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C-class Functions and Common Fixed Point Theorems Satisfying φ-weakly Contractive Conditions

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Abstract. In this paper, we discuss and extend some recent common fixed point results established by using $\varphi-$weakly contractive mappings. A very important step in the development of the fixed point theory was given by A.H. Ansari by the introduction of a $C-$class function. Using $C-$class functions, we generalize some known fixed point results. This type of functions is a very important class of functions which contains almost all known type contraction starting from 1922. year, respectively from famous Banach contraction principle. Three common fixed point theorems for four mappings are presented. The obtained results generalizes several existing ones in literature.We finally propose three open problems.

1. Introduction and Preliminaries

Start of development of the theory of fixed points is tied to the end of the 19th century. Method of successive approximations is used in order to prove the existence and uniqueness of the solution, at the beginning in differential and integral equations. This branch of nonlinear analysis has been developed through various classes of spaces, such as topological spaces, metric spaces, probabilistic metric spaces, fuzzy metric spaces and the others. Achievements in the development of the theory of fixed points are applied in various sciences, such as optimization, economics and approximation theory.

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One of the most important and most applicable results in the theory of fixed points is certainly the Banach contraction principle. The abovementioned principle motivates scientists around the world to prove different generalizations, both in metric spaces, and also, in the spaces which represent a generalization of metric spaces. Nice generalization is given in the Rhoades paper [25], where the notion of φ-weakly contractive mappings is introduced (for more details see: [1–3, 5–11, 13–17, 19–24]).

Theorem 1.1 ([25]). *Let (X, d) be a complete metric space, and let $T : X \to X$ be a mapping such that*

$$d(Tx, Ty) \leq d(x, y) - \varphi(d(x, y)), \quad \text{for all } x, y \in X,$$

where $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ is continuous and nondecreasing, and $\varphi(t) = 0$ if and only if $t = 0$. Then T has a unique fixed point.

In the literature, there are many generalizations of Rhoades results (for more details see [1, 2, 5–10, 13, 15, 16, 19–23]).

Motivated by these results, we consider the following contraction conditions for the four self-mappings A, B, S and T defined on the metric space (X, d) :

$$d(Tx, Sy) \leq \varphi_2 \left(M_i(x, y)\right), \quad \text{for all } x, y \in X, \tag{1.1}$$

$$\varphi_1 \left(d(Tx, Sy)\right) \leq \varphi_1 \left(M_i(x, y)\right) - \varphi_2 \left(M_i(x, y)\right), \quad \text{for all } x, y \in X, \tag{1.2}$$

where $i \in \{1, 2, 3\}$, $\varphi_1 \in \Phi_1$, $\varphi_2 \in \Phi_2$,

(1.3)

$$\begin{aligned}M_1(x, y) = \max\Big\{ & d(Ax, By), d(Ax, Tx), d(By, Sy), \\ & \frac{1}{2}\left[d(Ax, Sy) + d(Tx, By)\right], \frac{d(Ax, Sy)d(Tx, By)}{1 + d(Ax, By)}, \\ & \frac{d(Ax, Tx)d(By, Sy)}{1 + d(Ax, By)}, \frac{1 + d(Ax, Sy) + d(Tx, By)}{1 + d(Ax, Tx) + d(By, Sy)} \\ & \times d(Ax, Tx)\Big\},\end{aligned}$$

(1.4)

$$\begin{aligned}M_2(x, y) = \max\Big\{ & d(Ax, By), d(Ax, Tx), d(By, Sy), \\ & \frac{1}{2}\left[d(Ax, Sy) + d(Tx, By)\right], \frac{1 + d(Ax, Tx)}{1 + d(Ax, By)} d(By, Sy), \\ & \frac{1 + d(By, Sy)}{1 + d(Ax, By)} d(Ax, Tx),\end{aligned}$$

