbb{R}}, X)$ is a boundary for $A_{\mathbb{R}}$ with respect to X by (ii) since $\operatorname{Re} A$ separates the points of X. □

Lemma 2.3. *Let X and Y be compact Hausdorff spaces, A be a real linear subspace of $C(X)$ containing 1_X, B be a real linear subspace of $C(Y)$ containing 1_Y and $T : A \longrightarrow B$ be a surjective real-linear uniform isometry with $T1_X = 1_Y$. Then the following statements hold.*

(i) $T^\star(K_{\mathbb{R}}(B)) = K_{\mathbb{R}}(A)$, *where $T^\star : B^\star \longrightarrow A^\star$ is the adjoint operator of T defined by $T^\star\Lambda = \Lambda \circ T \quad (\Lambda \in B^\star)$.*

(ii) $T^\star(\operatorname{Ext}(K_{\mathbb{R}}(B))) = \operatorname{Ext}(K_{\mathbb{R}}(A))$.

Proof. Since $T : A \longrightarrow B$ is a surjective real-linear uniform isometry, we conclude that $T^\star : B^\star \longrightarrow A^\star$ is a surjective real-linear isometry from $(B^\star, \|\cdot\|_{op})$ to $(A^\star, \|\cdot\|_{op})$. We first assume that $\psi \in K_{\mathbb{R}}(A)$. Then $\psi \in A^\star$, $\psi(1_X) = 1 = \|\psi\|_{op}$. Surjectivity of $T^\star$ implies that there exists $\eta \in B^\star$ such that $\psi = T^\star\eta$. Hence, $\|T^\star\eta\|_{op} = \|\psi\|_{op} = 1$ and

$$\eta(1_Y) = \eta(T(1_X)) = (\eta \circ T)(1_X) = (T^\star\eta)(1_X) = \psi(1_X) = 1.$$

Thus, $\eta \in K_{\mathbb{R}}(B)$. Therefore,

$$K_{\mathbb{R}}(A) \subseteq T^\star(K_{\mathbb{R}}(B)). \tag{2.8}$$

We now assume that $\psi \in T^{\star}(K_{\mathbb{R}}(B))$. Then there exists $\eta \in K_{\mathbb{R}}(B)$ such that $\psi = T^{\star}\eta$. This implies that

$$\begin{aligned}\psi(1_X) &= (T^{\star}\eta)(1_X)\\ &= (\eta \circ T)(1_X)\\ &= \eta(T1_X)\\ &= \eta(1_Y)\\ &= 1\\ &= \|\eta\|_{op}\\ &= \|T^{\star}\eta\|_{op}\\ &= \|\psi\|_{op}.\end{aligned}$$

Hence, $\psi \in K_{\mathbb{R}}(A)$. Therefore,

$$T^{\star}(K_{\mathbb{R}}(B)) \subseteq K_{\mathbb{R}}(A). \tag{2.9}$$

From (2.8) and (2.9), we conclude that (i) holds.

To prove (ii), we first assume that $\psi \in \mathrm{Ext}(K_{\mathbb{R}}(A))$. Then $\psi \in K_{\mathbb{R}}(A)$. By (i), there exists $\eta \in K_{\mathbb{R}}(B)$ such that $\psi = T^{\star}\eta$. Suppose that $\eta = (1-t)\eta_1 + t\eta_2$, where $t \in (0,1)$ and $\eta_1, \eta_2 \in K_{\mathbb{R}}(B)$. Then $T^{\star}\eta_1, T^{\star}\eta_2 \in K_{\mathbb{R}}(A)$ by (i). Since

$$\begin{aligned}\psi(f) &= (T^{\star}\eta)(f)\\ &= (\eta \circ T)(f)\\ &= \eta(Tf)\\ &= ((1-t)\eta_1 + t\eta_2)(Tf)\\ &= (1-t)\eta_1(Tf) + t\eta_2(Tf)\\ &= (1-t)(\eta_1 \circ T)(f) + t(\eta_2 \circ T)(f)\\ &= (1-t)(T^{\star}\eta_1)(f) + t(T^{\star}\eta_2)(f)\\ &= ((1-t)T^{\star}\eta_1 + tT^{\star}\eta_2)(f),\end{aligned}$$

for all $f \in A$, we deduce that $\psi = (1-t)T^{\star}\eta_1 + tT^{\star}\eta_2$. This implies that $T^{\star}\eta = \psi = T^{\star}\eta_1 = T^{\star}\eta_2$. Hence, $\eta = \eta_1 = \eta_2$ since $T^{\star}$ is injective. So $\eta \in \mathrm{Ext}(K_{\mathbb{R}}(B))$. Therefore,

$$\mathrm{Ext}(K_{\mathbb{R}}(A)) \subseteq T^{\star}(\mathrm{Ext}(K_{\mathbb{R}}(B))). \tag{2.10}$$

We now assume that $\psi \in T^{\star}(\mathrm{Ext}(K_{\mathbb{R}}(B)))$. Then there exist $\eta \in \mathrm{Ext}(K_{\mathbb{R}}(B))$ such that $\psi = T^{\star}\eta$. Suppose that $\psi = (1-t)\psi_1 + t\psi_2$, where $t \in (0,1)$ and $\psi_1, \psi_2 \in K_{\mathbb{R}}(A)$. By (i), there exists $\eta_1, \eta_2 \in K_{\mathbb{R}}(B)$ such that $\psi_1 = T^{\star}\eta_1, \psi_2 = T^{\star}\eta_2$. Now, we have

$$T^{\star}\eta = (1-t)T^{\star}\eta_1 + tT^{\star}\eta_2 = T^{\star}((1-t)\eta_1 + t\eta_2).$$

This implies that $\eta = (1-t)\,\eta_1 + t\eta_2$ since $T^\star$ is injective. Hence, $\eta = \eta_1 = \eta_2$ since $\eta \in \operatorname{Ext}(K_{\mathbb{R}}(B))$, and so $\psi = \psi_1 = \psi_2$. Thus, $\psi \in \operatorname{Ext}(K_{\mathbb{R}}(A))$. Therefore,

$$T^\star(\operatorname{Ext}(K_{\mathbb{R}}(B))) \subseteq \operatorname{Ext}(K_{\mathbb{R}}(A)). \tag{2.11}$$

From (2.10) and (2.11), we deduce that (ii) holds. □

Lemma 2.4. *Let X and Y be compact Hausdorff spaces, A be a real linear subspace of $C(X)$ containing 1_X, B be a real linear subspace of $C(Y)$ containing 1_Y and $T : A \longrightarrow B$ be a surjective real-linear uniform isometry with $T1_X = 1_Y$. Suppose that $\operatorname{Re} A$ separates the points of X, $\operatorname{Re} B$ separates the points of Y and $y \in \operatorname{Ch}(B, Y)$. Then there exists $x \in \operatorname{Ch}(A, X)$ such that $\operatorname{Re}(Tf)(y) = \operatorname{Re} f(x)$ for all $f \in A$.*

Proof. Let $y \in \operatorname{Ch}(B, Y)$. Then $\operatorname{Re} e_{B,y} \in \operatorname{Ext}(K_{\mathbb{R}}(B))$. By part (ii) of Lemma 2.3, $T^\star(\operatorname{Re} e_{B,y}) \in \operatorname{Ext}(K_{\mathbb{R}}(A))$. Since $\operatorname{Re} A$ separates the points of X, we deduce that

$$T^\star(\operatorname{Re} e_{B,y}) = \operatorname{Re} e_{A,x}, \tag{2.12}$$

for some $x \in \operatorname{Ch}(A, X)$ by part (i) of Theorem 1.3. Let $f \in A$. Then by (2.12), we have

$$\begin{aligned}
\operatorname{Re} f(x) &= \operatorname{Re} e_{A,x}(f) \\
&= (\operatorname{Re} e_{A,x})(f) \\
&= T^\star(\operatorname{Re} e_{B,y})(f) \\
&= ((\operatorname{Re} e_{B,y}) \circ T)(f) \\
&= \operatorname{Re} e_{B,y}(Tf) \\
&= \operatorname{Re}((Tf)(y)) \\
&= \operatorname{Re}(Tf)(y).
\end{aligned}$$

Hence, the proof is complete. □

Lemma 2.5. *Let X and Y be compact Hausdorff spaces, A be a complex function space on X, B be a complex function space on Y and $T : A \longrightarrow B$ be a surjective real-linear uniform isometry with $T1_X = 1_Y$. Suppose that $\operatorname{ER}(A, X) = \operatorname{Ch}(A, X)$ and $y \in \operatorname{ER}(B, Y)$. Then there exists a unique $x \in \operatorname{Ch}(A, X)$ such that $(Tf)(y) = f(x)$ for all $f \in A$ or $(Tf)(y) = \overline{f(x)}$ for all $f \in A$. In particular, $(T(i1_X))(y) \in \{i, -i\}$.*

Proof. Since A and B are complex function spaces on X and Y, respectively, we deduce that $\operatorname{Ch}(A_{\mathbb{R}}, X)$ is nonempty, $\operatorname{Ch}(A_{\mathbb{R}}, X) = \operatorname{Ch}(A, X)$, $\operatorname{Ch}(B_{\mathbb{R}}, Y)$ is nonempty, and $\operatorname{Ch}(B, Y) = \operatorname{Ch}(B_{\mathbb{R}}, Y)$, by Lemma 2.2. Since B is a complex function space on Y and $y \in \operatorname{ER}(B, Y)$, we deduce that $y \in \mathrm{Ch}\,(B,Y)$ and so, by part (ii) of Lemma 2.2, $y \in \mathrm{Ch}\,(B_{\mathbb{R}},Y)$. By Lemma 2.4, there exists $x \in \mathrm{Ch}\,(A_{\mathbb{R}},X)$ such that

$$\mathrm{Re}\,(Tf)(y) = \mathrm{Re}\, f(x),$$

for all $f \in A_{\mathbb{R}}$. Since $x \in \mathrm{ER}\,(A,X)$ and $y \in \mathrm{ER}\,(B,Y)$, we deduce that $(Tf)(y) = f(x)$ for all $f \in A_{\mathbb{R}} = A$ or $(Tf)(y) = \overline{f(x)}$ for all $f \in A_{\mathbb{R}} = A$ by Lemma 2.1.

