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# Inequality Problems of Equilibrium Problems with Application

Ayed Eleiwis Hashoosh[1 *], Mohsen Alimohammady[2], and Haiffa Mohsen Buite[3]

Abstract. This paper aims at establishing the existence of results for a nonstandard equilibrium problems ($EP_N$). The solutions of this inequality are discussed in a subset $K$ (either bounded or unbounded) of a Banach spaces $X$. Moreover, we enhance the main results by application of some differential inclusion.

## 1. Introduction and Preliminaries

Equilibrium problems has played a crucial role in pure and applied mathematics problems, such as variational inequality problems, minimax inequality problems, optimization problems and fixed point problems. Moreover, it has been widely applied to study and develop other sciences, such as economics, mechanics and engineering science. Some of these in the recent decades, many results concerning the existence of solutions for equilibrium problems and vector equilibrium problems have been investigated. (see [1, 3, 4, 5]). Several problems of equilibrium problems and mixed equilibrium problems have also been investigated and generalized by a number of authors, (see e.g. [1, 12, 14, 19]). In this paper, unless otherwise specified, authors assume that $E$ is a Banach space and $E^*$ is a topological dual space of $E$, while $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$ denote the duality pairing between $E$ and $E^*$ and the norm in $E^*$, respectively. Assume that $K$ is a nonempty subset of a real reflexive Banach space $E$. Let $F, \Gamma, \alpha : K \times K \to \mathbb{R}$ be three real-valued functions, and let $A : K \to E^*$ be a nonlinear mapping. We consider the following, a

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* Corresponding author.

nonstandard equilibrium problem (for short, $(EP_N)$) which is to find a $x \in K$ such that

$$F(x,y) + \Gamma(x,y) + \langle Ax, y-x\rangle \geq 0, \quad \forall y \in K. \tag{1.1}$$

$F$ is $(EP_N)$ in which $F(x,x) = 0$. In order to highlight the generality of a nonstandard equilibrium problem $(EP_N)$, we recall several special cases, as below:

(i) If $\Gamma \equiv A \equiv 0$ then problem (1.1) is reduces to the classical equilibrium problem (for short, (EP)), which is to find $x \in K$ such that $F(x,y) \geq 0. \quad \forall y \in K$ see [5].

(ii) If $A \equiv 0$ and $\Gamma(x,y) = \Gamma(y) - \Gamma(x), \quad \forall y \in K$, then problem (1.1) is reduces to the mixed equilibrium problem (for short, (MEP)) see [17].

(iii) If $\Gamma(x,y) \equiv 0, \quad \forall y \in K$, then problem (1.1) is reduces to the generalized equilibrium problem (for short, (GEP)) see [15].

(iv) If $A \equiv 0$ then problem (1.1) is reduces to the new type of generalized equilibrium problem (for short, $(EP_\Gamma)$) see [13].

For the convenience of the reader, we indicate here some definitions and results that need to be imposed in order to prove our main results. In 2016 Hashoosh et. al. [13] introduced a new type of of monotone bifunction. They called it $\alpha$-monotone bifunction, as follows:

**Definition 1.1.** A bifunction $F : K \times K \to \mathbb{R}$ is called $\alpha$- monotone if

$$F(x,y) + F(y,x) + \alpha(x,y) \leq 0, \quad \forall x,y \in K.$$

**Example 1.2.** Let $E = \mathbb{R}, K = \mathbb{R}$ and let $F : K \times K \to \mathbb{R}$ be the bifunction defined by

$$F(u,v) = \cos(u-v)^2 + (u-v)^2,$$

for all $u, v \in K$. Then

$$F(u,v) + F(v,u) = 2\cos(u-v)^2 + 2(u-v)^2 \nleq 0,$$

where $u \neq v$. Therefore $F$ is not monotone bifunction. But, it easy to see that $F$ is $\alpha$-monotone bifunction with $\alpha(u,v) = -5(u-v)^2$. In fact,

$$\begin{aligned} F(u,v) + F(v,u) &= 2\cos(u-v)^2 + 2(u-v)^2 \\ &\leq 5(u+v)^2 \\ &= -\alpha(u,v). \end{aligned}$$