$$\left.\frac{1+d(Ax,Sy)+d(Tx,By)}{1+d(Ax,Tx)+d(By,Sy)}d(By,Sy)\right\},$$

and

$$M_3(x,y)=\max\left\{d(Ax,By),d(Ax,Tx),d(By,Sy),\right.$$
$$\left.\frac{1}{2}\left[d(Ax,Sy)+d(Tx,By)\right]\right\}, \tag{1.5}$$

while,

$$\Phi_1=\{\varphi:\mathbb{R}^+\to\mathbb{R}^+\colon \varphi \text{ is continuous and nondecreasing and } \varphi(t)=0 \text{ if and only if } t=0\},$$

and

$$\Phi_2=\{\varphi:\mathbb{R}^+\to\mathbb{R}^+\colon \varphi \text{ is lower semi-continuous and } \varphi(t)=0 \text{ if and only if } t=0\}.$$

Motivated by already some known results, our contribution in this paper is a generalization of the contractive conditions given in the paper by Liu et.al. [15], by introduction of the C-class functions.

In 2014, the concept of C-class functions (see Definition 1.2) was introduced by A.H. Ansari. This is a very important class of functions which contains almost all known types of contractions, starting from 1922, i.e. of Banach contractions.

Definition 1.2 ([18])**.** A mapping $H:[0,\infty)^2\to\mathbb{R}$ is called a C-*class* function if it is continuous and satisfies the following axioms, for all $u,v\in[0,\infty)$:

(h1) $H(u,v)\le u$;
(h2) $H(u,v)=u$ implies that either $u=0$ or $v=0$.

We denote C-class functions as $\mathcal{C}$.

Example 1.3 ([18])**.** The following functions $H:[0,\infty)^2\to\mathbb{R}$ are elements of the class $\mathcal{C}$:

- $H(u,v)=u-v$;
- $H(u,v)=\lambda u$, $0<\lambda<1$;
- $H(u,v)=u\beta(u)$, $\beta:[0,\infty)\to[0,1)$;
- $H(u,v)=u-\varphi(u)$, where $\varphi:[0,\infty)\to[0,\infty)$ is a continuous function such that $\varphi(u)=0$ if and only if $u=0$.

Below, the things that we need for the proof of our main results are listed.

Definition 1.4 ([12, 14])**.** A pair of self mappings f and g in a nonempty set X is weakly compatible if $f(g(t))=g(f(t))$ whenever $f(t)=g(t)$.

Lemma 1.5 ([11, 23])**.** *Let (X,d) be a metric space and let $\{y_n\}$ be a sequence in X such that $d(y_n,y_{n+1})$ is nonincreasing and $d(y_n,y_{n+1})\to 0$ as $n\to\infty$. If $\{y_{2n}\}$ is not a Cauchy sequence, then there exist $\varepsilon>0$ and sequences $\{m_k\}$ and $\{n_k\}$ of positive integers such that the following sequences tend to ε when $k\to\infty$:*

$$d\left(y_{2m_k},y_{2n_k}\right),d\left(y_{2m_k},y_{2n_k+1}\right),d\left(y_{2m_k-1},y_{2n_k}\right),d\left(y_{2m_k-1},y_{2n_k+1}\right),\\ d\left(y_{2m_k+1},y_{2n_k+1}\right),\ldots.$$

2. Common Fixed Point Results

Our main results are as follows.

Theorem 2.1. *Let A,B,S and T be self mappings in a metric space $(X,\,d)$ such that*

$$\{A,T\} \text{ and } \{B,S\} \text{ are weakly compatible;} \tag{2.1}$$

$$T(X)\subseteq B(X) \text{ and } S(X)\subseteq A(X); \tag{2.2}$$

$$\text{one of } A(X),B(X),S(X) \text{ and } T(X) \text{ is complete}; \tag{2.3}$$

$$d(Tx,Sy)\le H(M_1(x,y),\varphi(M_1(x,y))),\quad \text{for all } x,y\in X, \tag{2.4}$$

where $H\in\mathcal{C}$, $\varphi\in\Phi_1$ and M_1 is defined by (1.3). Then A,B,S and T have a unique common fixed point in X.

