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Richardson and Chebyshev Iterative Methods by Using G-frames

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ABSTRACT. In this paper, we design some iterative schemes for solving operator equation $Lu = f$, where $L : H \to H$ is a bounded, invertible and self-adjoint operator on a separable Hilbert space H. In this concern, Richardson and Chebyshev iterative methods are two outstanding as well as long-standing ones. They can be implemented in different ways via different concepts.

In this paper, these schemes exploit the almost recently developed notion of g-frames which result in modified convergence rates compared with early computed ones in corresponding classical formulations.

In fact, these convergence rates are formed by the lower and upper bounds of the given g-frame. Therefore, we can determine any convergence rate by considering an appropriate g-frame.

1. Introduction and Preliminaries

In this paper, we will use g-frames to get some approximate solutions for the operator equation

$$Lu = f, \tag{1.1}$$

where $L : H \to H$ is a bounded, invertible and self-adjoint linear operator on a separable Hilbert space H. Technically, numerical linear algebra approaches the problem in different iteration schemes, which in general case produces a sequence of vectors converging to the exact solution of equation (1.1). In the meantime, Richardson iterative method and Chebyshev polynomials are two prominent approaches which can be

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implemented in a different way [1, 6, 11]. In [11, 7], one can see some developments of adaptive numerical methods for solving the problem (1.1) by using frames. G-frames naturally are the most recent generalizations of frames and provide more choices on analyzing functions from frame expansion coefficients. To recall the notion of g-frames, we refer to Sun [12, 13]. Let J be a countable index set and let $\{\Lambda_j\}_{j\in J}$ be a set of bounded operators from a separable Hilbert space H to another separable Hilbert spaces V_j for $j \in J$. The sequence $\{\Lambda_j\}_{j\in J}$ is called a *g-frame* for H with respect to $\{V_j\}_{j\in J}$, if there are two positive constants A and B such that

$$A\|f\|^2 \leq \sum_{j\in J} \|\Lambda_j f\|^2 \leq B\|f\|^2, \quad \forall f \in H. \tag{1.2}$$

The constants A and B are called the lower and upper g-frame bounds, respectively. If $A = B$ then $\{\Lambda_j\}_{j\in J}$ is called A-tight g-frame. The g-frame operator S for a given g-frame $\{\Lambda_j\}_{j\in J}$, for H with respect to $\{V_j\}_{j\in J}$ with bounds A and B, is defined by

$$Sf = \sum_{j\in J} \Lambda_j^* \Lambda_j f, \quad \forall f \in H. \tag{1.3}$$

The g-frame operator S is a bounded, invertible and self-adjoint operator and

$$AI \leq S \leq BI, \qquad \frac{1}{B} I \leq S^{-1} \leq \frac{1}{A} I,$$

and reconstruction formula holds as follows

$$f = \sum_{j\in J} \Lambda_j^* \tilde{\Lambda_j} f = \sum_{j\in J} \tilde{\Lambda_j}^* \Lambda_j f, \quad \forall f \in H, \tag{1.4}$$

where $\tilde{\Lambda_j}$ is called the canonical dual of Λ_j and their lower and upper bounds are $1/A$ and $1/B$, respectively [12]. Also, S^{-1} is the g-frame operator of $\tilde{\Lambda_j}$. For more details, we refer the reader to [5, 12].

2. Using g-frames in Richardson Iterative Method

In this section, by using g-frames, we want to solve the operator equation (1.1) via a Richardson-based iterative method. We review some given results in [8, 9] which are needed for the next studying. For the rest of the manuscript, suppose that $\{\Lambda_j\}_{j\in J}$ is a g-frame for H with respect to $\{V_j\}_{j\in J}$. Firstly, we see the exact solution by using a given g-frame.