Clearly, one can have the to the classical definition of monotone bifunction if take $\alpha \equiv 0$. Recently, a number of authors have proposed many essential generalizations of monotonicity, such as $\alpha$- monotonicity, relaxed monotonicity, relaxed $\eta - \alpha -$ monotonicity and quasimononicity (see e.g. [2, 11, 16, 21]. The following notions of a KKM mapping and the well-known intersection Lemma [9] play an important role in the proof or the main results.

**Definition 1.3** ([16])**.** Let $K$ be a nonempty subset of a Hausdorff topological vector space $E$. A mapping $\Lambda : K \to 2^E$ is said to be a KKM mapping if for any finite subset $\{u_1, u_2, \dots, u_n\}$ of $K$, we have

$$co\{u_1, u_2, \dots, u_n\} \subset \bigcup_{i=1}^{n} \Lambda(u_i),$$

where $co\{u_1, u_2, \dots, u_n\}$ denotes the convex hull of $\{u_1, \dots, u_n\}$.

**Lemma 1.4** ([9])**.** *Let $K$ be a nonempty subset of a Hausdorff topological vector space $E$ and let $\Lambda : K \multimap E$ be a KKM mapping. If $\Lambda(x)$ is closed in $E$ for every $x \in K$ and compact for some $u_0 \in K$, then $\bigcap_{u \in K} \Lambda(u) \neq \phi$.*

**Definition 1.5** ([6])**.** A real-valued function $g$, defined on a convex subset $K$ of $X$, is said to be hemicontinuous, if

$$\lim_{t \to 0^+} g(tx + (1-t)y) = g(y), \quad \forall x, y \in K.$$

**Definition 1.6.** Let $X$ be a Banach space. A mapping $\Lambda : X \to \mathbb{R}$ is said to be

(i) lower semicontinuous (for short, (l.s.c)) at $x_0 \in X$, if

$$\Lambda(x_0) \leq \liminf_n \Lambda(x_n),$$

(ii) upper semicontinuous (for short, (u.s.c)) at $x_0 \in X$, if

$$\Lambda(x_0) \geq \limsup_n \Lambda(x_n),$$

for any sequence $\{x_n\}$ of X such that $x_n \to x_0$.

**Definition 1.7** ([13])**.** Assume that $X$ is a Banach space, and $f : X \to \mathbb{R} \cup \{+\infty\}$ a proper function. We say that $x^* \in X^*$ is a $\alpha$-subdifferential of $f$ in $x \in domf = \{x : f(x) < \infty\}$, if

$$\partial_\alpha f(x) = \left\{ x^* \in X^* : f(y) - \frac{\alpha\,(y, x)}{2} \geq f(x) + \langle x^*, y - x \rangle \, (\forall y \in X) \right\}.$$

Throughout this paper, let us assume that $\forall r \in [0, 1]$

$$\lim_{r \to 0} \frac{\alpha(x, x_r)}{r} = 0, \tag{1.2}$$

$$\alpha(x, y) \leq \lim_{r \to 0} \frac{r-1}{r} \left[ \Gamma(x, x) + \alpha(x, x) \right]. \tag{1.3}$$

The main purpose of this work is to give a new contribution in this area. In particular, we establish the existence and uniqueness of solutions for new type of equilibrium problem. It is worth mentioning that we do not deal with a classical technique to proof our results. Thus, several difficulties occur in finding an application to the main results since the classical methods fail to be applied directly. In order to achieve the aim, the study is divided into three sections. In Section 2, we prove the existence of solutions for the problem. The proof of the first result is based on arguments of $\alpha$-monotone bifunctions. However, in the second result in this section, we provide sufficient conditions for existence solutions for the case of unbounded sets. Under suitable conditions, we show that the solution of $(EP_N)$ is unique. In Section 3, we illustrate the applicability of our approach by a differential inclusion of our main results. We point out the fact that the results of this work can be viewed as a generalization of many known results (see e.g. [10, 18, 20]).

