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On Approximate Birkhoff-James Orthogonality and Approximate $*$-orthogonality in C^*-algebras

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Abstract. We offer a new definition of ε-orthogonality in normed spaces, and we try to explain some properties of which. Also we introduce some types of ε-orthogonality in an arbitrary C^*-algebra $\mathcal{A}$, as a Hilbert C^*-module over itself, and investigate some of its properties in such spaces. We state some results relating range-kernel orthogonality in C^*-algebras.

1. Introduction

Let $\mathcal{X}$ be a complex normed space. An element x of $\mathcal{X}$ is orthogonal to another element y of $\mathcal{X}$, in the sense of Birkhoff–James, denoted by $x \perp_{BJ} y$, if

$$\|x + \lambda y\| \geq \|x\|, \quad (\lambda \in \mathbb{C}).$$

This type of orthogonality was first introduced by Birkhoff for real normed spaces in [4]. Several properties of this relation were given in the work of James in [11]. Since then this type of orthogonality was investigated by many authors who paid attention to a certain normed space such as spaces of square complex matrices or more generally bounded operators on a Hilbert space, C^*-algebra and Hilbert C^*-modules. In some works, some types of approximate Birkhoff-James orthogonality are introduced. In an inner product space $(\mathcal{X}, \langle ., . \rangle)$, it seems that the natural approach to this concept is as follows:

$$x \perp^{\varepsilon} y \quad \Leftrightarrow \quad |\langle x, y\rangle| \leq \varepsilon \|x\| \|y\|, \quad (x, y \in \mathcal{X}).$$

Two notions of approximate BJ-orthogonality in general normed spaces which are not necessarily inner product spaces were considered. First

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one is proposed in [7] as follows:

$$x \perp_{BJD}^{\varepsilon} y \quad \Leftrightarrow \quad \|x+\lambda y\| \geq (1-\varepsilon)\|x\|, \quad \forall \lambda \in \mathbb{C},$$

and the second one is introduced in [6] by:

$$x \perp_{BJC}^{\varepsilon} y \quad \Leftrightarrow \quad \|x+\lambda y\|^2 \geq \|x\|^2 - 2\varepsilon\|x\|\|\lambda y\|, \quad \forall \lambda \in \mathbb{C}.$$

In this note, we give another definition of ε-Birkhoff-James-orthogonality and we provide some results concerned with it.

The notion of Hilbert C^*-module is a generalization of that of Hilbert space which the role of complex scalers are played by elements of a C^*-algebra $\mathcal{A}$.

We speak about the approximate BJ-orthogonality in the second section of this paper.

In the last section of the paper, considering the absolute value rather than norm in C^*-algebra $\mathcal{A}$, we present some results similar to the results stated above related to a type of orthogonality in $\mathcal{A}$, as a Hilbert C^*-module over itself.

Throughout the paper $\mathbb{B}(\mathscr{H})$ and $\mathbb{K}(\mathscr{H})$, denote the C^*-algebra of all bounded linear operators and the two-sided ideal of all compact operators on $\mathscr{H}$, respectively, and I is the identity operator. Let $\mathcal{A}$ be a (unital) C^*-algebra (with unit 1). An element a in $\mathcal{A}$ is said to be normal if $a^*a = aa^*$ and to be isometry if $a^*a = 1$. An element a in $\mathcal{A}$ is called $*$-orthogonal to $b \in \mathcal{A}$, denoted by $a \perp_{BJ}^{*} b$, if

$$|a+\lambda b|^2 \geq |a|^2, \quad (\lambda \in \mathbb{C}).$$

Given subspaces $\mathcal{M}$ and $\mathcal{N}$ of $\mathcal{A}$, we write $\mathcal{M} \perp_{BJ}^{*} \mathcal{N}$ to mean $|m+n|^2 \geq |m|^2$ for any $m \in \mathcal{M}$ and $n \in \mathcal{N}$. This type of orthogonality may be compared with the Birkhoff–James Orthogonality.