Theorem 2.1 ([9])**.** *Let $L : H \to H$ be a bounded and invertible operator and $\{\Lambda_j\}_{j\in J}$ be a g-frame for H. Then $\{\Lambda_j L\}_{j\in J}$ is also a g-frame for H.*

Remark 2.2. If L is a bounded and invertible on H and $\{\Lambda_j\}_{j\in J}$ and $\{\Lambda_j L\}_{j\in J}$ are g-frames with g-frame operators S and S' (respectively), then for all $v \in H$, we have $S' = LSL$ [9].

Now the following theorem gives the exact solution of the equation (1.1) based on a g-frame.

Theorem 2.3 ([8]). *Let u be the solution of the equation (1.1) and let $\{\Lambda_j\}_{j\in J}$ be a g-frame for H with g-frame operator S. Then*

$$u = S'^{-1} LSf,$$

where S' is the g-frame operator of the g-frame $\{\Lambda_j L\}_{j\in J}$.

Note that, by the given assumptions in Theorem 2.3 if $\{\Lambda_j L\}_{j\in J}$ is A-tight g-frame, then $u = \frac{1}{A} LSf$. Therefore, in this case the exact solution u could be obtained only by using the already known operators L and S.

2.1. **Richardson Iterative Method by Using G-frames.** Reformulating the equation (1.1) as a linear fixed-point iteration, regularly, gives rise to an equation helps deriving an iteration sequence which converges to the exact solution of (1.1). To this concern, Richardson iterative method, discussing below, plays an important role in numerical linear algebra. For more details, we refer the reader to [10].

Firstly, we write $u = (I - L)u + f$ in the equation $Lu = f$. Let $u_0 \in H$ be a given element of H, where I is the identity operator on H. We define the sequence $\{u_k\}$ as follows.

$$u_{k+1} := (I - L)u_k + f, \quad \forall k \in \mathbb{N}. \tag{2.1}$$

Since $Lu - f = 0$, we can see

$$\begin{aligned} u_{k+1} - u &= (I - L)u_k + f - u - (f - Lu) \\ &= (I - L)u_k - u + Lu \\ &= (I - L)(u_k - u). \end{aligned}$$

Thus

$$\| u_{k+1} - u \|_H \leq \| I - L \| \| u_k - u \|_H .$$

So we can easily see that the sequence (2.1) converges if

$$\| I - L \| < 1. \tag{2.2}$$

To improve the convergence rate of the sequence (2.1), we might multiply both sides of equation (1.1) in a matrix M,

$$MLu = Mf. \tag{2.3}$$

This is a common and effective technique for solving differential equations, integral equations, and related problems [2, 3]. For this purpose, we usually find a matrix M which approximates L^{-1} i.e. $M \approx L^{-1}$ or $ML \approx I$ [10], which in this case, (2.2) would be satisfied much more as well i.e. $\| I - ML \| \ll 1$. Here, we want to seek M by using g-frames. First of all, we consider the following theorem.

Theorem 2.4. *Let $\{\Lambda_j\}_{j\in J}$ be a g-frame with g-frame operator S, and let A and B be the g-frame bounds of the g-frame $\{\Lambda_j L\}_{j\in J}$. Then*

$$\left\| I - \frac{2}{A+B} LSL \right\| \leq \frac{B-A}{B+A}. \tag{2.4}$$

Proof. For all $v \in H$, we have

$$\begin{aligned}
\left\langle \left(I - \frac{2}{A+B} LSL \right) v, v \right\rangle &= \|v\|^2 - \frac{2}{A+B} \langle LSLv, v\rangle \\
&= \|v\|^2 - \frac{2}{A+B} \langle Sv, v\rangle \\
&\leq \|v\|^2 - \frac{2}{A+B} A\|v\|^2 \\
&= \|v\|^2 \left(1 - \frac{2A}{A+B} \right) \\
&= \|v\|^2 \left(\frac{B-A}{B+A} \right).
\end{aligned}$$