To prove the uniqueness of x, let $x' \in \mathrm{Ch}\,(A,X)$ such that $(Tf)(y) = f(x')$ for all $f \in A$ or $(Tf)(y) = \overline{f(x')}$ for all $f \in A$. We claim that $x' = x$. Otherwise, there exists $f_0 \in A$ such that $f_0(x') \neq f_0(x)$ since A separates the points of X. Without loss of generality, we can assume that $(Tf_0)(y) = f_0(x)$ and $(Tf_0)(y) = \overline{f_0(x')}$. Hence, $(Tf)(y) = f(x)$ for all $f \in A$ and $Tf(y) = \overline{f(x')}$ for all $f \in A$. Since $i1_X \in A$, we deduce that

$$i = (i1_X)(x) = (T(i1_X))(y) = \overline{(i1_X)(x')} = \bar{i} = -i,$$

that is a contradiction. Hence, our claim is justified.

By the above argument, we have

$$(T(i1_X))(y) \in \left\{ (i1_X)(x), \overline{(i1_X)(x)} \right\} = \{i, -i\}.$$

Hence, the proof is complete. □

Theorem 2.6. *Let X and Y be compact Hausdorff spaces, A be a complex function space on X, B be a complex function space on Y and $T : A \longrightarrow B$ be a surjective real-linear uniform isometry with $T1_X = 1_Y$. Suppose that* $\mathrm{ER}\,(A,X) = \mathrm{Ch}\,(A,X)$ *and* $\mathrm{ER}\,(B,Y) = \mathrm{Ch}\,(B,Y)$. *Then there exist a continuous function $\varphi : \mathrm{Ch}\,(B,Y) \longrightarrow \mathrm{Ch}\,(A,X)$ and a function α in B with $\|\alpha\|_Y = 1$ and $\alpha(y) \in \{-1,1\}$ for all $y \in \mathrm{Ch}\,(B,Y)$ such that*

$$(Tf)(y) = \mathrm{Re}\, f(\varphi(y)) + i\alpha(y)\,\mathrm{Im}\, f(\varphi(y)),$$

for all $f \in A$ and $y \in \mathrm{Ch}\,(B,Y)$.

Proof. Since A (B, respectively) is a complex function space on X (Y, respectively), we deduce that $\mathrm{Re}\,A$ ($\mathrm{Re}\,B$, respectively) separates the points of X (Y, respectively). Define the map $\varphi : \mathrm{Ch}\,(B,Y) \longrightarrow \mathrm{Ch}\,(A,X)$ by $\varphi(y) = x$, where $y \in \mathrm{Ch}\,(B,Y)$ and x is the unique element of $\mathrm{Ch}\,(A,X)$ such that $(Tf)(y) = f(x)$ for all $f \in A$ or $(Tf)(y) = \overline{f(x)}$ for all $f \in A$. By Lemma 2.5, φ is well-defined. Set $\alpha = iT(i1_X)$. Clearly $\alpha \in B$ and

$$\|\alpha\|_Y = \|iT(i1_Y)\|_Y = \|T(i1_Y)\|_Y = \|i1_X\|_X = \|1_X\|_X = 1.$$

On the other hand, $T(i1_X)(y) \in \{-i, i\}$ for all $y \in \mathrm{Ch}(B, Y)$ by Lemma 2.5. This implies that $\alpha(y) = 1$ if $T(i1_X)(y) = i$ and $\alpha(y) = -1$ if $T(i1_X)(y) = -i$, where $y \in \mathrm{Ch}(B, Y)$.

Let $f \in A$ and $y \in \mathrm{Ch}(B, Y)$. Then $(Tf)(y) = f(\varphi(y))$ or $(Tf)(y) = \overline{f(\varphi(y))}$. If $(Tf)(y) = f(\varphi(y))$, then $T(i1_X)(y) = i1_X(\varphi(y)) = i$ and so $\alpha(y) = 1$. Therefore,

$$\begin{aligned}(Tf)(y) &= f(\varphi(y)) \\ &= \mathrm{Re}\,(f(\varphi(y))) + i\mathrm{Im}\, f(\varphi(y)) \\ &= \mathrm{Re}\, f(\varphi(y)) + i\alpha(y)\,\mathrm{Im}\, f(\varphi(y)).\end{aligned}$$

If $(Tf)(y) = \overline{f(\varphi(y))}$, then $T(i1_X)(y) = \overline{i1_X(\varphi(y))} = -i$ and so $\alpha(y) = -1$. Therefore

$$\begin{aligned}(Tf)(y) &= \overline{f(\varphi(y))} \\ &= \mathrm{Re}\, f(\varphi(y)) - i\mathrm{Im} f(\varphi(y)) \\ &= \mathrm{Re}\, f(\varphi(y)) + i\alpha(y)\,\mathrm{Im} f(\varphi(y)).\end{aligned}$$

Thus, $(Tf)(y) = \mathrm{Re}\, f(\varphi(y)) + i\alpha(y)\,\mathrm{Im}\, f(\varphi(y))$ for all $f \in A$ and $y \in \mathrm{Ch}(B, Y)$.

To prove the continuity of φ, let $y \in \mathrm{Ch}(B, Y)$ and $\{y_\gamma\}_{\gamma\in\Gamma}$ be a net in $\mathrm{Ch}(B, Y)$ with

$$\lim_\gamma y_\gamma = y. \tag{2.13}$$

Let $f \in A$. Then

$$\lim_\gamma \mathrm{Re}\,(Tf)(y_\gamma) = \mathrm{Re}\,(Tf)(y),$$

by (2.13) and $\mathrm{Re}\,(Tf) \in C(Y)$. Since $\alpha(\mathrm{Ch}(B, Y)) \subseteq \{-1, 1\}$, we have $\mathrm{Re}\,(Tf)(y) = \mathrm{Re}\, f(\varphi(y))$ and $\mathrm{Re}\,(Tf)(y_\gamma) = \mathrm{Re}\, f(\varphi(y_\gamma))$ for all $\gamma \in \Gamma$. Hence,

$$\lim_\gamma (\mathrm{Re}\, f)(\varphi(y_\gamma)) = (\mathrm{Re}\, f)(\varphi(y)). \tag{2.14}$$

Since $\mathrm{Re}\, A \subseteq C(X)$, $1_X \in \mathrm{Re}\, A$ and $\mathrm{Re}\, A$ separates the points of X, by [5, Proposition 2.2.14], we deduce that the topology of X equals to the weak topology of X with respect to the functions $x \longmapsto g(x)$, $g \in \mathrm{Re}\, A$. This fact implies that

$$\lim_\gamma \varphi(y_\gamma) = \varphi(y),$$

since (2.14) holds for all $f \in A$. Therefore, φ is continuous and the proof is complete. □

Theorem 2.7. *Let X and Y be compact Hausdorff spaces, A be a complex function space on X, B be a complex function space on Y and $T : A \longrightarrow B$ be a surjective real-linear mapping. Suppose that*

$\mathrm{Ch}\,(A,X) = \mathrm{ER}\,(A,X)$ *and* $\mathrm{Ch}\,(B,Y) = \mathrm{ER}\,(B,Y)$. *Then the following statements are equivalent.*

(i) T *is a uniform isometry and* $T1_X = 1_Y$.

(ii) *There exist a homeomorphism* $\varphi : \mathrm{Ch}\,(B,Y) \longrightarrow \mathrm{Ch}\,(A,X)$ *and a function* α *in* B *with* $\|\alpha\|_Y = 1$ *and* $\alpha\,(y) \in \{-1,1\}$ *for all* $y \in \mathrm{Ch}\,(B,Y)$ *such that*

$$(Tf)\,(y) = \mathrm{Re}\, f\,(\varphi\,(y)) + i\alpha\,(y)\, \mathrm{Im}\, f\,(\varphi\,(y))\,,$$

for all $f \in A$ *and* $y \in \mathrm{Ch}\,(B,Y)$.

(iii) *There exist a homeomorphism* $\varphi : \mathrm{Ch}\,(B,Y) \longrightarrow \mathrm{Ch}\,(A,X)$ *and a clopen subset* K *of* $\mathrm{Ch}\,(B,Y)$, *possibly empty, such that*

$$(Tf)\,(y) = \begin{cases} f\,(\varphi\,(y))\,, & y \in K, \\ \overline{f\,(\varphi\,(y))}, & y \in \mathrm{Ch}\,(B,Y) \setminus K, \end{cases}$$

for all $f \in A$.

(iv) *There exist a surjective mapping* $\varphi : \mathrm{Ch}\,(B,Y) \longrightarrow \mathrm{Ch}\,(A,X)$ *and a subset* K *of* $\mathrm{Ch}\,(B,Y)$, *possibly empty, such that*

$$(Tf)\,(y) = \begin{cases} f\,(\varphi\,(y))\,, & y \in K, \\ \overline{f\,(\varphi\,(y))}, & y \in \mathrm{Ch}\,(B,Y) \setminus K, \end{cases}$$

for all $f \in A$.

Proof. (i) $\Rightarrow$ (ii). By Theorem 2.6, there exist a continuous function $\varphi : \mathrm{Ch}\,(B,Y) \longrightarrow \mathrm{Ch}\,(A,X)$ and a function α in B with $\|\alpha\|_Y = 1$ and $\alpha\,(y) \in \{-1,1\}$ for all $y \in \mathrm{Ch}\,(B,Y)$ such that

$$(Tf)\,(y) = \mathrm{Re}\, f\,(\varphi\,(y)) + i\alpha\,(y)\, \mathrm{Im}\, f\,(\varphi\,(y))\,, \tag{2.15}$$

for all $f \in A$ and $y \in \mathrm{Ch}\,(B,Y)$.