Related to definitions of approximate types of orthogonality $\perp_{BJC}^{\varepsilon}$ and $\perp_{BJD}^{\varepsilon}$, we could consider the following definitions in C^*-algebra $\mathcal{A}$ with absolute value instead of norm:

$$x \perp_{BJD}^{*\varepsilon} y \quad \Leftrightarrow \quad |x+\lambda y|^2 \geq (1-\varepsilon)|x|^2, \quad \forall \lambda \in \mathbb{C},$$

and

$$x \perp_{BJC}^{*\varepsilon} y \quad \Leftrightarrow \quad |x+\lambda y|^2 \geq |x|^2 - 2\varepsilon|x|\|\lambda y\|, \quad \forall \lambda \in \mathbb{C}.$$

In [10, Proposition 2.1] Birkhoff–James orthogonality was generalized to inner product $\mathcal{A}$-modules. In [2], it is shown that for a and b in a C^*-algebra $\mathcal{A}$, $a \perp b$ is equivalent to $a^*b = 0$. It follows from this and [3, Theorem 3.1] that $\mathbb{C}$ is the only unital C^*-algebra in which the Birkhoff–James orthogonality $a \perp_{BJ} b$ coincides with $a \perp_{BJ}^{*} b$, for all elements $a, b \in \mathcal{A}$. Utilizing this fact, one can see if $\|a\| \geq \|b\|$ implies $|a| \geq |b|$, for any $a, b \in \mathcal{A}$, then $\mathcal{A}$ is isomorphic to $\mathbb{C}$. Indeed in this case from

$$\||a+\lambda b|^2\| = \|a+\lambda b\|^2 \geq \|a\|^2 = \||a|^2\|,$$

we get

$$|a+\lambda b|^2 \geq |a|^2,$$

for any $\lambda \in \mathbb{C}$ and any $a, b \in \mathcal{A}$ which implies $\mathcal{A} = \mathbb{C}1$. In the last section, utilizing the method of [2], we present some similar results for approximate $*$-orthogonality stated in previous sections. Also in this section using the fact concluded from [2] we present some results concerned with the well-known Anderson's theorem. This theorem originally deals with the range-kernel orthogonality of inner derivations. For a and b in an algebra $\mathcal{A}$, we denote $\delta_{a,b}$ the generalized inner derivation defined by

$$\delta_{a,b}(x) := ax - xb, \quad (x \in \mathcal{A}).$$

If $a = b$, we write δ_a instead of $\delta_{a,b}$. Anderson showed that if $A \in \mathbb{B}(\mathscr{H})$ is either an isometry or a normal operator, then $\mathscr{R}(\delta_A) \perp_{BJ} \mathscr{N}(\delta_A)$, when $\mathscr{R}$ and $\mathscr{N}$ denote the range and the kernel, respectively [1]. Utilizing Berberian's 2 by 2 matrix trick, one can extend this result to generalized inner derivations by a routine argument. There are many extensions of this result and its converse; see [5, 9, 12, 14, 16] and references therein.

For instance the following theorem shows an interesting relationship between range-kernel orthogonality and Fuglede-Putnam property. First we shall say that an ordered pair (A, B) of operators in $\mathbb{B}(\mathscr{H})$ has Fuglede–Putnam property relative to $\mathbb{K}(\mathscr{H})$, if whenever $T \in \mathbb{K}(\mathscr{H})$ and $AT = TB$, we have $A^*T = TB^*$. The reader is referred to [15] and references therein for further information about the Fuglede–Putnam property.

Theorem 1.1. [5] *Let A and B be in $\mathbb{B}(\mathscr{H})$. Then (A, B) has the Fuglede–Putnam property relative to $\mathbb{K}(\mathscr{H})$ if and only if*

$$\|\delta_{A,B}X + Y\| \geq \|X\|,$$

where $Y \in \mathscr{N}(\delta_{A,B}) \cap \mathbb{K}(\mathscr{H})$ and $X \in \mathbb{K}(\mathscr{H})$.