## 2. Main Results

In this section, we establish some existence and uniqueness results for a nonstandard equilibrium problem $(EP_N)$. It is worth mentioning that through the results of this section, we prove the existence of a solution of the problem $(EP_N)$ without the assumption of boundedness of the set $K$. In the next lemme, we assume that $K$ is a nonempty convex subset of a real reflexive Banach space $X$.

**Lemma 2.1.** *Suppose that $F : K\times K \to \mathbb{R}$ is an $\alpha$-monotone bifunction, hemicontinuous in the first argument and convex in the second argument. Let $\Gamma, \alpha : K\times K \to \mathbb{R}$ be convex in the second argument, and let $A : K \to E^*$ be an arbitrary nonlinear operator. Then a nonstandard equilibrium problem $(EP_N)$ is equivalent to the following problem: Find $x \in K$ such that*

$$\langle Ax, x-y\rangle + F(y,x) + \alpha(x,y) \le \Gamma(x,y), \quad \forall y \in K. \tag{2.1}$$

*Proof.* Suppose that the $(EP_N)$ has a solution. There exists $x \in K$ such that

$$F(x,y) + \Gamma(x,y) + \langle Ax, y-x\rangle \ge 0, \quad \forall y \in K. \tag{2.2}$$

Since $F$ is $\alpha$-monotone bifunction, then

$$F(y,x) + \alpha(x,y) \le -F(x,y), \quad \forall x, y \in K,$$

and so by (2.2)

$$\begin{aligned} F(y,x) + \alpha(x,y) &\le -F(x,y) \\ &\le \Gamma(x,y) - \langle Ax, x-y\rangle. \end{aligned}$$

Therefore, $x \in K$ is a solution of problem (2.1). Conversely, assume that $x \in K$ is a solution of problem (2.1) and fix $y \in K$. Letting

$x_\lambda = x - \lambda(x-y), \lambda \in ]0,1[$. Then $x_\lambda \in K$, since $K$ is convex, so,

$$F(x_\lambda, x) + \alpha(x, x_\lambda) - \Gamma(x, x_\lambda) \leq \langle Ax, x_\lambda - x\rangle = \lambda \langle Ax, y - x\rangle . \tag{2.3}$$

Since $F$ is convex in the second argument, then

$$0 = F(x_\lambda, x_\lambda) \leq F(x_\lambda, x) - \lambda \left[F(x_\lambda, x) - F(x_\lambda, y)\right],$$

so,

$$\lambda \left[F(x_\lambda, x) - F(x_\lambda, y)\right] \leq F(x_\lambda, x). \tag{2.4}$$

By the convexity of $\Gamma$ and $\alpha$ in the second argument, one can have

$$\alpha(x, x_\lambda) \leq \alpha(x,x) - \lambda\left[\alpha(x,x) - \alpha(x,y)\right], \tag{2.5}$$

$$\Gamma(x, x_\lambda) \leq \Gamma(x,x) - \lambda\left[\Gamma(x,x) - \Gamma(x,y)\right], \tag{2.6}$$

Then, from (2.3), (2.4), (2.5) and (2.6), one can obtain

$$\begin{aligned}\lambda &\left[F(x_\lambda, x) - F(x_\lambda, y) + \alpha(x,x) - \alpha(x,y) + \Gamma(x,x) - \Gamma(x,y)\right] \\ &\leq F(x_\lambda, x) + \alpha(x,x) - \alpha(x, x_\lambda) + \Gamma(x,x) - \Gamma(x, x_\lambda) \\ &\leq \lambda \langle Ax, y-x\rangle - 2\alpha(x, x_\lambda) + \alpha(x,x) + \Gamma(x,x).\end{aligned}$$