By hypothesises, $T^{-1} : B \longrightarrow A$ is a surjective real-linear uniform isometry and $T^{-1}1_Y = 1_X$. Applying Theorem 2.6, we deduce that there exist a continuous function $\psi : \mathrm{Ch}\,(A,X) \longrightarrow \mathrm{Ch}\,(B,Y)$ and a function β in A with $\|\beta\|_X = 1$ and $\beta\,(x) \in \{-1,1\}$ for all $x \in \mathrm{Ch}\,(A,X)$ such that

$$\left(T^{-1}g\right)(x) = \mathrm{Re}\, g\,(\psi\,(x)) + i\beta\,(x)\, \mathrm{Im}\, g\,(\psi\,(x))\,, \tag{2.16}$$

for all $g \in B$ and $x \in \mathrm{Ch}\,(A,X)$.

Let $x \in X$. Since $\beta(\mathrm{Ch}(A,X)) \subseteq \mathbb{R}$ and $\alpha(\mathrm{Ch}(B,Y)) \subseteq \mathbb{R}$, by applying (2.16) and (2.15), we conclude that

$$\begin{aligned}\mathrm{Re}\, f(x) &= \mathrm{Re}\, \left(T^{-1}(Tf)\right)(x)\\ &= \mathrm{Re}\, (Tf)(\psi(x))\\ &= \mathrm{Re}\, f(\varphi(\psi(x)))\\ &= \mathrm{Re}\, f(\varphi \circ \psi)(x),\end{aligned}$$

for all $f \in A$. This implies that $x = (\varphi \circ \psi)(x)$ since $\mathrm{Re}\, A$ separates the points of X. Hence, $\varphi \circ \psi = I_{\mathrm{Ch}(A,X)}$.

Similarly, one can show that $\psi \circ \varphi = I_{\mathrm{Ch}(B,X)}$. Hence, φ is bijective and $\varphi^{-1} = \psi$. Therefore, φ is a homeomorphism and so (ii) holds.

(ii) $\Rightarrow$ (iii). Set $K = \{y \in \mathrm{Ch}(B,Y) : \alpha(y) = 1\}$. This implies that K is a closed subset of $\mathrm{Ch}(B,Y)$, possibly empty. Since $\alpha(\mathrm{Ch}(B,Y)) \subseteq \{-1,1\}$, we deduce that $\mathrm{Ch}(B,Y) \setminus K = \{y \in \mathrm{Ch}(B,Y) : \alpha(y) = -1\}$ and so $\mathrm{Ch}(B,Y) \setminus K$ is a closed subset of $\mathrm{Ch}(B,Y)$. Therefore, K is a clopen subset of $\mathrm{Ch}(B,Y)$, possibly empty. It is easy to see that

$$(Tf)(y) = \begin{cases} f(\varphi(y)), & y \in K,\\ \overline{f(\varphi(y))}, & y \in \mathrm{Ch}(B,Y) \setminus K,\end{cases}$$

for all $f \in A$. Hence, (iii) holds.

(iii) $\Rightarrow$ (iv). It is obvious.

(iv)$\Rightarrow$ (i). Since $1_X \in A$, we deduce that $T1_X|_{\mathrm{Ch}(B,Y)} = 1_Y|_{\mathrm{Ch}(B,Y)}$ by (iv). On the other hand, $\mathrm{Ch}(B,Y)$ is a boundary for B with respect to Y. Therefore, $T1_X = 1_Y$.

Let $f \in A$. Since $Tf \in B$ and $\mathrm{Ch}(B,Y)$ is a boundary for B with respect to Y, there exists $y_0 \in \mathrm{Ch}(B,Y)$ such that

$$\|Tf\|_Y = |(Tf)(y_0)|. \tag{2.17}$$

By (iv), we have

$$|(Tf)(y_0)| = |f(\varphi(y_0))|. \tag{2.18}$$

From (2.17) and (2.18), we get

$$\|Tf\|_Y \leq \|f\|_X. \tag{2.19}$$

Since $\mathrm{Ch}(A,X)$ is a boundary for A with respect to X, there exists $x_1 \in \mathrm{Ch}(A,X)$ such that

$$\|f\|_X = |f(x_1)|. \tag{2.20}$$

Surjectivity of $\varphi : \mathrm{Ch}(B,Y) \longrightarrow \mathrm{Ch}(A,X)$ implies that there exists $y_1 \in \mathrm{Ch}(B,Y)$ such that

$$x_1 = \varphi(y_1). \tag{2.21}$$

Applying (iv), (2.20) and (2.21), we get

$$\|f\|_X = |f(\varphi(y_1))| = |(Tf)(y_1)| \le \|Tf\|_Y. \tag{2.22}$$

From (2.19) and (2.22), we deduce $\|Tf\|_Y = \|f\|_X$. Hence, T is norm preserving. This implies that T an is isometry since T is a real-linear mapping. Thus, (i) holds. □

Corollary 2.8. *Let the hypotheses of Theorem 2.7 hold. Suppose that* $T(i1_X) = i1_Y$. *Then the following statements are equivalent.*

(i) *T is an isometry and* $T1_X = 1_Y$

(ii) *There exists a homeomorphism* $\varphi : \mathrm{Ch}(B,Y) \longrightarrow \mathrm{Ch}(A,X)$ *such that* $(Tf)(y) = f(\varphi(y))$ *for all* $f \in A$ *and* $y \in \mathrm{Ch}(B,Y)$.

(iii) *There exists a surjective mapping* $\varphi : \mathrm{Ch}(B,Y) \longrightarrow \mathrm{Ch}(A,X)$ *such that* $(Tf)(y) = f(\varphi(y))$ *for all* $f \in A$ *and* $y \in \mathrm{Ch}(B,Y)$.

Proof. (i) ⇒ (ii). By Theorem 2.7, there exists a homeomorphism $\varphi : \mathrm{Ch}(B,Y) \longrightarrow \mathrm{Ch}(A,X)$ and a subset K of $\mathrm{Ch}(B,Y)$, possibly empty, such that

$$(Tf)(y) = \begin{cases} f(\varphi(y)), & y \in K, \\ \\ \overline{f(\varphi(y))}, & y \in \mathrm{Ch}(B,Y) \setminus K, \end{cases}$$

for all $f \in A$. Since $T(i1_X) = i1_Y$, we deduce that $K = \mathrm{Ch}(B,Y)$ and so $(Tf)(y) = f(\varphi(y))$ for all $f \in A$ and $y \in \mathrm{Ch}(B,Y)$. Hence, (ii) holds.

(ii) ⇒ (iii). It is obvious.

(iii) ⇒ (i). It is sufficient to apply the given argument in the proof of (iv) ⇒ (i) in Theorem 2.7. □

Remark 2.9. If $T : A \longrightarrow B$ is a complex linear mapping in Theorem 2.7, then the statements (i), (ii) and (iii) in Corollary 2.8 are equivalent.

We recall that if A is a complex function space on a compact Hausdorff space X, then $\mathrm{ER}(A,X) \subseteq \mathrm{Ch}(A,X)$. We now give an example that the equality holds in the mentioned inclusion.

Example 2.10. Let (X,d) be a compact metric space. It is easy to see that if $t \in X$ and $\delta > 0$ then the function $h_{t,\delta} : X \longrightarrow \mathbb{C}$ defined by

$$h_{t,\delta}(x) = \max\left\{0, \frac{\delta - d(x,t)}{\delta}\right\}, \quad (x \in X),$$

belongs to $C(X)$. Let A be the complex linear span of the set $\{1_X\} \cup \left\{h_{t,\frac{1}{n}} : t \in X, n \in \mathbb{N}\right\}$. Suppose that $x, y \in X$ with $x \neq y$. Then there exists $m \in \mathbb{N}$ such that $\frac{1}{m} < d(x,y)$. It is clear that $h_{x,\frac{1}{m}} \in A$ with

$h_{x,\frac{1}{m}}(x) = 1$ and $h_{x,\frac{1}{m}}(y) = 0$. Hence, A separates the points of X and so A is a complex function space on X.

We now show that

$$\mathrm{ER}\,(A, X) = X = \mathrm{Ch}\,(A, X)\,. \tag{2.23}$$

Let $x \in X$. Suppose that $\varepsilon > 0$ and U is an open neigborhood of x in (X, d). Then there exists a positive integer m such that

$$B_d\left(x, \frac{1}{m}\right) = \left\{y \in X : d\,(y, x) < \frac{1}{m}\right\} \subseteq U.$$

It is clear that $h_{x,\frac{1}{m}} \in A$ with $h_{x,\frac{1}{m}}(x) = 1 = \|h_{x,\frac{1}{m}}\|_X$ and $|h_{x,\frac{1}{m}}(y)| = 0 < \varepsilon$ for all $y \in X \setminus U$. Hence, $x \in \mathrm{ER}\,(A, X)$. Therefore,

$$X \subseteq \mathrm{ER}\,(A, X)\,. \tag{2.24}$$

On the other hand, we have

$$\mathrm{ER}\,(A, X) \subseteq \mathrm{Ch}\,(A, X) \subseteq X. \tag{2.25}$$

From (2.25) and (2.24), we deduce that (2.23) holds.