Obviously the inequality

$$|\delta_{A,B}X + Y|^2 \geq |X|^2, \quad \left(Y \in \mathscr{N}(\delta_{A,B}) \cap \mathbb{K}(\mathscr{H}), X \in \mathbb{K}(\mathscr{H})\right),$$

yields

$$\|\delta_{A,B}X + Y\| \geq \|X\|, \quad \left(Y \in \mathscr{N}(\delta_{A,B}) \cap \mathbb{K}(\mathscr{H}), X \in \mathbb{K}(\mathscr{H})\right).$$

Therefore by Theorem 1.1, it ensures that (A, B) has Fuglede–Putnam property relative to $\mathbb{K}(\mathscr{H})$.

2. ε-ORTHOGONALITY

First, we see a new definition of approximate BJ-orthogonality,

Definition 2.1. Let $(\mathcal{X}, \|.\|)$ be a normed space and x and y be two non-zero elements in $\mathcal{X}$. We say that x is ε- Birkhoff-James-orthogonal(or ε-BJ-orthogonal) to the vector y, denoted by $x \perp_{BJ}^{\varepsilon} y$, if there exists a real number θ, such that

$$\frac{x}{\|x\|} + \varepsilon e^{i\theta} \frac{y}{\|y\|} \perp_{BJ} y,$$

that is

$$\left\| \frac{x}{\|x\|} + \varepsilon e^{i\theta} \frac{y}{\|y\|} + \lambda y \right\| \geq \left\| \frac{x}{\|x\|} + \varepsilon e^{i\theta} \frac{y}{\|y\|} \right\|,$$

for any $\lambda \in \mathbb{C}$.

In fact $x \perp_{BJ}^{\varepsilon} y$ tells when x is changed for ε value, in direction y, we reach to a vector BJ-orthogonal to y. Motivating by Definition 2.1, we may consider the following ε-$*$-orthogonality in C^*-algebras,

$$x \perp_{BJ}^{*\varepsilon} y \quad \Leftrightarrow \quad \exists \theta \in \mathbb{R}; \quad \frac{x}{\|x\|} + \varepsilon e^{i\theta} \frac{y}{\|y\|} \perp_{BJ}^{*} y.$$

Note that ε-BJ-orthogonality is homogeneous. Also, when $\varepsilon \in [0, 1)$, we see that $\perp_{BJ}^{\varepsilon} \subset \perp_{BJD}^{\varepsilon}$. Let $x \perp_{BJ}^{\varepsilon} y$ in some normed space. Then

$$\left\| \frac{x}{\|x\|} + \lambda y \right\| \geq \left\| \frac{x}{\|x\|} + \varepsilon e^{i\theta} \frac{y}{\|y\|} \right\|,$$

for any $\lambda \in \mathbb{C}$ which implies that

$$\frac{1}{\|x\|} \|x + \lambda y\| \geq \left\| \frac{x}{\|x\|} + \varepsilon e^{i\theta} \frac{y}{\|y\|} \right\|,$$

for some θ and α. Hence

$$\|x + \lambda y\| \geq \left\| \frac{x}{\|x\|} + \varepsilon e^{i\theta} \frac{y}{\|y\|} \right\| \|x\| \geq (1 - \varepsilon)\|x\|,$$

for any $\lambda \in \mathbb{C}$. However, the converse is not true in general. For example, consider $\mathcal{X} = \mathbb{C}^2$ equipped with the maximum norm $\|(x, y)\| = \max\{|x|, |y|\}$. Now, let $x = (0, 1)$ and $y = (1, 1)$ and $\varepsilon = \frac{2}{3}$. Then

$$\|x + \lambda y\| = \max\{|1 - \lambda|, |\lambda|\} \geq \frac{1}{2} > 1 - \varepsilon = \frac{1}{3},$$

thus $x \perp_{BJD}^{\varepsilon} y$. But $x \perp_{BJ}^{\varepsilon} y$ does not occur for all complex numbers λ, because

$$\|x + \lambda y\| = \max\{|1 - \lambda|, |\lambda|\},$$

and

$$\left\| x + \varepsilon e^{i\theta} y \right\| = \max\left\{ \left| 1 + \frac{2}{3} e^{i\theta} \right|, \left| \frac{2}{3} e^{i\theta} \right| \right\} \geq \frac{2}{3}.$$