Similarly, we have for $v \in H$,

$$\begin{aligned}
\left\langle \left(I - \frac{2}{A+B} LSL \right) v, v \right\rangle &= \|v\|^2 - \frac{2}{A+B} \langle LSLv, v\rangle \\
&= \|v\|^2 - \frac{2}{A+B} \langle Sv, v\rangle \\
&\geq \|v\|^2 - \frac{2}{A+B} B\|v\|^2 \\
&= \|v\|^2 \left(1 - \frac{2B}{A+B} \right) \\
&= \|v\|^2 \left(\frac{A-B}{A+B} \right),
\end{aligned}$$

which together with previous relation yields the result. □

By preceding theorem and the fact that $\frac{B-A}{A+B} < 1$, we can put $M := \frac{2}{A+B} LS$ to precondition (1.1). Thus, by performing the Richardson iteration (2.1) on preconditioned linear equation (2.3), we therefore arrive at the following theorem.

Theorem 2.5. *Let $\{\Lambda_j\}_{j\in J}$ be a g-frame with g-frame operator S, and let A and B be g-frame bounds of the g-frame $\{\Lambda_j L\}_{j\in J}$. Put $u_0 = 0$ and for $k \geq 1$,*

$$u_k = u_{k-1} + \frac{2}{A+B} LS(f - Lu_{k-1}),$$

then

$$\|u - u_k\| \leq \left(\frac{B-A}{B+A}\right)^k \|u\|.$$

In particular, the vectors u_k converge to u as $k \to \infty$.

Proof. We have

$$\begin{aligned} u - u_k &= u - u_{k-1} - \frac{2}{A+B} LSL(u - u_{k-1}) \\ &= \left(I - \frac{2}{A+B} LSL\right)(u - u_{k-1}) \\ &= \left(I - \frac{2}{A+B} LSL\right)\left(u - u_{k-2} - \frac{2}{A+B} LSL\right)(u - u_{k-2}) \\ &= \left(I - \frac{2}{A+B} LSL\right)^2 (u - u_{k-2}) \\ &\ \ \vdots \\ &= \left(I - \frac{2}{A+B} LSL\right)^k (u - u_0), \end{aligned}$$

thus

$$\|u - u_k\| \leq \left\|I - \frac{2}{A+B} LSL\right\|^k \|u\|. \tag{2.5}$$

By inequalities (2.4) and (2.5), we obtain

$$\|u - u_k\| \leq \left(\frac{B-A}{B+A}\right)^k \|u\|,$$

as desired. □

The convergence rate obtained by this approach is directly computed via the bounds of the primary g-frame. Clearly, the optimum case (exact solution) occurs when A-tight g-frame is applied as discussed after Theorem 2.1.

In the sequel, concerning the preceding theorem we establish an algorithm in which an approximate solution with prescribed accuracy of the equation (1.1) is obtained. To do this, let $\{\Lambda_j\}_{j\in J}$ be a g-frame for H with g-frame operator S and let A and B be bounds of the g-frame $\{\Lambda_j L\}_{j\in J}$.

Algorithm 1. $[A, B, \epsilon] \to u_\epsilon$

(i) Let $\alpha_0 = \frac{B-A}{A+B}$
(ii) $k := 0,\ u_k := 0$
(iii) $k := k + 1$
  (1) $u_k = u_{k-1} + \frac{2}{A+B} LS\left(f - Lu_{k-1}\right)$
  (2) $\alpha_k := (\alpha_0)^k \, \|f\|$
(iv) If $\alpha_k \leq \epsilon$ stop and set $u_\epsilon := u_k$, if else go to (iii).

3. Chebyshev Method by Using g-frames

Before introducing our next method, we want to state some basic facts about Chebyshev polynomials. These polynomials are defined by

(3.1)
$$c_n(x) = \begin{cases} \cos(ncos^{-1}(x)), & |x| \leq 1; \\ \cosh(n\cosh^{-1}(x)) = \frac{1}{2}\left((x + \sqrt{x^2-1})^n + (x + \sqrt{x^2-1})^{-n}\right), & |x| > 1, \end{cases}$$

and are satisfied the following recurrence relation.

$$c_0(x) = 1, \qquad c_1(x) = x, \qquad c_n(x) = 2xc_{n-1}(x) - c_{n-2}(x), \quad \forall n \geq 2.$$

In the first place, the following lemma is illustrated.