Theorem 2.11. *Let X and Y be compact Hausdorff spaces, A and B be complex uniform function algebras on X and Y, respectively, and $T : A \longrightarrow B$ be a surjective real-linear mapping. Then the following statements are equivalent.*

(i) *T is a uniform isometry and $T1_X = 1_Y$.*
(ii) *T is a real-algebra isomorphism from $A_{\mathbb{R}}$ onto $B_{\mathbb{R}}$.*

Proof. Since A (B, respectively) is a complex uniform function algebra on X (Y, respectively), we deduce that $\mathrm{ER}\,(A, X) = \mathrm{Ch}\,(A, X)$ ($\mathrm{ER}\,(B, Y) = \mathrm{Ch}\,(B, Y)$, respectively) by Theorem 1.5.

To prove $(i) \Rightarrow (ii)$, it is sufficient to show that T is multiplicative. By (i) and Theorem 2.7, there exists a surjective mapping $\varphi : \mathrm{Ch}\,(B, Y) \longrightarrow \mathrm{Ch}\,(A, X)$ and a subset K of $\mathrm{Ch}\,(B, Y)$, possibly empty, such that

$$(Tf)\,(y) = \begin{cases} f\,(\varphi\,(y))\,, & y \in K, \\ \overline{f\,(\varphi\,(y))}, & y \in \mathrm{Ch}\,(B, Y) \setminus K, \end{cases}$$

for all $f \in A$ and $y \in \mathrm{Ch}\,(B, Y)$.

Let $f, g \in A$. Then for each $y \in K$ we have

$$\begin{aligned} (T\,(fg))\,(y) &= (fg)\,(\varphi\,(y)) \\ &= f\,(\varphi\,(y))\,g\,(\varphi\,(y)) \\ &= (Tf)\,(y)\,(Tg)\,(y) \\ &= ((Tf)\,(Tg))\,(y)\,, \end{aligned}$$

and, for each $y \in \mathrm{Ch}(B,Y) \setminus K$ we have

$$\begin{aligned} T(fg)(y) &= \overline{(fg)(\varphi(y))} \\ &= \overline{f(\varphi(y))\, g(\varphi(y))} \\ &= \overline{f(\varphi(y))}\overline{g(\varphi(y))} \\ &= (Tf)(y)\,(Tg)(y) \\ &= (Tf)(Tg)(y)\,. \end{aligned}$$

Hence, $T(fg)(y) = (Tf)(Tg)(y)$ for all $y \in \mathrm{Ch}(B,Y)$. This implies that $T(fg)(y) = (Tf)(Tg)(y)$ for all $y \in Y$ since $\mathrm{Ch}(B,Y)$ is a boundary for B with respect to Y. Thus, $T(fg) = (Tf)(Tg)$. Therefore, T is multiplicative and so (ii) holds.

To prove $(ii) \Rightarrow (i)$, we first show that $T1_X = 1_Y$. Since $1_Y \in B$ and T is surjective, there exists a function f_1 in A such that $Tf_1 = 1_Y$. This implies that

$$T1_X = (Tf_1)(T1_X) = T(f_1 1_X) = Tf_1 = 1_Y.$$

Since $T : A_{\mathbb{R}} \to B_{\mathbb{R}}$ is a real-algebra isomorphism from the real unital Banach algebra $(A_{\mathbb{R}}, \|\cdot\|_X)$ to the real Banach algebra $(B_{\mathbb{R}}, \|\cdot\|_X)$, we deduce that $r_{B_{\mathbb{R}}}(Tf) = r_{A_{\mathbb{R}}}(f)$ for all $f \in A$ by [7, Remark 1.1.22], where $r_A(a)$ is the spectral radius of $a \in A$. On the other hand, $r_{B_{\mathbb{R}}}(Tf) = \|Tf\|_Y$ and $r_{A_{\mathbb{R}}}(f) = \|f\|_X$, for all $f \in A$. Therefore, $\|Tf\|_Y = \|f\|_X$ for all $f \in A$. This implies that T is an isometry since T is additive and norm preserving. Hence, (i) holds. □

In the rest of this section, it is not assumed that T is unit-preserving.

Lemma 2.12. *Let X be a compact Hausdorff space and A be a complex function algebra on X such that $\mathrm{ER}(A,X) = \mathrm{Ch}(A,X)$. Suppose that $f \in A$ with $\|f\|_X = 1$. Then the following statements are equivalent.*

(i) $|f(x)| = 1$ *for all* $x \in \mathrm{Ch}(A,X)$.

(ii) *For each $\varepsilon > 0$, there exists $\delta > 0$ such that for each $g \in A$ with $\|g\|_X \geq \varepsilon$, we have*

$$\max\{\|f+g\|_X, \|f-g\|_X\} > 1 + \delta.$$

Proof. (i) ⇒ (ii). Let $\varepsilon > 0$ and $g \in A$ with $\|g\|_X \geq \varepsilon$. Since $\mathrm{Ch}(A,X)$ is a boundary for A with respect to X, there exists $x \in \mathrm{Ch}(A,X)$ such that $|g(x)| \geq \varepsilon$. This implies that

$$|f(x) + g(x)|^2 + |f(x) - g(x)|^2 = 2\left(|f(x)|^2 + |g(x)|^2\right) \geq 2\left(1 + \varepsilon^2\right),$$

and so

$$\max\{|f(x) + g(x)|, |f(x) - g(x)|\} \geq \sqrt{1+\varepsilon^2}.$$

Therefore,

$$\max\{\|f+g\|_X, \|f-g\|_X\} \geq \sqrt{1+\varepsilon^2}.$$

If $0 < \delta < \sqrt{1+\varepsilon^2} - 1$, then

$$\max\{\|f+g\|_X, \|f-g\|_X\} > 1+\delta.$$

Hence, (ii) holds.

(ii) $\Rightarrow$ (i). Assume that there exists $x_0 \in \mathrm{Ch}(A,X)$ with $|f(x_0)| < 1$. Choose $\varepsilon > 0$ with $|f(x_0)| < 1 - 2\varepsilon$. Set

$$V = \{x \in X : |f(x)| < 1 - \varepsilon\}.$$

Then V is an open neighborhood of x_0 in X. Let $\delta > 0$ be arbitrary. Since $x_0 \in \mathrm{Ch}(A,X) = \mathrm{ER}(A,X)$, there exists $g \in A$ with $\|g\|_X = \varepsilon = g(x_0)$ and $|g(x)| < \delta$ for all $x \in X \setminus V$. If $x \in V$, then

$$\begin{aligned} \max\{|(f+g)(x)|, |(f-g)(x)|\} &\le |g(x)| + |f(x)| \\ &< \varepsilon + 1 - \varepsilon \\ &< 1+\delta. \end{aligned}$$

If $x \in X \setminus V$, then

$$\begin{aligned} \max\{|(f+g)(x)|, |(f-g)(x)|\} &\le |f(x)| + |g(x)| \\ &\le \|f\|_X + |g(x)| \\ &< 1+\delta. \end{aligned}$$

Therefore,

$$\max\{\|f+g\|_X, \|f-g\|_X\} \le 1+\delta.$$

Hence, (ii) implies (i). □

Lemma 2.13. *Let X and Y be compact Hausdorff spaces, A be a complex function algebra on X, B be a complex function algebra on Y, $T : A \longrightarrow B$ be a surjective real-linear uniform isometry and $h = T1_X$. Suppose that $\mathrm{ER}(A,X) = \mathrm{Ch}(A,X)$ and $\mathrm{ER}(B,Y) = \mathrm{Ch}(B,Y)$ Then the following statements hold.*

(i) *$|h(y)| = 1$ for all $y \in \mathrm{Ch}(B,Y)$.*
(ii) *h is invertible in B.*
(iii) *The map $T' : A \longrightarrow B$, defined by $T'f = h^{-1}Tf \qquad (f \in A)$, is a surjective real-linear uniform isometry with $T'1_X = 1_Y$.*

Proof. (i). Let $\varepsilon > 0$ be arbitrary. Since $1_X \in A$ and $\|1_X\|_X = 1$, by Lemma 2.12, there exists $\delta > 0$ such that for all $g \in A$ with $\|g\|_X \ge \varepsilon$ we have

$$\max\{\|1_X+g\|_X, \|1_X-g\|_X\} > 1+\delta.$$

Assume that $k \in B$ with $\|k\|_Y \ge \varepsilon$. Then there is $g \in A$ such that $Tg = k$ and so $\|g\|_X = \|Tg\|_Y = \|k\|_Y \ge \varepsilon$. By the exists above argument,

$$\max\{\|1_X+g\|_X, \|1_X-g\|_X\} > 1+\delta.$$

Since $h = T1_X$ and T is a uniform isometry, we have

$$\begin{aligned}\max\{\|h+k\|_Y, \|h-k\|_Y\} &= \max\{\|1_X+g\|_X, \|1_X-g\|_X\}\\ &> 1+\delta.\end{aligned}$$

Therefore, $|h(y)| = 1$ for all $y \in \mathrm{Ch}(B,Y)$ by Lemma 2.12.

(ii). By part (iv) of Theorem 1.2, $\Gamma(B,Y)$ is the closure of $\mathrm{Ch}(B,Y)$ in Y and a closed boundary for B with respect to Y. By (i) and the continuity of h on X, we deduce that $|h(y)| = 1$ for all $y \in \Gamma(B,Y)$. Define

$$B_0 = \left\{\left(\bar{h}k\right)|_{\Gamma(B,Y)} : k \in B\right\}.$$

It is easy to see that B_0 is a complex linear subspace of $C(\Gamma(B,Y))$. Since $h \in B$, we deduce that $\bar{h}h|_{\Gamma(B,Y)} \in B_0$. Hence, $1_{\Gamma(B,Y)} \in B_0$. We claim that B_0 separates the points of $\Gamma(B,Y)$. Assume that $y, y' \in \Gamma(B,Y)$ with $y \neq y'$. Then there exists $k_0 \in B$ such that $k_0(y) = 0$ and $k_0(y') = 1$. Thus $\left(\bar{h}k_0\right)(y) = 0$ and $\left(\bar{h}k_0\right)(y') = \bar{h}(y') \neq 0$. Moreover, $\bar{h}k_0 \in B_0$. Hence, our claim is justified. Therefore B_0 is a complex function space on $\Gamma(B,Y)$. We claim that $B|_{\Gamma(B,Y)} \subseteq B_0$. Let $k \in B$. Then $\left(\bar{h}hk\right)|_{\Gamma(B,Y)} \in B_0$ since $hk \in B$. If $y \in \Gamma(B,Y)$, then

$$\left(\bar{h}hk\right)(y) = |h(y)|^2 k(y) = k(y).$$

This implies that $\left(\bar{h}hk\right)|_{\Gamma(B,Y)} = k|_{\Gamma(B,Y)}$. Hence, $k|_{\Gamma(B,Y)} \in B_0$ and so our claim is justified.