Since $F$ is hemicontinuous in first argument, then

$$\begin{aligned}\lambda\left[-F(x,y) - \Gamma(x,y) - \langle Ax, y-x\rangle\right] \leq &-2\alpha(x, x_\lambda) + \lambda\alpha(x,y) \\ &+ (1-\lambda)\left[\Gamma(x,x) + \alpha(x,x)\right],\end{aligned}$$

so

$$\begin{aligned}F(x,y) + \Gamma(x,y) + \langle Ax, y-x\rangle \geq &\frac{2\alpha(x,x_\lambda)}{\lambda} - \alpha(x,y) \\ &+ \frac{(\lambda-1)}{\lambda}\left[\Gamma(x,x) + \alpha(x,x)\right].\end{aligned}$$

From (1.2) and (1.3), we have

$$F(x,y) + \Gamma(x,y) + \langle Ax, y-x\rangle \geq 0 \quad \forall y \in K.$$

Therefore, $(EP_N)$ admits at least one solution. □

**Theorem 2.2.** *Let $K$ be a nonempty closed bounded convex subset of a real reflexive Banach space $X$. Assume that*

(i) *$F : K \times K \to \mathbb{R}$ is $\alpha-$ monotone bifunction, hemicontinuous in first argument, and l.s.c , convex in second argument,*
(ii) *$\Gamma : K \times K \to \mathbb{R}$ is convex in second argument, u.s.c in first argument,*
(iii) *$\alpha : K \times K \to \mathbb{R}$ is l.s.c in first argument and convex in second argument,*
(iv) *$x \mapsto \langle Ax, y-x\rangle$ is u.s.c on $K$ with respect to weak$^*$-topology of $E^*$.*

*Then* $(EP_N)$ *admits at least one solution.*

*Proof.* Define two set valued mappings $\Upsilon, \Gamma : K \multimap K$ as follows:

$$\Upsilon(y) = \{x \in K : F(x,y) + \Gamma(x,y) + \langle Ax, y-x\rangle \geq 0, \ \forall y \in K\},$$
$$\Gamma(y) = \{x \in K : F(y,x) + \alpha(x,y) + \langle Ax, x-y\rangle \leq \Gamma(x,y), \ \forall y \in K\}.$$

Then, for any $x \in K$ the problem $(EP_N)$ has a solution iff

$$\bigcap_{y\in K} \Upsilon(y) \neq \phi$$

Now, we show that $\Upsilon$ is a KKM-mapping. On the contrary, if $\Upsilon$ is not a KKM-mapping. There exist a finite subset $\{x_1, x_2, \ldots, x_n\}$ of $K$ and $\lambda_i \geq 0$ for every $i = \overline{1,n}$ with

$$\sum_{i=1}^{n} \lambda_i = 1,$$

such that

$$x_0 = \sum_{i=1}^{n} \lambda_i x_i \notin \bigcup_{i=1}^{n} \Upsilon(x_i).$$

Then

$$F(x_0, x_i) + \Gamma(x_0, x_i) + \langle Ax_0, x_i - x_0\rangle < 0, \tag{2.7}$$

for every $i = \overline{1,n}$. From (2.7), convexity of $F$ and $\Gamma$, one can get

$$\begin{aligned}
0 &= F(x_0, x_0) + \Gamma(x_0, x_0)\\
&= F(x_0, \sum_{i=1}^{n} \lambda_i x_i) + \Gamma(x_0, \sum_{i=1}^{n} \lambda_i x_i)\\
&\leq \sum_{i=1}^{n} \lambda_i \left[F(x_0, x_i) + \Gamma(x_0, x_i)\right]\\
&< \sum_{i=1}^{n} \lambda_i \left[\langle Ax_0, x_0 - x_i\rangle\right]\\
&= \langle Ax_0, x_0 - x_0\rangle,
\end{aligned}$$