Lemma 3.1 ([4]). *Given any constants $a \leq b \leq 1$, set*

$$P_n(x) = \frac{c_n\left(\frac{2x-a-b}{b-a}\right)}{c_n\left(\frac{2-a-b}{b-a}\right)}$$

for $x \in [a, b]$, then

$$\max_{a\leq x\leq b} \mid P_n(x) \mid \leq \max_{a\leq x\leq b} \mid Q_n(x) \mid,$$

for all polynomials of degree n, Q_n, with the condition $Q_n(1) = 1$. Furthermore

$$\max_{a\leq x\leq b} \mid P_n(x) \mid = \frac{1}{c_n\left(\frac{2-a-b}{b-a}\right)}.$$

Now, let

$$h_n = \sum_{k=1}^{n} a_{n_k} u_k,$$

be a polynomial such that

$$\sum_{k=1}^{n} a_{n_k} = 1,$$

where $\{u_k\}_{k\in\mathbb{N}}$ is the approximate solution induced by the iterative method presented in Theorem 2.5. Similar to the proof of Theorem 2.5, we have

$$\begin{aligned} u - h_n &= \sum_{k=1}^{n} a_{n_k} u - \sum_{k=1}^{n} a_{n_k} u_k \\ &= \sum_{k=1}^{n} a_{n_k} (u - u_k) \\ &= \sum_{k=1}^{n} a_{n_k} \left(I - \frac{2}{A+B} LSL \right)^k (u - u_0). \end{aligned}$$

Set $R = I - \frac{2}{A+B} LSL$ and

$$Q_n(x) = \sum_{k=1}^{n} a_{n_k} x^k,$$

we obtain

$$u - h_n = \sum_{k=1}^{n} a_{n_k} R^k (u - u_0) = Q_n(R)(u - u_0), \tag{3.2}$$

which means that the error $u - h_n$ is a polynomial in R applied to the initial error $u - u_0$.

By the given inequalities in the proof of Theorem 2.5, for $v \in H$, one obtains

$$-\frac{B-A}{B+A} \|v\|^2 \leq \left\langle \left(I - \frac{2}{A+B} LSL \right) v, v \right\rangle \leq \frac{B-A}{B+A} \|v\|^2.$$

It concludes that the spectrum of R is a subset of the interval $[-\rho, \rho]$ where $\rho = \frac{B-A}{B+A}$. Since L is an invertible and self-adjoint operator and also S is positive definite, LSL would be a positive definite operator. Thus, the spectral theorem yields

$$\|u - h_n\| \leq \|Q_n(R)\| \|u - u_0\| \leq \max_{|x| \leq \rho} \mid Q_n(x) \mid \|u - u_0\|. \tag{3.3}$$

In order to minimize the error $\|u - h_n\|$, we consider the following minimization problem

$$\min_{Q_n(1)=1} \max_{|x| \leq \rho} \mid Q_n(x) \mid, \tag{3.4}$$

where the minimum is taken over all polynomials of degree less than or equal to n, with the property $Q_n(1) = 1$. By Lemma 3.1, this problem can be solved in terms of the Chebyshev polynomials.

Firstly, we set $a = -\frac{B-A}{B+A}$ and $b = \frac{B-A}{B+A}$ in Lemma 3.1, and obtain the following polynomial

$$P_n(x) = \frac{c_n\left(\frac{2x+\frac{B-A}{B+A}-\frac{B-A}{B+A}}{\frac{B-A}{B+A}+\frac{B-A}{B+A}}\right)}{c_n\left(\frac{2+\frac{B-A}{B+A}-\frac{B-A}{B+A}}{\frac{B-A}{B+A}+\frac{B-A}{B+A}}\right)} = \frac{c_n\left(\frac{x}{\rho}\right)}{c_n\left(\frac{1}{\rho}\right)}, \tag{3.5}$$