Now, we show that

$$\mathrm{Ch}(B,Y) \subseteq \mathrm{ER}(B_0, \Gamma(B,Y)). \tag{2.26}$$

Let $y \in \mathrm{Ch}(B,Y)$. Then $y \in \mathrm{ER}(B,Y)$ and $y \in \Gamma(B,Y)$. Let $\varepsilon > 0$ be given and $V \subseteq \Gamma(B,Y)$ be a neighborhood of y in $\Gamma(B,Y)$. Then there exists a neighborhood U of y in Y such that $V = U \cap \Gamma(B,Y)$. Since $y \in \mathrm{ER}(B,Y)$, there exists a function $k \in B$ with $\|k\|_Y = k(y) = 1$ and $|k(z)| < \varepsilon$ for each $z \in Y \setminus U$. Set $k_0 = k|_{\Gamma(B,Y)}$. Then $k_0 \in B_0$ since $k \in B$ and $B|_{\Gamma(B,Y)} \subseteq B_0$. Moreover,

$$k_0(y) = k(y) = 1 = \|k\|_Y.$$

On the other hand, $\|k\|_Y = \|k\|_{\Gamma(B,Y)} = \|k_0\|_{\Gamma(B,Y)}$ since $\Gamma(B,Y)$ is a boundary for B with respect to Y. If $z \in \Gamma(B,Y) \setminus V$, then $z \in Y \setminus U$ and so

$$|k_0(z)| = |k(z)| < \varepsilon.$$

Hence, $y \in \mathrm{ER}(B_0, \Gamma(B,Y))$. Therefore, (2.26) holds.

Define the map $T_0 : A \longrightarrow B_0$ by

$$T_0 f = \left(\bar{h}Tf\right)|_{\Gamma(B,Y)}, \qquad (f \in A).$$

Clearly, T_0 is well-defined and it is a surjective real-linear mapping. We claim that T_0 is a uniform isometry. Let $f \in A$. Then

$$\begin{aligned} \|T_0 f\|_{\Gamma(B,Y)} &= \|\bar{h}Tf|_{\Gamma(B,Y)}\|_{\Gamma(B,Y)} \\ &\leq \|\bar{h}Tf\|_Y \\ &\leq \|\bar{h}\|_Y \|Tf\|_Y \\ &= \|Tf\|_Y \\ &= \|f\|_X. \end{aligned} \tag{2.27}$$

Since $Tf \in B$ and $\Gamma(B,Y)$ is a boundary for B with respect to Y, there exists $y_0 \in \Gamma(B,Y)$ such that

$$\|Tf\|_Y = |(Tf)(y_0)|. \tag{2.28}$$

From $\|f\|_X = \|Tf\|_Y$, (2.28) and $|\bar{h}(y_0)| = |\overline{h(y_0)}| = |h(y_0)| = 1$, we have

$$\begin{aligned} \|f\|_X &= |\bar{h}(y_0)||(Tf)(y_0)| \\ &= |\left(\bar{h}Tf\right)(y_0)| \\ &= |(T_0 f)(y_0)| \\ &\leq \|T_0 f\|_{\Gamma(B,Y)}. \end{aligned} \tag{2.29}$$

Applying (2.27) and (2.29), we deduce that $\|T_0 f\|_{\Gamma(B,Y)} = \|f\|_X$. This implies that T_0 is a uniform isometry.

Since $1_Y \in B$ and $T : A \longrightarrow B$ is surjective, there exists $g \in A$ such that $Tg = 1_Y$. Let $y \in \mathrm{Ch}(B,Y)$ be arbitrary. Then $y \in \mathrm{ER}(B_0, \Gamma(B,Y))$ by (2.26). By Lemma 2.5, there exists a unique $x \in \mathrm{Ch}(A,Y)$ such that $(T_0 f)(y) = f(x)$ for all $f \in A$ or $(T_0 f)(y) = \overline{f(x)}$ for all $f \in A$. Since $|h(y)| = 1$ and $g, g^2 \in A$, we have

$$\begin{aligned} (Tg)(y) &= h(y)(T_0 g)(y), \\ \left(Tg^2\right)(y) &= h(y)\left(T_0 g^2\right)(y). \end{aligned}$$

If $(T_0 g)(y) = g(x)$, then $\left(T_0 g^2\right)(y) = g^2(x)$ and so

$$\begin{aligned}
\left(hTg^2\right)(y) &= h(y)\left(Tg^2\right)(y) \\
&= h(y)\left(h(y)\left(T_0 g^2\right)(y)\right) \\
&= (h(y))^2 g^2(x) \\
&= (h(y))^2 (g(x))^2 \\
&= (h(y)\, g(x))^2 \\
&= (h(y)\,(T \cdot g)(y))^2 \\
&= ((Tg)(y))^2 \\
&= (1_Y(y))^2 \\
&= 1 = 1_Y(y).
\end{aligned}$$

If $(T_0 g)(y) = \overline{g(x)}$, then $\left(T_0 g^2\right)(y) = g^2(x)$ and so

$$\begin{aligned}
\left(hTg^2\right)(y) &= h(y)\left(Tg^2\right)(y) \\
&= h(y)^2\left(\left(T_0 g^2\right)(y)\right) \\
&= h(y)\,\overline{g^2(x)} \\
&= \left(h(y)\,\overline{g(x)}\right)^2 \\
&= (h(y)\,(T_0 g)(y))^2 \\
&= (1_Y(y))^2 \\
&= 1 \\
&= 1_Y(y).
\end{aligned}$$

Therefore,

$$\left(hTg^2 - 1_Y\right)(y) = 0. \tag{2.30}$$

Since $hTg^2 - 1_Y \in B$, (2.30) holds for each $y \in \mathrm{Ch}(B, Y)$ and $\mathrm{Ch}(B, Y)$ is a boundary for B with respect to Y, we deduce that (2.30) holds for each $y \in Y$. Hence, $hTg^2 = 1_Y$ and so h is invertible in B.

(iii). This follows from (i) and (ii). □

Theorem 2.14. *Let X and Y be compact Hausdorff spaces, A be a complex function algebra on X, B be a complex function algebra on Y and $T : A \longrightarrow B$ be a surjective real-linear uniform isometry. Suppose that $\mathrm{ER}(A, X) = \mathrm{Ch}(A, X)$ and $\mathrm{ER}(B, Y) = \mathrm{Ch}(B, Y)$. Then the following statements hold.*

(i) $|\,(T1_X)(y)\,| = 1$ *for all* $y \in \mathrm{Ch}(B, Y)$ *and* $T1_X$ *is invertible in* B.

(ii) *There exist a homeomorphism* $\varphi : \mathrm{Ch}\,(B, Y) \longrightarrow \mathrm{Ch}\,(A, X)$ *and a function* $\alpha \in B$ *with* $\|\alpha\|_Y = 1$ *and* $\alpha\,(y) \in \{-1, 1\}$ *for all* $y \in \mathrm{Ch}\,(B, Y)$ *such that*

$$(Tf)\,(y) = (T1_X)\,(y)\,(\mathrm{Re}\,\,(f \circ \varphi)\,(y) + i\alpha\,(y)\,\mathrm{Im}\,\,(f \circ \varphi)\,(y))\,,$$

for all $f \in A$ *and* $y \in \mathrm{Ch}\,(B, Y)$.

(iii) *There exist a homeomorphism* $\varphi : \mathrm{Ch}\,(B, Y) \longrightarrow \mathrm{Ch}\,(A, X)$ *and a clopen subset* K *of* $\mathrm{Ch}\,(B, Y)$, *possible empty, such that*

$$(Tf)\,(y) = \begin{cases} (T1_X)\,(y)\,f\,(\varphi\,(y))\,, & y \in K, \\ \\ (T1_X)\,(y)\,\overline{f\,(\varphi\,(y))}, & y \in \mathrm{Ch}\,(B, Y) \setminus K, \end{cases}$$

for all $f \in A$.

Proof. (i). Let $h = T1_X$. By Lemma 2.13, $|h\,(y)| = 1$ for all $y \in Y$ and h is invertible on B. Hence, (i) holds.

(ii). Define the map $T' : A \longrightarrow B$ by

$$T'f = h^{-1}Tf, \quad (f \in A)\,.$$

By Lemma 2.13, T' is a surjective real-linear uniform isometry with $T'1_X = 1_Y$. By Theorem 2.7, there exist a homeomorphism φ from $\mathrm{Ch}\,(B, Y)$ onto $\mathrm{Ch}\,(A, X)$ and a function α in B with $\|\alpha\|_Y = 1$ and $\alpha\,(y) \in \{-1, 1\}$ for all $y \in \mathrm{Ch}\,(B, Y)$ such that

$$\left(T'f\right)(y) = \mathrm{Re}\,f\,(\varphi\,(y)) + i\alpha\,(y)\,\mathrm{Im}\,f\,(\varphi\,(y))\,,$$

for all $f \in A$ and $y \in \mathrm{Ch}\,(B, Y)$. This implies that

$$(Tf)\,(y) = (T1_X)\,(y)\,(\mathrm{Re}\,\,(f \circ \varphi)\,(y) + i\alpha\,(y)\,\mathrm{Im}\,\,(f \circ \varphi)\,(y))\,,$$

for all $f \in A$ and $y \in \mathrm{Ch}\,(B, Y)$. So (ii) holds.