Thus for $\lambda = -\frac{1}{2}$, we see that

$$\|x + \lambda y\| = \frac{1}{2} < \frac{2}{3} \leq \left\|x + \varepsilon e^{i\theta} y\right\|.$$

This debate arises naturally for the issue of comparing the approximate orthogonality relations $\perp_{BJC}^{\varepsilon}$ and $\perp_{BJ}^{\varepsilon}$. In general normed spaces, there are examples which show that neither $\perp_{BJ}^{\varepsilon} \subset \perp_{BJC}^{\varepsilon}$ nor $\perp_{BJC}^{\varepsilon} \subset \perp_{BJ}^{\varepsilon}$ holds when $\varepsilon \in [0,1)$. Consider $\mathcal{X} = \mathbb{C}^2$ with maximum norm. Let $x = (1,0), y = \left(\frac{1}{2}, 1\right)$ and $\varepsilon = \frac{1}{2}$. It is easy to verify that $x \perp_{BJC}^{\varepsilon} y$ but not $x \perp_{BJ}^{\varepsilon} y$.

On the other hand for $x = \left(1, \frac{1}{2}\right)$, $y = (1,0)$ and $\varepsilon = \frac{\sqrt{3}}{2}$, we have $x \perp_{BJ}^{\varepsilon} y$ but not $x \perp_{BJC}^{\varepsilon} y$.

Now, we want to see the relationship between the notions of ε-orthogonality and ε-BJ-orthogonality in an inner product space.

Proposition 2.2. *For any nonzero vectors x and y in an inner product space $\mathcal{X}$ and $\varepsilon \in [0,1)$, there is a positive number $\varepsilon_0 \leq \varepsilon$ such that $x \perp^{\varepsilon} y$ implies that $x \perp_{BJ}^{\varepsilon_0} y$.*

Proof. The relation $x \perp^{\varepsilon} y$ means that $|\langle x, y\rangle| \leq \varepsilon\|x\|\|y\|$. Thus for $\alpha \in [0,1]$ and real number θ, we have $\langle x, y\rangle = \varepsilon\alpha e^{i\theta}\|x\|\|y\|$ that is $\left\langle \frac{x}{\|x\|}, y\right\rangle - \varepsilon\alpha e^{i\theta}\|y\| = 0$. Hence

$$\left\langle \frac{x}{\|x\|}, y\right\rangle - \left\langle \frac{\varepsilon\alpha e^{i\theta} y}{\|y\|}, y\right\rangle = 0,$$

which implies that $\left\langle \frac{x}{\|x\|} - \varepsilon\alpha e^{i\theta}\frac{y}{\|y\|}, y\right\rangle = 0$. Now using Proposition 3.1 of [6] we see that

$$\frac{x}{\|x\|} - \varepsilon\alpha e^{i\theta}\frac{y}{\|y\|} \perp_{BJ} y.$$

$\square$

Proposition 2.3. *If $\mathcal{X}$ is an inner product space and x and y are two non-zero elements of it such that $x \perp_{BJ}^{\varepsilon} y$, then $x \perp_{s}^{\varepsilon} y$.*

Proof. The relation$x \perp_{BJ}^{\varepsilon} y$ means that

$$\frac{x}{\|x\|} + \varepsilon e^{i\theta}\frac{y}{\|y\|} \perp_{BJ} y,$$

for some $\theta \in \mathbb{R}$. Since $\mathcal{X}$ is an inner product space we have

$$\left\langle \frac{x}{\|x\|} + \varepsilon e^{i\theta}\frac{y}{\|y\|}, y\right\rangle = 0.$$

Hence $\langle x, y\rangle = \varepsilon e^{i\theta}\|x\|\|y\|$ which implies that $|\langle x, y\rangle| \leq \varepsilon\|x\|\|y\|$. $\square$

3. $*$-orthogonality and Approximate $*$-orthogonality in C^*-algebras

We start this section with the following theorem which is about BJD-ε-$*$-orthogonality.