which solves (3.4). By (3.5) and the definition of $c_n(x)$, for $n \geq 2$, we have

$$\begin{aligned} c_n\left(\frac{1}{\rho}\right)P_n(x) &= c_n\left(\frac{x}{\rho}\right) \\ &= \frac{2x}{\rho}c_{n-1}\left(\frac{x}{\rho}\right) - c_{n-2}\left(\frac{x}{\rho}\right) \\ &= \frac{2x}{\rho}c_{n-1}\left(\frac{1}{\rho}\right)P_{n-1}(x) - c_{n-2}\left(\frac{1}{\rho}\right)P_{n-2}(x). \end{aligned}$$

Now, if x is replaced by R, and applying the resulting operator identity to $(u - u_0)$, we get

$$\begin{aligned} &c_n\left(\frac{1}{\rho}\right)P_n(R)(u-u_0) = \\ &\qquad \frac{2R}{\rho}c_{n-1}\left(\frac{1}{\rho}\right)P_{n-1}(u-u_0) - c_{n-2}\left(\frac{1}{\rho}\right)P_{n-2}(R)(u-u_0), \end{aligned}$$

and since $P_n(x)$ is the solution of minimization problem (3.4), by virtue of (3.2) one implicates

$$c_n\left(\frac{1}{\rho}\right)(u-h_n)\frac{2}{\rho}c_{n-1}\left(\frac{1}{\rho}\right)R(u-h_{n-1}) - c_{n-2}\left(\frac{1}{\rho}\right)(u-h_{n-2}).$$

By $R = I - \frac{2}{A+B}LSL$, we can see

$$\begin{aligned} &c_n\left(\frac{1}{\rho}\right)(u-h_n) = \\ &\quad \frac{2}{\rho}c_{n-1}\left(\frac{1}{\rho}\right)\left(I - \frac{2}{A+B}LSL\right)(u-h_{n-1}) - c_{n-2}\left(\frac{1}{\rho}\right)(u-h_{n-2}), \end{aligned}$$

and

$$\begin{aligned} c_n\left(\frac{1}{\rho}\right)u - c_n\left(\frac{1}{\rho}\right)h_n &= \frac{2}{\rho}c_{n-1}\left(\frac{1}{\rho}\right)u \\ &\quad + \frac{2}{\rho}c_{n-1}\left(\frac{1}{\rho}\right)\left(-h_{n-1} - \frac{2}{A+B}LSL(u-h_{n-1})\right) \\ &\quad - c_{n-2}\left(\frac{1}{\rho}\right)u + c_{n-2}\left(\frac{1}{\rho}\right)h_{n-2}, \end{aligned}$$

now, by the definition of c_n, for $n \geq 2$,

$$c_n\left(\frac{1}{\rho}\right)h_n = \frac{2}{\rho}c_{n-1}\left(\frac{1}{\rho}\right)\left(h_{n-1} + \frac{2}{A+B}LSL(u-h_{n-1})\right) - c_{n-2}\left(\frac{1}{\rho}\right)h_{n-2},$$

or equivalently,

$$h_n = \frac{2}{\rho}\frac{c_{n-1}(\frac{1}{\rho})}{c_n(\frac{1}{\rho})}\left(h_{n-1} + \frac{2}{A+B}LSL(u-h_{n-1})\right) - \frac{c_{n-2}(\frac{1}{\rho})}{c_n(\frac{1}{\rho})}h_{n-2}.$$