for every $i = \overline{1,n}$. This is a contradiction. Therefore, $\Upsilon$ is a KKM-map. Next, from Lemma 2.1. $\Upsilon(y) \subset \Gamma(y)$ for any $y \in K$. Therefore, $\Gamma(y)$ is a KKM-map. Since $\alpha(.,y)$ and $F(y,.)$ are l.s.c and $\Gamma(.,y)$, $x \mapsto \langle Ax, y-x\rangle$

is u.s.c, then

$$\begin{aligned} F(y,x)+\alpha(x,y) &\le \liminf_n \left[F(y,x_n)+\alpha(x_n,y)\right] \\ &\le \limsup_n \left[\Gamma(x_n,y)+\langle Ax_n, y-x_n\rangle\right] \\ &\le \Gamma(x,y)+\langle Ax, y-x\rangle . \end{aligned}$$

Therefore, $\Gamma(y)$ is weakly closed $\forall y \in K$. Since $K$ is nonempty, bounded, closed and convex and $X$ is real reflexive, it follows that $K$ is weakly compact. Hence, $\Gamma(y)$ is weakly compact for any $y \in K$. From Lemma 1.4 and Lemma 2.1.

$$\bigcap_{y\in K} \Upsilon(y) = \bigcap_{y\in K} \Gamma(y) \neq \phi.$$

So, any element of this intersection is a solution. Therefore, the problem $(EP_N)$ has a solution. □

Under appropriate conditions, we show that the problem $(EP_N)$ has one solution without the assumption of boundedness of $K$.

**Theorem 2.3.** *Suppose that the same hypotheses as in Theorem 2.2 satisfy without the assumption of boundedness of $K$. Suppose in addition, the following conditions hold:*

(i) $\Gamma(x,x)=0, \quad \forall x \in K$.

(ii) *there exists $x_0 \in K$ such that*

$$F(x,x_0)+\Gamma(x,x_0)+\langle Ax, x_0-x\rangle < 0,$$

*where $x \in K$ and $\|x\|$ is large enough.*

*Then the problem $(EP_N)$ admits a solution.*

*Proof.* Set $K_n = \{y \in K : \|x\| \le n, n > 0\}$. Let us consider the following problem. Find $x_n \in K_n$ such that for any $y \in K_n$

$$F(x_n,y)+\Gamma(x_n,y)+\langle Ax_n, y-x_n\rangle \ge 0. \tag{2.8}$$

From the boundedness of $K_n$ and the Lemma 2.1, we get that the problem $(EP_N)$ has at least one solution. Choose $\|x_0\| < n_0$. From (2.8), we get

$$F(x_{n_0},x_0)+\Gamma(x_{n_0},x_0)+\langle Ax_{n_0}, x_0-x_{n_0}\rangle \ge 0. \tag{2.9}$$

So $\|x_{n_0}\| \le n_0$ since $x_{n_0} \in K_{n_0}$. Let us choose $\|x_{n_0}\| = n_0$, and that $n_0$ large enough so that by condition $(ii)$, we have

$$F(x_{n_0},x_0)+\Gamma(x_{n_0},x_0)+\langle Ax_{n_0}, x_0-x_{n_0}\rangle < 0,$$

which contradicts (2.9). Hence, $\|x_{n_0}\| < n_0$. For each $y \in K$, one can take $\lambda \in (0,1)$ small enough such that $z = \lambda y + (1-\lambda)x_{n_0} \in K_{n_0}$. From (2.8), for each $y \in K$, one can obtain

$$\begin{aligned}\lambda \langle Ax_{n_0}, x_{n_0} - y\rangle &= \langle Ax_{n_0}, x_{n_0} - z\rangle \\ &\leq F(x_{n_0}, z) + \Gamma(x_{n_0}, z) \\ &= F(x_{n_0}, \lambda y + (1-\lambda)x_{n_0}) + \Gamma(x_{n_0}, \lambda y + (1-\lambda)x_{n_0}) \\ &\leq \lambda F(x_{n_0}, y) + (1-\lambda)F(x_{n_0}, x_{n_0}) \\ &\quad + \lambda\Gamma(x_{n_0}, y) + (1-\lambda)\Gamma(x_{n_0}, x_{n_0}),\end{aligned}$$

it means that,

$$\langle Ax_{n_0}, x_{n_0} - y\rangle \leq F(x_{n_0}, y) + \Gamma(x_{n_0}, y).$$