Theorem 3.1. *Let a and b be two elements of a C^*-algebra $\mathcal{A}$. If $a \perp^{*\varepsilon}_{BJD} b$ for some $\varepsilon \in [0,1)$, then*

$$|\mathrm{Re}(a^*b)| \leq \sqrt{\varepsilon}\|a\|\|b\|, \qquad |\mathrm{Im}(a^*b)| \leq \sqrt{\varepsilon}\|a\|\|b\|.$$

Especially

$$\|a^*b\| \leq 2\sqrt{\varepsilon}\|a\|\|b\|.$$

Proof. The relation $a \perp^{*\varepsilon}_{BJD} b$ means that

$$|a - \lambda b|^2 \geq (1-\varepsilon)|a|^2,$$

for all complex numbers λ. Thus by a routine computation we have

$$-\lambda a^*b - \overline{\lambda} b^*a + |\lambda|^2|b|^2 \geq -\varepsilon|a|^2. \tag{3.1}$$

For $t > 0$ set $\lambda = t$ and $\lambda = -t$ in (3.1) we see that

$$|a^*b + b^*a| \leq \frac{t^2|b|^2 + \varepsilon|a|^2}{t} \leq \frac{t^2\|b\|^2 + \varepsilon\|a\|^2}{t}. \tag{3.2}$$

Now consider the function $f : \mathbb{R}^+ \to \mathbb{R}^+$ defined by

$$f(t) = \frac{t^2\|b\|^2 + \varepsilon\|a\|^2}{t}.$$

This function attains its minimum at $t = \sqrt{\varepsilon}\frac{\|a\|}{\|b\|}$ which is $2\sqrt{\varepsilon}\|a\|\|b\|$. Therefore from (3.2) we have

$$|a^*b + b^*a| \leq 2\sqrt{\varepsilon}\|a\|\|b\|. \tag{3.3}$$

The following equation can be achieved in the similar method to the above by setting $\lambda = -it$ and $\lambda = it$ for $t > 0$ in (3.1)

$$|a^*b - b^*a| \leq 2\sqrt{\varepsilon}\|a\|\|b\|. \tag{3.4}$$

Thus $|\mathrm{Re}(a^*b)| \leq \sqrt{\varepsilon}\|a\|\|b\|$ and $|\mathrm{Im}(a^*b)| \leq \sqrt{\varepsilon}\|a\|\|b\|$. Now what we want is obtained from these inequalities. □

We prove the following theorem using some ideas from [2].

Theorem 3.2. *Let a and b be two elements of a C^*-algebra $\mathcal{A}$. If $a \perp^{*\varepsilon}_{BJC} b$ for some $\varepsilon \in [0,1)$, then*

$$|\mathrm{Re}(a^*b)| \leq \frac{1}{2}\varepsilon\|a\|\|b\|, \qquad |\mathrm{Im}(a^*b)| \leq \frac{1}{2}\varepsilon\|a\|\|b\|.$$

Especially

$$\|a^*b\| \leq \varepsilon\|b\|\|a\|.$$

Proof. Since $a \perp_{BJC}^{*\varepsilon} b$, we have

$$|a+\lambda b|^2 \geq |a|^2 - \varepsilon|a|\|\lambda b\| \tag{3.5}$$

for all $\lambda \in \mathbb{C}$. A routine computation shows that

$$-\lambda a^*b - \bar{\lambda} b^*a \leq |\lambda|^2|b|^2 + \varepsilon|\lambda||a|\|b\|. \tag{3.6}$$

Let $t > 0$. Taking $\lambda = t$, $\lambda = -t$, $\lambda = \mathrm{i}t$ and $\lambda = -\mathrm{i}t$ and letting $t \to 0$, we reach to the following inequalities respectively

$$\begin{aligned} & a^*b + b^*a \leq \varepsilon|a|\|b\|, \\ & -(a^*b + b^*a) \leq \varepsilon|a|\|b\|, \\ & a^*b - b^*a \leq \varepsilon|a|\|b\|, \\ & -a^*b + b^*a \leq \varepsilon|a|\|b\|. \end{aligned} \tag{3.7}$$

These inequalities yield $|\mathrm{Re}(a^*b)| \leq \frac{1}{2}\varepsilon|a|\|b\|$ and $|\mathrm{Im}(a^*b)| \leq \frac{1}{2}\varepsilon|a|\|b\|$ and these ones give us what we want as well. □

In the following, we assume that the maximum takes place in the first and third inequalities of inequalities (3.7) and we see that the converse of Theorem 3.2 is valid for this spacial case.