(iii). Set $K = \{y \in \mathrm{Ch}\,(B, Y) : \alpha\,(y) = 1\}$. By the given argument in the proof of Theorem 2.7, K is a clopen subset of $\mathrm{Ch}\,(B, Y)$, possibly empty. Applying (ii), we get

$$(Tf)\,(y) = \begin{cases} (T1_X)\,(y)\,(f \circ \varphi)\,(y)\,, & y \in K, \\ \\ (T1_X)\,(y)\,\overline{f \circ \varphi\,(y)}, & y \in \mathrm{Ch}\,(B, Y) \setminus K, \end{cases}$$

for all $f \in A$ and $y \in \mathrm{Ch}\,(B, Y)$. So (iii) holds. □

By Theorem 1.4, $\mathrm{ER}\,(A, X) = \mathrm{Ch}\,(A, X)$ where X is a compact Hausdorff space and A is a complex uniform function algebra on X. We now give two examples of complex function algebras which are not uniformly closed and one of them is extremely regular at each point of its Choquet boundary.

Example 2.15. Let $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$, $\overline{\mathbb{D}} = \{z \in \mathbb{C} : |z| \leq 1\}$ and $P_0\left(\overline{\mathbb{D}}\right)$ denotes the set of all $f \in C\left(\overline{\mathbb{D}}\right)$ such that $f = p|_{\overline{\mathbb{D}}}$, where p is a polynomial of one variable with coefficients in $\mathbb{C}$. Clearly, $P_0\left(\overline{\mathbb{D}}\right)$ is a complex function algebra on $\overline{\mathbb{D}}$ and $P_0\left(\overline{\mathbb{D}}\right) \neq \overline{P_0\left(\overline{\mathbb{D}}\right)}$. Let $\lambda \in \mathbb{T}$ and define the function $f : \overline{\mathbb{D}} \longrightarrow \mathbb{C}$ by

$$f(z) = \frac{1}{2}\left(1 + \overline{\lambda} z\right), \quad \left(z \in \overline{\mathbb{D}}\right). \tag{2.31}$$

It is easy to see that $f \in P_0\left(\overline{\mathbb{D}}\right)$, $f(\lambda) = 1$ and $|f(z)| < 1$ for all $z \in \overline{\mathbb{D}} \setminus \{\lambda\}$. Hence, $\mathbb{T} \subseteq S_0\left(P_0\left(\overline{\mathbb{D}}\right), \overline{\mathbb{D}}\right)$ and so $\mathbb{T}$ is a subset of every boundary for $P_0\left(\overline{\mathbb{D}}\right)$ with respect to $\overline{\mathbb{D}}$. On the other hand,

$$\left\{z \in \overline{\mathbb{D}} : |g(z)| = \|g\|_{\overline{\mathbb{D}}}\right\} \subseteq \mathbb{T},$$

for all $g \in P_0\left(\overline{\mathbb{D}}\right) \setminus \mathbb{C}|_{\overline{\mathbb{D}}}$ by the maximum modulus principle. This implies that $\mathbb{T}$ is a boundary for $P_0\left(\bar{\mathbb{D}}\right)$ and $S_0\left(P_0\left(\bar{\mathbb{D}}\right), \bar{\mathbb{D}}\right) \subseteq \mathbb{T}$. Therefore, $S_0\left(P_0\left(\overline{\mathbb{D}}\right), \overline{\mathbb{D}}\right) = \mathbb{T}$ and so $\mathbb{T}$ is the intersection of all boundaries for $P_0\left(\overline{\mathbb{D}}\right)$ with respect to $\overline{\mathbb{D}}$. This implies that

$$\mathbb{T} \subseteq \mathrm{Ch}\left(P_0\left(\overline{\mathbb{D}}\right), \overline{\mathbb{D}}\right). \tag{2.32}$$

Since $\mathbb{T}$ is a boundary for $P_0\left(\overline{\mathbb{D}}\right)$ with respect to $\overline{\mathbb{D}}$ and $\mathbb{T}$ is a closed subset of $\overline{\mathbb{D}}$, we deduce that $\mathbb{T}$ is a closed boundary for $P_0\left(\overline{\mathbb{D}}\right)$ with respect to $\overline{\mathbb{D}}$. Hence,

$$\mathrm{Ch}\left(P_0\left(\overline{\mathbb{D}}\right), \overline{\mathbb{D}}\right) \subseteq \mathbb{T}. \tag{2.33}$$

From (2.32) and (2.33), we have

$$\mathrm{Ch}\left(P_0\left(\overline{\mathbb{D}}\right), \overline{\mathbb{D}}\right) = \mathbb{T}.$$

Let $\lambda \in \mathbb{T}$, U be an open neighborhood of λ in $\overline{\mathbb{D}}$ and $\varepsilon > 0$ be given. Define the function $f : \overline{\mathbb{D}} \longrightarrow \mathbb{C}$ by (2.31). Then $f \in P_0\left(\overline{\mathbb{D}}\right)$, $\|f\|_{\overline{\mathbb{D}}} = 1 = f(\lambda)$. Since $\|f\|_{\overline{\mathbb{D}} \setminus U} < 1$, there exists $N \in \mathbb{N}$ such that $\|f^N\|_{\overline{\mathbb{D}} \setminus U} < \varepsilon$. Set $g = f^N$. Then $g \in P_0\left(\overline{\mathbb{D}}\right)$, $\|g\|_{\overline{\mathbb{D}}} = 1 = g(\lambda)$ and $|g(z)| < \varepsilon$ for all $z \in \overline{\mathbb{D}} \setminus U$. Therefore, $P_0\left(\overline{\mathbb{D}}\right)$ is extremely regular at λ.

Example 2.16. Let (X, d) be a compact metric space and Lip (X, d) denotes the set of all complex-valued functions f on X for which

$$\sup\left\{\frac{|f(x) - f(y)|}{d(x, y)} : x, y \in X, x \neq y\right\} < \infty.$$

It is known that Lip (X, d) is a complex function algebra on (X, d) such that $\overline{\mathrm{Lip}\,(X, d)} = C(X)$ and Lip $(X, d) \neq C(X)$ whenever X is infinite.

This algebra is called Lipschitz algebra on (X, d) and was first studied by Sherbert in [9].

Suppose that X is an infinite set. For each $x \in X$, the function $f : X \longrightarrow \mathbb{C}$ defined by

$$f(y) = 1 - \frac{d(x,y)}{\text{diam}(X)}, \quad (y \in X), \tag{2.34}$$

belongs to Lip (X,d), $f(x) = 1$ and $|f(y)| < 1$ for all $y \in X \setminus \{x\}$. Therefore, $S_0(\text{Lip}(X,d), X) = X$. This implies that X is the only boundary for Lip (X,d) with respect to X and so $\text{Ch}(\text{Lip}(X,d), X) = X$.

Let $x \in X$, U be an open neighborhood of x in (X,d) and $\varepsilon > 0$ be given. Define the function $f : X \longrightarrow \mathbb{C}$ by (2.34). Then $f \in \text{Lip}(X,d)$, $\|f\|_X = f(x) = 1$. Since $\|f\|_{X \setminus U} < 1$, we deduce that there exists $N \in \mathbb{N}$ such that $\|f^N\|_{X \setminus U} < \varepsilon$. Set $g = f^N$. Then $g \in \text{Lip}(X,d)$, $\|g\|_X = g(x) = 1$ and $|g(y)| < \varepsilon$ for all $y \in X \setminus U$. Therefore, Lip (X,d) is extremely regular at x.

Remark 2.17. Let (X,d) be a compact metric space and A be the complex linear span of $\{1_X\} \cup \left\{h_{x,\frac{1}{n}} : x \in X, n \in \mathbb{N}\right\}$, where for each $x \in X$ and every $n \in \mathbb{N}$ the function $h_{x,\frac{1}{n}} : X \longrightarrow \mathbb{C}$ is defined as the one in Example 2.10. It is easy to see that A is a self-adjoint complex linear subspace of $\text{Lip}(X,d)$ containing 1_X. This fact implies that A is not a uniformly closed subalgebra of $C(X)$ whenever X is an infinite set. Otherwise, $A = C(X)$ by the Stone-Weierestrass theorem and so $\text{Lip}(X,d) = C(X)$ which is a contradiction.

Definition 2.18. Let (X,d) and (Y,ρ) be metric spaces.

(i) A map $\varphi : Y \longrightarrow X$ is called a Lipschitz mapping from (Y,ρ) to (X,d) if there exists a constant $M > 0$ such that

$$d(\varphi(y_1), \varphi(y_2)) \leq M\rho(y_1, y_2),$$

for all $y_1, y_2 \in Y$.

(ii) A bijective map $\varphi : Y \longrightarrow X$ is called a Lipschitz homeomorphism from (Y,ρ) to (X,d) if φ is a Lipschitz mapping from (Y,ρ) to (X,d) and φ^{-1} is a Lipschitz mapping from (X,d) to (Y,ρ).

Applying Theorem 2.14 and the mentioned facts in Example 2.16, we obtain a description of surjective real-linear uniform isometries between complex Lipschitz algebras on compact metric spaces.

Theorem 2.19. *Let (X,d) and (Y,ρ) be compact metric spaces and $T : \text{Lip}(X,d) \longrightarrow \text{Lip}(Y,\rho)$ be a surjective real-linear uniform isometry with $T1_X = 1_Y$. Then there exist a Lipschitz homeomorphism φ from*

(X,d) to (Y,ρ) and a function α in $\mathrm{Lip}\,(Y,\rho)$ with $\alpha\,(y)\in\{-1,1\}$ for all $y\in Y$ such that

$$(Tf)\,(y)=\mathrm{Re}\,(f\circ\varphi)\,(y)+i\alpha\,(y)\,\mathrm{Im}\;(f\circ\varphi)\,(y)\,,$$

for all $f\in\mathrm{Lip}\,(X,d)$ and $y\in Y$.