Therefore, the problem $(EP_N)$ admits a solution. □

As for uniqueness of solutions authors present the next result.

**Theorem 2.4.** *Suppose that the same hypotheses as in Theorem 2.2 hold. In addition, let $F$ and $\Gamma$ be two monotone bifunctions and let that $A$ be stronge monotone (i.e., there exists $M > 0$ such that $\langle Ay - Ax, y - x\rangle \geq M\|y - x\|^2$). Then $(EP_N)$ has a unique solution.*

*Proof.* Towards to a contradiction, let us assume that $y_1, y_2 \in K$ be two solutions to $(EP_N)$. So, if we write in $(EP_N)$ for $y_1$ with $x = y_2$, we have

$$F(y_2, y_1) + \Gamma(y_2, y_1) + \langle Ay_2, y_1 - y_2\rangle \geq 0, \tag{2.10}$$

and then for $y_2$ with $x = y_1$, we have

$$F(y_1, y_2) + \Gamma(y_1, y_2) + \langle Ay_1, y_2 - y_1\rangle \geq 0. \tag{2.11}$$

By multiplying each of the equations (2.4) and (2.10) by $-1$ and summing together, one can get

$$\begin{aligned}0 \geq &- [F(y_2, y_1) + F(y_1, y_2)] - [\Gamma(y_2, y_1) + \Gamma(y_1, y_2)] \\ &+ \langle Ay_2 - Ay_1, y_2 - y_1\rangle \\ \geq &\, M\|y_2 - y_1\|^2.\end{aligned}$$

a contradiction. Therefore, $(EP_N)$ has a unique solution. □

## 3. Application in Partial Differential Inclusion Problems

It is important to say that there has been an increased interest in differential problems governed by higher order operators. In this section, we apply our main result, expressed in Theorem 2.2, to some partial differential inclusion problems. Let us consider the usual Sobolev space $W^{1,p}(\Omega)$ and the Banach $W^{-1,p'}(\Omega)$ as it's dual space, where $\frac{1}{p} + \frac{1}{p'} = 1$.

The $p-$ Laplacian operator $-\Delta_p : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$, by $-\Delta_p u = div(|\nabla u|^{p-2}\nabla u),\ p > 1$ is a real constant, and $\Omega$ is a bounded domain of $\mathbb{R}^N, N \geq 1$ with smooth boundary $\partial\Omega$. In what follows, let us consider the partial differential inclusion problem

$$\begin{cases} -\Delta_p u - g(x) \in \partial_{-2\alpha} f(u), & x \in \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{3.1}$$

such that $g : \Omega \to \mathbb{R}$ is continuous with compact support. $f : K \to \mathbb{R}$ is a continuous concave function and $K$ is a bounded convex subset of Sobolev space $W^{1,p}(\Omega)$. For technical reasons, let us define $\Gamma : W_0^{1,p}(\Omega) \times W_0^{1,p}(\Omega) \to \mathbb{R}$ as follows:

$$\Gamma(\xi, \Gamma) := \int_\Omega g(x)(\xi - \Gamma)(x)dx.$$

We suppose that $K$ is a nonempty, closed, bounded and convex subset of Sobolev space $W_0^{1,p}(\Omega)$.