Proposition 3.3. *Let a and b be two elements of a C^*-algebra $\mathcal{A}$. If $a^*b = \frac{(1+\mathrm{i})}{2}\varepsilon|a|\|b\|$ for some $\varepsilon \in [0,1)$, then $a \perp_{BJC}^{*\varepsilon} b$.*

Proof. Let $\lambda = \lambda_1 + \mathrm{i}\lambda_2$. Thus

$$\begin{aligned} -\lambda a^*b - \bar{\lambda} b^*a &= -\frac{\varepsilon}{2}|a|\|b\|(\lambda_1 - \lambda_2) \\ &\leq \varepsilon|\lambda||a|\|b\| \\ &\leq |\lambda|^2|b|^2 + \varepsilon|\lambda||a|\|b\|, \end{aligned}$$

which yields (3.5). □

Theorem 3.4. *Let a and b be two non-zero elements of a C^*-algebra $\mathcal{A}$. Then $a \perp_{BJ}^{*\varepsilon} b$ if and only if*

$$\frac{a^*b}{\|a\|} = \varepsilon e^{\mathrm{i}\theta}\frac{|b|^2}{\|b\|}$$

for some $\theta \in \mathbb{R}$. In addition in this case, we have $\varepsilon \leq 1$.

Proof. The relation $a \perp_{BJ}^{*\varepsilon} b$ means that

$$\left|\frac{a}{\|a\|} + \varepsilon e^{\mathrm{i}\theta}\frac{b}{\|b\|} + \lambda b\right|^2 \geq \left|\frac{a}{\|a\|} + \varepsilon e^{\mathrm{i}\theta}\frac{b}{\|b\|}\right|^2,$$

for some $\theta \in \mathbb{R}$ and all $\lambda \in \mathbb{C}$. This happens if and only if

$$\left(\frac{a}{\|a\|} + \varepsilon e^{\mathrm{i}\theta}\frac{b}{\|b\|}\right)^* b = 0,$$

if and only if

$$\frac{a^*b}{\|a\|} = -\varepsilon e^{\mathrm{i}\theta}\frac{b^*b}{\|b\|} = \varepsilon e^{\mathrm{i}(\theta+\pi)}\frac{b^*b}{\|b\|}.$$

Takeing norm from the sides of these equalities, we reach to $\varepsilon\|a\|\|b\| = \|a^*b\| \leq \|a\|\|b\|$ which yields $\varepsilon \leq 1$ as we want. $\square$

Now, we want to state the following theorem concerning the range-kernel orthogonality of generalized inner derivations. It is proved in the similar method of [2].

Theorem 3.5. *Let a and b be two elements in a C^*-algebra $\mathcal{A}$. The following assertions are equivalent.*

(i) $\mathscr{R}(\delta_{a,b}) \perp \mathscr{N}(\delta_{a,b})$.

(ii) $(xa^* - b^*x)t = 0$ *for any $x \in \mathcal{A}$ and any $t \in \mathscr{N}(\delta_{a,b})$.*