Proof. By the mentioned facts in Example 2.16, $\mathrm{Lip}\,(X,d)$ and $\mathrm{Lip}\,(Y,\rho)$ are complex function algebras on compact metric spaces (X,d) and (Y,ρ), respectively, $\mathrm{ER}\,(\mathrm{Lip}\,(X,d)\,,X)=\mathrm{Ch}\,(\mathrm{Lip}\,(X,d)\,,X)=X$ and $\mathrm{ER}\,(\mathrm{Lip}\,(Y,\rho)\,,Y)=\mathrm{Ch}\,(\mathrm{Lip}\,(Y,\rho)\,,Y)=Y$. Hence, by hypotheses and Theorem 2.14, there exist a homeomorphism φ from Y with the generated topology by ρ onto X with the generated topology by d and a function α in $\mathrm{Lip}\,(Y,\rho)$ with $\alpha\,(y)\in\{-1,1\}$ for all $y\in Y$ such that

$$(Tf)\,(y)=\mathrm{Re}\;(f\circ\varphi)\,(y)+i\alpha\,(y)\,\mathrm{Im}\;(f\circ\varphi)\,(y)\,,$$

for all $f\in\mathrm{Lip}\,(X,d)$ and $y\in\mathrm{Ch}\,(\mathrm{Lip}\,(Y,\rho)\,,Y)=Y$.

To complete the proof, it is sufficient to show that φ is a Lipschitz mapping from (Y,ρ) to (X,d) and φ^{-1} is a Lipschitz mapping from (X,d) to (Y,ρ).

Let $f\in\mathrm{Lip}\,(X,d)$. Then

$$Tf=\mathrm{Re}\;(f\circ\varphi)+i\alpha\mathrm{Im}\;(f\circ\varphi)\,. \tag{2.35}$$

From (2.35) and $\alpha\,(y)\subseteq\mathbb{R}$, we deduce that $\mathrm{Re}\;(Tf)=\mathrm{Re}\;(f\circ\varphi)$ and $\mathrm{Im}\;(Tf)=\alpha\mathrm{Im}\;(f\circ\varphi)$. Since $\mathrm{Lip}\,(Y,\rho)$ is a self-adjoint complex linear subspace of $C\,(Y)$, we conclude that $\mathrm{Re}\;(f\circ\varphi)$, $\alpha\mathrm{Im}\;(f\circ\varphi)\in\mathrm{Lip}\,(Y,\rho)$. The invertibility of α in $\mathrm{Lip}\,(Y,\rho)$, implies that $\mathrm{Im}\;(f\circ\varphi)\in\mathrm{Lip}\,(Y,\rho)$. Hence, $f\circ\varphi\in\mathrm{Lip}\,(Y,\rho)$. Define the map $S:\mathrm{Lip}\,(X,d)\longrightarrow\mathrm{Lip}\,(Y,\rho)$ by

$$S\,(f)=f\circ\varphi,\quad(f\in\mathrm{Lip}\,(X,d))\,.$$

By the above argument above, S is well-define. Clearly, S is an algebra homomorphism with $S1_X=1_Y$. By [9, Theorem 5.1], there exists a function $\psi:Y\longmapsto X$ which is a Lipschitz mapping from (Y,ρ) to (X,d) such that $S\,(f)=f\circ\psi$ for all $f\in\mathrm{Lip}\,(X,d)$. Hence, $f\circ\varphi=f\circ\psi$ for all $f\in\mathrm{Lip}\,(X,d)$. This implies that $\varphi=\psi$ since $\mathrm{Lip}\,(X,d)$ separates the points of X. Therefore, φ is a Lipschitz mapping from $\mathrm{Lip}\,(Y,\rho)$ to $\mathrm{Lip}\,(X,d)$.

On the other hand, $T^{-1}:\mathrm{Lip}\,(Y,\rho)\longrightarrow\mathrm{Lip}\,(X,d)$ is a surjective real-linear uniform isometry with $T^{-1}1_Y=1_X$. By the above argument, there exists a function $\eta:X\longrightarrow Y$ which is a Lipschitz mapping from (X,d) to (Y,ρ) and a function β in $\mathrm{Lip}\,(X,d)$ with $\beta\,(x)\in\{-1,1\}$ for all $x\in X$ such that

$$\left(T^{-1}g\right)(x)=\mathrm{Re}\;(g\circ\eta)\,(x)+i\beta\,(x)\,\mathrm{Im}\;(g\circ\eta)\,(x)\,,$$

for all $g \in \mathrm{Lip}\,(Y, \rho)$ and $x \in X$. By the given argument in the proof of $(i) \Rightarrow (ii)$ in Theorem 2.7, one can show that $\varphi^{-1} = \eta$. Therefore, φ^{-1} is a Lipschitz mapping from $\mathrm{Lip}\,(X, d)$ to $\mathrm{Lip}\,(Y, \rho)$. This completes the proof. □

Theorem 2.20. *Let (X, d) and (Y, ρ) be compact metric spaces and $T : \mathrm{Lip}\,(X, d) \longrightarrow \mathrm{Lip}\,(Y, \rho)$ be a surjective real-linear uniform isometry. Then $|\,(T1_X)\,(y)\,| = 1$ for all $y \in Y$, $T1_X$ is invertible in $\mathrm{Lip}\,(Y, \rho)$ and there exist a Lipschitz homeomorphism φ from (Y, ρ) to (X, d) and a function α in $\mathrm{Lip}\,(Y, \rho)$ with $\alpha\,(y) \in \{-1, 1\}$ for all $y \in Y$ such that*

$$(Tf)\,(y) = (T1_X)\,(y)\,(\mathrm{Re}\,\,(f \circ \varphi)\,(y) + i\alpha\,(y)\,\mathrm{Im}\,\,(f \circ \varphi)\,(y))\,,$$

for all $f \in \mathrm{Lip}\,(X, d)$ and $y \in Y$.

Proof. By the mentioned facts in Example 2.16, $\mathrm{Lip}\,(X, d)$ and $\mathrm{Lip}\,(Y, \rho)$ are complex function algebras on X and Y, respectively,

$$\mathrm{ER}\,(\mathrm{Lip}\,(X, d)\,, X) = \mathrm{Ch}\,(\mathrm{Lip}\,(X, d)\,, X) = X,$$

and

$$\mathrm{ER}\,(\mathrm{Lip}\,(Y, \rho)\,, Y) = \mathrm{Ch}\,(\mathrm{Lip}\,(Y, \rho)\,, Y) = Y.$$

Hence, by hypotheses and Theorem 2.14, $|\,(T1_X)\,(y)\,| = 1$ for all $y \in Y$ and $T1_X$ is invertible in $\mathrm{Lip}\,(Y, \rho)$. Define the map $T' : \mathrm{Lip}\,(X, d) \longrightarrow \mathrm{Lip}\,(Y, \rho)$ by

$$T'f = (T1_X)^{-1}\,Tf, \quad (f \in \mathrm{Lip}\,(X, d))\,.$$

Clearly, T' is a surjective real-linear uniform isometry with $T'1_X = 1_Y$. By Theorem 2.19, there exist α Lipschitz homeomorphism φ from (Y, ρ) to (X, d) and a function α in $\mathrm{Lip}\,(Y, \rho)$ with $\alpha\,(y) \in \{-1, 1\}$ for all $y \in Y$ such that

$$\left(T'f\right)(y) = \mathrm{Re}\,\,(f \circ \varphi)\,(y) + i\alpha\,(y)\,\mathrm{Im}\,\,(f \circ \varphi)\,,$$

for all $f \in \mathrm{Lip}\,(X, d)$ and $y \in Y$. Therefore,

$$(Tf)\,(y) = (T1_X)\,(y)\,(\mathrm{Re}\,\,(f \circ \varphi)\,(y) + i\alpha\,(y)\,\mathrm{Im}\,\,(f \circ \varphi)\,(y))\,,$$

for all $f \in \mathrm{Lip}\,(X, d)$ and $y \in Y$. Hence, the proof is complete. □

Applying Theorem 2.14, we obtain a description of surjective real-linear uniform isometries between complex uniform function algebras on compact Hausdorff spaces.

Theorem 2.21. *Let X and Y be compact Hausdorff spaces, A be a complex uniform function algebra on X, B be a a complex uniform function algebra on Y and $T : A \longrightarrow B$ be a surjective real-linear uniform isometry. Then the following statements hold.*

(i) $|\,(T1_X)\,(y)\,| = 1$ *for all* $y \in \mathrm{Ch}\,(\mathrm{B}, \mathrm{Y})$ *and* $T1_X$ *is invertible in* B.

(ii) *There exist a homeomorphism* $\varphi : \mathrm{Ch}\,(B, Y) \longrightarrow \mathrm{Ch}\,(A, X)$ *and a function* $\alpha \in B$ *with* $\|\alpha\|_Y = 1$ *and* $\alpha\,(y) \in \{-1, 1\}$ *for all* $y \in \mathrm{Ch}\,(B, Y)$ *such that*

$$(Tf)\,(y) = (T1_X)\,(y)\,(\mathrm{Re}\, f\,(\varphi\,(y)) + i\alpha\,(y)\,\mathrm{Im}\, f\,(\alpha\,(y)))\,,$$

for all $f \in A$ *and* $y \in \mathrm{Ch}\,(B, Y)$.