Now by setting

$$\beta_n = \frac{2}{\rho}\frac{c_{n-1}(\frac{1}{\rho})}{c_n(\frac{1}{\rho})},$$

we observe that

$$1-\beta_n = -\frac{c_{n-2}(\frac{1}{\rho})}{c_n(\frac{1}{\rho})},$$

and finally it concludes that the following recurrence formula holds

$$h_n = \frac{2}{\rho}\beta_n\left(h_{n-1} + \frac{2}{A+B}LSL(u-h_{n-1})\right) + (1-\beta_n)h_{n-2},$$

which connects three successive h_n polynomials. We note also here that, by the definition of c_n, we see

$$\begin{aligned}\beta_n &= \left(\frac{\rho}{2}\frac{c_n(\frac{1}{\rho})}{c_{n-1}(\frac{1}{\rho})}\right)^{-1} \\ &= \left(\frac{\rho}{2}\frac{\frac{2}{\rho}c_{n-1}(\frac{1}{\rho}) - c_{n-2}(\frac{1}{\rho})}{c_{n-1}(\frac{1}{\rho})}\right)^{-1} \\ &= \left(1-\frac{\rho^2}{4}\beta_{n-1}\right)^{-1}.\end{aligned}$$

Therefore, an approximate solution can be written based on the above argument and Chebyshev polynomials. For this, let $\{\Lambda_j\}_{j\in J}$ be a g-frame for H with g-frame operator S and let A and B be the bounds of the g-frame $\{\Lambda_j L\}_{j\in J}$.

Algorithm 2. $[A, B, \epsilon] \to u_\epsilon$

(i) put $\rho = \frac{B-A}{B+A}$, $\sigma = \frac{\sqrt{B}+\sqrt{A}}{\sqrt{B}-\sqrt{A}}$ set $h_0 = 0$, $h_1 = \frac{2}{A+B}LSf$, $\beta_1 = 2$, $n = 1$

(ii) while $\frac{2\sigma^n}{1+\sigma^{2n}}\frac{\|f\|}{m} > \epsilon$

(1) $n = n+1$

(2) $\beta_n = (1 - \frac{\rho^2}{4}\beta_{n-1})^{-1}$
(3) $h_n = \frac{2}{\rho}\beta_n(h_{n-1} + \frac{2}{A+B}LS(f - Lh_{n-1})) + (1-\beta_n)h_{n-2}$
(iii) $u_\epsilon := h_n$.

The following theorem verifies the convergence of this algorithm.

Theorem 3.2. *If u is the exact solution of the equation (1.1), then the approximate solution h_n satisfies*

$$\|u - h_n\| \le \frac{2\sigma^n}{1+\sigma^{2n}}\frac{\|f\|}{m}.$$

Consequently, the output u_ϵ of Algorithm2 satisfies

$$\|u - u_\epsilon\| < \epsilon.$$

Proof. By Lemma 3.1 and relation (3.3), one sees for $n \in \mathbb{N}$,

$$\begin{aligned}
\|u - h_n\| &\le \frac{1}{c_n(\frac{2+\rho-\rho}{\rho+\rho})} \\
&= \frac{1}{c_n(\frac{1}{\rho})} \\
&= \frac{1}{c_n(\frac{B+A}{B-A})} \\
&= \frac{1}{\frac{1}{2}[(\frac{B+A}{B-A} + \sqrt{(\frac{B+A}{B-A})^2 - 1})^n + (\frac{B+A}{B-A} + \sqrt{(\frac{B+A}{B-A})^2 - 1})^{-n}]} \\
&= \frac{1}{\frac{\sigma^{2n}+1}{2\sigma^n}} \\
&= \frac{2\sigma^n}{1+\sigma^{2n}}.
\end{aligned}$$

Hence,

$$\|u - h_n\| \le \frac{2\sigma^n}{1+\sigma^{2n}}\|u\| \le \frac{2\sigma^n}{1+\sigma^{2n}}\frac{\|Lu\|}{m} = \frac{2\sigma^n}{1+\sigma^{2n}}\frac{\|f\|}{m}.$$

□

Remark 3.3. It is obvious that for every $n > 1$, $\frac{2\sigma^n}{1+\sigma^{2n}} \le \rho^n$. Therefore, this algorithm presents an iterative method that its convergence rate is faster than that of Richardson iterative method which is presented in Theorem 2.5.

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