Proof. Let $\mathscr{R}(\delta_{a,b}) \perp \mathscr{N}(\delta_{a,b})$. Hence $|ax - xb + t|^2 \geq |t|^2$ for any $x \in \mathcal{A}$ and $t \in \mathscr{N}(\delta_{a,b})$. Thus

$$|\lambda(ax - xb) + t|^2 \geq |t|^2,$$

for any , $t \in \mathscr{N}(\delta_{a,b})$ and $\lambda \in \mathbb{C}$, because $\lambda x \in \mathcal{A}$, whenever $x \in \mathcal{A}$ and $\lambda \in \mathbb{C}$. We may assume that $ax \neq xb$. For any number λ, we have

$$\begin{aligned} |t|^2 &\leq |t + \lambda(ax - xb)|^2 \\ &= (t + \lambda(ax - xb))^*(t + \lambda(ax - xb)) \\ &= |t|^2 + \lambda t^*(ax - xb) + \overline{\lambda}(ax - xb)^*t + |\lambda(ax - xb)|^2, \end{aligned}$$

which implies

$$-\lambda t^*(ax - xb) - \overline{\lambda}(ax - xb)^*t \leq |\lambda|^2|(ax - xb)|^2. \tag{3.8}$$

Let $r > 0$. Setting $\lambda = r, \lambda = -r, \lambda = \mathrm{i}r$ and $\lambda = -\mathrm{i}r$ in (3.8), we reach to the following inequalities respectively;

$$\begin{aligned} -t^*(ax - xb) - (ax - xb)^*t &\leq r|(ax - xb)|^2, \\ t^*(ax - xb) + (ax - xb)^*t &\leq r|(ax - xb)|^2, \\ -\mathrm{i}t^*(ax - xb) + \mathrm{i}(ax - xb)^*t &\leq r|(ax - xb)|^2, \end{aligned}$$

and

$$\mathrm{i}t^*(ax - xb) - \mathrm{i}(ax - xb)^*t \leq r|(ax - xb)|^2.$$

Letting $r \to 0$ in these inequalities, we get the following;

$$-(t^*(ax - xb) + (ax - xb)^*t) \leq 0, \tag{3.9}$$

$$t^*(ax - xb) + (ax - xb)^*t \leq 0, \tag{3.10}$$

$$-\mathrm{i}t^*(ax - xb) + \mathrm{i}(ax - xb)^*t \leq 0 \tag{3.11}$$

$$\mathrm{i}t^*(ax - xb) - \mathrm{i}(ax - xb)^*t \leq 0. \tag{3.12}$$

It is inferred from (3.9) and (3.10) that

$$t^*(ax - xb) = -(ax - xb)^* t, \tag{3.13}$$

and from (3.11) and (3.12) that

$$t^*(ax - xb) = (ax - xb)^* t. \tag{3.14}$$

Thus (ii) is concluded from (3.13) and (3.14).

Now assume that $(xa^* - b^*x)t = 0$ for any $x \in \mathcal{A}$ and any $t \in \mathscr{N}(\delta_{a,b})$. Then

$$|t|^2 \le |t|^2 + |ax - xb|^2 = |t + ax - xb|^2,$$

for any $x \in \mathcal{A}$ and any $t \in \mathscr{N}(\delta_{a,b})$ which is (i). □

Corollary 3.6. *Let u and v be two unitary elements of a unital C^*-algebra $\mathcal{A}$ such that $z_1 + z_2$ not be in $\mathrm{sp}(z_1uv^* + z_2u^*v)$ for some $z_1, z_2 \in \mathbb{C}$. If $\mathscr{N}(\delta_{u,v}) \perp \mathscr{R}(\delta_{u,v})$, then $\mathscr{N}(\delta_{u,v}) = 0$.*

Proof. From our assumption and Theorem 3.5 we have

$$t^*(ux - xv) = 0, \quad (x \in \mathcal{A}, t \in \mathscr{N}(\delta_{u,v})). \tag{3.15}$$

Let z_1 and z_2 be complex numbers such that $z_1 + z_2$ not be in $\mathrm{sp}(z_1uv^* + z_2u^*v)$. Setting $x = z_1v^* - z_2u^*$ in (3.15), we obtain the desired result. □

Corollary 3.7. *Let $\mathcal{A}$ be a unital C^*-algebra with unit 1 and let $a \in \mathcal{A}$. The following assertions are equivalent.*

(i) $\mathscr{N}(\delta_a) \perp \mathscr{R}(\delta_a)$.
(ii) $\mathscr{N}(\delta_a) = \mathcal{A}$.