**Definition 3.1.** $u \in K$ is called a K-weak subsolution of the problem (2.11) if,

$$\langle -\Delta_p u - g(x), u - v\rangle \leq f(u) - f(v) - \alpha(u, v), \quad \forall v \in K.$$

Here, we show the K-weak solvability of (3.1). By the integral form of $\langle ., .\rangle$, one can get that

$$\int_\Omega |\nabla u|^{p-2}\nabla u \nabla(u - v)dx - \int_\Omega g(x)(u - v)(x)dx + \alpha(u, v)f(u) - f(v).$$

Set $A := -\Delta_p$ ,

$$F(v, u) := f(v) - f(u),$$

and

$$\Gamma(u, v) := \int_\Omega g(x)(u - v)(x)dx.$$

Therefore, we obtain

$$\langle Au, u - v\rangle + F(v, u) + \alpha(u, v) \leq \Gamma(u, v), \quad \forall v \in K.$$

Considering $\alpha(u, v) = -(v - u)^2$. Then $F$ is $\alpha$-monotone bifunction. In Lemma 2.1, we proved that (2.1) and (2.2) are equivalent under some assumptions. Therefore, we must prove that $A, F$ and $\Gamma$ satisfy all the assumptions of Theorem 2.2. It is clear that the bifunction $\alpha(u, v) = -(v - u)^2$ satisfies all assumptions in Theorem 2.2. Claim 1: $\Gamma$ is convex in second argument, Let $z = tv_1 + (1-t)v_2 \in K$, $t \in [0,1]$, so

$$\begin{aligned}\Gamma(u,z) &= \Gamma(u, tv_1 + (1-t)v_2)\\ &= \int_\Omega g(x)(u - tv_1 - (1-t)v_2))(x)dx\\ &= \int_\Omega g(x)u(x)dx - \int_\Omega g(x)(tv_1 + (1-t)v_2)(x)dx\\ &= \int_\Omega g(x)(tu + (1-t)u)(x)dx - \int_\Omega g(x)tv_1(x)dx\\ &\quad - \int_\Omega g(x)(1-t)v_2(x)dx\\ &= t\int_\Omega g(x)(u(x) - v_1(x))dx + (1-t)\int_\Omega g(x)(u(x) - v_2(x))dx\\ &= t\Gamma(u,v_1) + (1-t)\Gamma(u,v_2).\end{aligned}$$

If $u_n \to u \in W_0^{1,p}(\Omega)$, by Sobolev embedding $u_n \to u \in L^p(\Omega)$,

$$\begin{aligned}\|\Gamma(u_n,v) - \Gamma(u,v)\| &= \left\|\int_\Omega g(x)\left(u_n(x) - u(x)\right)dx\right\|\\ &\leq \left(\int_\Omega |g(x)|^{p'}\right)^{\frac{1}{p'}} \cdot \left(\int_\Omega |u_n(x) - u(x)|^p\right)^{\frac{1}{p}}\\ &\leq M \parallel u_n - u \parallel_{L^p}\\ &\leq M \parallel u_n - u \parallel\\ &\rightharpoonup 0.\end{aligned}$$

This shows that $\Gamma$ is continuous. So, it is u.s.c in the first argument.

Notice that $F$ is hemicontinuous in first argument, l.s.c and convex in second argument, because f is a concave and continuous function. Moreover, $A = -\Delta_p$ is a u.s.c function, since $-\Delta_p$ is continuous (see e.g. [7]). Therefore, all conditions are satisfied.

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[1] Department of Mathematics, Faculty of Education for Pure Sciences, University of Thi-Qar, Nasiriyah, Iraq.
*E-mail address*: ayed197991@yahoo.com

[2] Department of mathematics, University of Mazandaran, Babolsar, Iran.
*E-mail address*: amohsen@umz.ac.ir

[3] Department of Mathematics, Faculty of Education for Pure Sciences, University of Thi-Qar, Nasiriyah, Iraq.

Department of mathematics, University of Mazandaran, Babolsar, Iran.
*E-mail address*: alrfayalrfay@gmail.com