(iii) *There exist a homeomorphism* $\varphi : \mathrm{Ch}\,(B, Y) \longrightarrow \mathrm{Ch}\,(A, X)$ *and a clopen subset* K *of* $\mathrm{Ch}\,(B, Y)$*, possibly empty, such that*

$$(Tf)\,(y) = \begin{cases} (T1_X)\,(y)\,(f \circ \varphi)\,(y)\,, & y \in K, \\ (T1_X)\,(y)\,\overline{(f \circ \varphi)\,(y)}, & y \in \mathrm{Ch}\,(B, Y) \setminus K, \end{cases}$$

for all $f \in A$.

Proof. Since A is a complex uniform function algebra on X and B is a complex uniform function algebra on Y, we deduce $\mathrm{ER}\,(A, X) = \mathrm{Ch}\,(A, X)$ and $\mathrm{ER}\,(B, Y) = \mathrm{Ch}\,(B, Y)$ by Theorem 1.5. So (i), (ii) and (iii) follow by Theorem 2.14. □

Corollary 2.22. *Let* X *and* Y *be compact Hausdorff spaces and* $T : C\,(X) \longrightarrow C\,(Y)$ *be a surjective real-linear uniform isometry. Then the following statements hold.*

(i) $|\,(T1_X)\,(y)\,| = 1$ *for all* $y \in Y$.

(ii) *There exist a homeomorphism* $\varphi : Y \longrightarrow X$ *and a function* $\alpha \in C\,(Y)$ *with* $|\alpha\,(y)\,| = 1$ *for all* $y \in Y$ *such that*

$$(Tf)\,(y) = (T1_X)\,(y)\,(\mathrm{Re}\, f\,(\varphi\,(y)) + i\alpha\,(y))\,\mathrm{Im}\, f\,(\varphi\,(y))\,,$$

for all $f \in C\,(X)$ *and* $y \in Y$.

(iii) *There exist a homeomorphim* $\varphi : Y \longrightarrow X$ *and a clopen subset* K *of* Y*, possibly empty, such that*

$$(Tf)\,(y) = \begin{cases} (T1_X)\,(y)\,(f \circ \varphi)\,(y)\,, & y \in K, \\ (T1_X)\,(y)\,\overline{(f \circ \varphi)\,(y)}, & y \in Y \setminus K, \end{cases}$$

for all $f \in C\,(X)$.

Proof. Since $C\,(X)$ is a uniform function algebra on X, $C\,(Y)$ is a uniform function algebra on Y, $\mathrm{Ch}\,(C\,(X)\,, X) = X$ and $\mathrm{Ch}\,(C\,(Y)\,, Y) = Y$, we deduce that (i), (ii) and (iii) hold by Theorem 2.21. □

We now omit the condition $\mathrm{ER}\,(A, X) = \mathrm{Ch}\,(A, X)$ and $\mathrm{ER}\,(B, Y) = \mathrm{Ch}\,(B, Y)$ in Theorem 2.14 and obtain the following result, which is a generalization of Theorem 2.21.

Theorem 2.23. *Let X and Y be compact Hausdorff spaces, A and B be complex function spaces on X and Y, respectively, and $T : A \longrightarrow B$ be a surjective real-linear uniform isometry. Then the following statements hold.*

(i) $|\,(T1_X)\,(y)\,| = 1$ *for all* $y \in \mathrm{Ch}\,(B, Y)$ *and* $T1_X$ *is invertible in* $\bar{B}$.

(ii) *There exist a homeomorphism* $\varphi : \mathrm{Ch}\,(B, Y) \longrightarrow \mathrm{Ch}\,(A, X)$ *and a function* $\alpha \in \bar{B}$ *with* $\|\alpha\|_Y = 1$ *and* $\alpha\,(y) \in \{-1, 1\}$ *for all* $y \in \mathrm{Ch}\,(B, Y)$ *such that*

$$(Tf)\,(y) = (T1_X)\,(y)\,(\mathrm{Re}\,(f \circ \varphi)\,(y) + i\alpha\,(y)\,\mathrm{Im}\,(f \circ \varphi)\,(y))\,,$$

for all $f \in Y$ *and* $y \in \mathrm{Ch}\,(B, Y)$.

(iii) *There exist a homeomorphism* $\varphi : \mathrm{Ch}\,(B, Y) \longrightarrow \mathrm{Ch}\,(A, X)$ *and a clopen subset* K *of* $\mathrm{Ch}\,(B, Y)$*, possibly empty, such that*

$$(Tf)\,(y) = \begin{cases} (T1_X)\,(y)\,(f \circ \varphi)\,(y)\,, & y \in K, \\ \\ (T1_X)\,(y)\,\overline{(f \circ \varphi)\,(y)}, & y \in \mathrm{Ch}\,(B, Y) \setminus K, \end{cases}$$

for all $f \in A$.

Proof. Define the map $\tilde{T} : \bar{A} \longrightarrow \bar{B}$ by

$$\tilde{T}f = \lim_{n\to\infty} Tf_n \quad (\text{in } \, (C\,(Y)\,, \|\cdot\|_Y))\,, \tag{2.36}$$

where $f \in \bar{A}$, $\{f_n\}_{n=1}^{\infty}$ is a sequence in A with

$$\lim_{n\to\infty} f_n = f \quad (\text{in } \, (C\,(X)\,, \|\cdot\|_X))\,. \tag{2.37}$$

If $f \in \bar{A}$, there exists a sequence $\{f_n\}_{n=1}^{\infty}$ in A satisfying (2.37). Since $T : A \longrightarrow B$ is a real-linear uniform isometry, we conclude that $\{Tf_n\}_{n=1}^{\infty}$ is a Cauchy sequences in $(C\,(Y)\,, \|\cdot\|_Y)$ and so $\lim\limits_{n\to\infty} Tf_n \in \bar{B}$. Moreover, one can easily show that if $\{f_n\}_{n=1}^{\infty}$ and $\{g_n\}_{n=1}^{\infty}$ are sequence in A with $f = \lim\limits_{n\to\infty} f_n = \lim\limits_{n\to\infty} g_n$ in $(C\,(X)\,, \|\cdot\|_X)$, then $\lim\limits_{n\to\infty} Tf_n = \lim\limits_{n\to\infty} Tg_n$ in $(C\,(Y)\,, \|\cdot\|_Y)$. Hence, $\tilde{T}$ is well-defined. It is easy to see the $\tilde{T}$ is a real-linear mapping. If $f \in \bar{A}$ and $\{f_n\}_{n=1}^{\infty}$ is a sequences in A satisfying (2.37) and (2.36), then

$$\|\tilde{T}f\|_Y = \lim_{n\to\infty} \|Tf_n\|_Y = \lim_{n\to\infty} \|f_n\|_X = \|f\|_X.$$

Hence, $\tilde{T}$ is uniform norm preserving and so $\tilde{T}$ is a uniform isometry since $\tilde{T}$ is real-linear.

Let $g \in \bar{B}$. Surjectivity of T implies that there exists a sequence $\{f_n\}_{n=1}^{\infty}$ in A such that

$$g = \lim_{n\to\infty} Tf_n \quad (\text{in } \, (C\,(Y)\,, \|\cdot\|_Y))\,.$$

This implies that $\{f_n\}$ is a Cauchy sequence in $(C(X), \|\cdot\|_X)$, since $T : A \longrightarrow B$ is a real-linear uniform isometry. Thus there exists a function $f \in C(X)$ such that

$$\lim_{n\to\infty} f_n = f, \quad (\text{in } (C(X), \|\cdot\|_X)).$$

Clearly, $f \in \bar{A}$ and so by definition of $\tilde{T}$, $\tilde{T}f = g$. Hence, $\tilde{T}$ is surjective. The definition of $\tilde{T}$ implies that $\tilde{T}f = Tf$ for all $f \in A$.

By Theorem 2.21, we deduce that the following statements hold.

(I). $\left|\left(\tilde{T}1_X\right)(y)\right| = 1$ for all $y \in \mathrm{Ch}\left(\bar{B}, Y\right)$ and $\tilde{T}1_X$ is invertible in $\bar{B}$.

(II). There exist a homeomorphism $\varphi : \mathrm{Ch}\left(\bar{B}, Y\right) \longrightarrow \mathrm{Ch}\left(\bar{A}, X\right)$ and a function $\alpha \in \bar{B}$ with $\|\alpha\|_Y = 1$ and $\alpha(y) \in \{-1, 1\}$ for all $y \in \mathrm{Ch}\left(\bar{B}, Y\right)$ such that

$$\left(\tilde{T}f\right)(y) = \left(\tilde{T}1_X\right)(y)\left(\mathrm{Re}\left(f\circ\varphi\right)(y) + i\alpha(y)\,\mathrm{Im}\left(f\circ\varphi\right)(y)\right),$$

for all $f \in \bar{A}$ and $y \in \mathrm{Ch}\left(\bar{B}, Y\right)$.

(III). There exist a homeomorphism $\varphi : \mathrm{Ch}\left(\bar{B}, Y\right) \longrightarrow \mathrm{Ch}\left(\bar{A}, X\right)$ and a clopen subset K of $\mathrm{Ch}(B, Y)$, possibly empty, such that

$$\left(\tilde{T}f\right)(y) = \begin{cases} \left(\tilde{T}1_X\right)(y)\left(f\circ\varphi\right)(y), & y \in K, \\ \\ \left(\tilde{T}1_X\right)(y)\overline{\left(f\circ\varphi\right)(y)}, & y \in \mathrm{Ch}\left(\bar{B}, Y\right)\setminus K, \end{cases}$$

for all $f \in \bar{A}$. Since $\tilde{T}f = Tf$ for all $f \in A$, $1_X \in A$ and by Theorem 1.4, $\mathrm{Ch}\left(\bar{A}, X\right) = \mathrm{Ch}(A, X)$ and $\mathrm{Ch}\left(\bar{B}, Y\right) = \mathrm{Ch}(B, Y)$, we deduce that (i), (ii) and (iii) follow by (I), (II) and (III). □

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