Proof. (i)⇒(ii) From $\mathscr{N}(\delta_a) \perp \mathscr{R}(\delta_a)$ and Theorem 3.5 we get $t^*ax = t^*xa$ for any $x \in \mathcal{A}$ and $t \in \mathscr{N}(\delta_a)$. Since $1 \in \mathscr{N}(\delta_a) = \mathcal{A}$, we have $xa = ax$ for any $x \in \mathcal{A}$. The converse is obvious. □

Corollary 3.8. *Let $A \in \mathbb{B}(\mathscr{H})$ be arbitrary and $\mathscr{N}(\delta_A) \perp \mathscr{R}(\delta_A)$. Then $A \in \mathbb{C}I$.*

Proof. It follows from Corollary 3.7. □

For elements a and b of a C^*-algebra $\mathcal{A}$, the condition $\mathscr{R}(\delta_{a,b}) \perp \mathscr{N}(\delta_{a,b})$ obviously implies $\mathscr{R}(\delta_{a,b}) \perp_{BJ} \mathscr{N}(\delta_{a,b})$, but the converse is not true in general. However, when $\mathcal{A}$ is a unital, commutative, von Neumann algebra, $\mathscr{R}(\delta_{a,b}) \perp \mathscr{N}(\delta_{a,b})$ and $\mathscr{R}(\delta_{a,b}) \perp_{BJ} \mathscr{N}(\delta_{a,b})$ coincide. In fact, in this case since every element of $\mathcal{A}$ is normal, Anderson's theorem ensures that $\mathscr{R}(\delta_{a,b}) \perp_{BJ} \mathscr{N}(\delta_{a,b})$ for any $a, b \in \mathcal{A}$. On the other hand, let $a, b \in \mathcal{A}$ and t be an element of $\mathcal{A}$ such that $at = tb$. Then by Fuglede–Putnam theorem $t^*a = bt^*$. Hence

$$t^*(ax - xb) = t^*ax - t^*xb = bt^*x - t^*xb = bt^*x - bt^*x = 0, \quad (x \in \mathcal{A}),$$

that is $\mathscr{R}(\delta_{a,b}) \perp \mathscr{N}(\delta_{a,b})$.

In the following, it is shown that the commutative von Neumann algebras are the only unital von Neumann algebras, in which these two types of range-kernel orthogonality coincide.

Proposition 3.9. *Let $\mathcal{A}$ be a unital, von Neumann algebra such that for any $a, b \in \mathcal{A}$, the condition $\mathscr{R}(\delta_{a,b}) \perp_{BJ} \mathscr{N}(\delta_{a,b})$ implies $\mathscr{R}(\delta_{a,b}) \perp \mathscr{N}(\delta_{a,b})$. Then $\mathcal{A}$ is commutative.*

Proof. Let a be a self-adjoint element of $\mathcal{A}$. Then by Anderson's theorem $\mathscr{R}(\delta_a) \perp_{BJ} \mathscr{N}(\delta_a)$ which by our assumption yields $\mathscr{R}(\delta_a) \perp \mathscr{N}(\delta_a)$. Hence by Theorem 3.7 $ax = xa$ for any $x \in \mathcal{A}$. Now, let a be an arbitrary element of $\mathcal{A}$. Since it can be written in the form $a = a_1 + \mathrm{i}a_2$ which a_1 and a_2 are self-adjoint elements of $\mathcal{A}$, we see that a commutes with any element of $\mathcal{A}$. This implies that $\mathcal{A}$ is commutative. □

Corollary 3.10. *Let $\mathcal{A} = \mathbb{B}(\mathscr{H})$ for some Hilbert space $\mathscr{H}$ and let the condition $\mathscr{R}(\delta_{a,b}) \perp_{BJ} \mathscr{N}(\delta_{a,b})$ implies $\mathscr{R}(\delta_{a,b}) \perp \mathscr{N}(\delta_{a,b})$, for any $a, b \in \mathcal{A}$. Then $\mathcal{A} = \mathbb{C}I$.*

Proof. It follows from Corollary 3.8 and the proof of Theorem 3.9. □

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