Proof. Let $x_0\in X$. Using (2.2), we conclude that there exist two sequences $\{y_n\}_{n\in\mathbb{N}}$ and $\{x_n\}_{n\in\mathbb{N}_0}$ in X such that

$$y_{2n+1}:=Bx_{2n+1}=Tx_{2n},\quad y_{2n+2}:=Ax_{2n+2}=Sx_{2n+1},\quad \text{for all } n\in\mathbb{N}. \tag{2.5}$$

If $y_k=y_{k+1}$ for some $k\in\mathbb{N}$, then it is not difficult to obtain that $\{y_n\}$ is a Cauchy sequence. Therefore, let $\delta_n=d(y_n,y_{n+1})>0$ for all $n\in\mathbb{N}$.

We will show that δ_n is nonincreasing as well as that

$$\lim_{n\to\infty}\delta_n=0. \tag{2.6}$$

Using (2.4) and (2.5), we get that

$$\delta_{2n}=d(Tx_{2n},Sx_{2n-1})\le H(M_1(x_{2n},x_{2n-1}),\varphi(M_1(x_{2n},x_{2n-1})), \tag{2.7}$$

for all $n\in\mathbb{N}$, where

$$M_1(x_{2n},x_{2n-1})=\max\Big\{d\left(Ax_{2n},Bx_{2n-1}\right),d\left(Ax_{2n},Tx_{2n}\right),\\ d\left(Bx_{2n-1},Sx_{2n-1}\right),\\ \frac{1}{2}\left[d\left(Ax_{2n},Sx_{2n-1}\right)+d\left(Tx_{2n},Bx_{2n-1}\right)\right],$$

$$
\begin{aligned}
&\qquad \frac{d(Ax_{2n}, Sx_{2n-1})d(Tx_{2n}, Bx_{2n-1})}{1 + d(Ax_{2n}, Bx_{2n-1})},\\
&\qquad \frac{d(Ax_{2n}, Tx_{2n})d(Bx_{2n-1}, Sx_{2n-1})}{1 + d(Ax_{2n}, Bx_{2n-1})},\\
&\qquad \frac{1 + d(Ax_{2n}, Sx_{2n-1}) + d(Tx_{2n}, Bx_{2n-1})}{1 + d(Ax_{2n}, Tx_{2n}) + d(Bx_{2n-1}, Sx_{2n-1})}\\
&\qquad \times d(Ax_{2n}, Tx_{2n})\Big\}\\
&= \max\Big\{d(y_{2n}, y_{2n-1}), d(y_{2n}, y_{2n+1}), d(y_{2n-1}, y_{2n}),\\
&\qquad \frac{1}{2}[d(y_{2n}, y_{2n}) + d(y_{2n+1}, y_{2n-1})],\\
&\qquad \frac{d(y_{2n}, y_{2n})d(y_{2n+1}, y_{2n-1})}{1 + d(y_{2n}, y_{2n-1})},\\
&\qquad \frac{d(y_{2n}, y_{2n+1})d(y_{2n-1}, y_{2n})}{1 + d(y_{2n}, y_{2n-1})},\\
&\qquad \frac{1 + d(y_{2n}, y_{2n}) + d(y_{2n+1}, y_{2n-1})}{1 + d(y_{2n}, y_{2n+1}) + d(y_{2n-1}, y_{2n})} d(y_{2n}, y_{2n+1})\Big\}\\
&= \max\Big\{\delta_{2n-1}, \delta_{2n}, \delta_{2n-1}, \frac{1}{2}d(y_{2n+1}, y_{2n-1}), 0, \frac{\delta_{2n}\delta_{2n-1}}{1 + \delta_{2n-1}},\\
&\qquad \frac{1 + d(y_{2n+1}, y_{2n-1})}{1 + \delta_{2n} + \delta_{2n-1}}\delta_{2n}\Big\}\\
&= \max\{\delta_{2n-1}, \delta_{2n}\}, \quad \text{for all } n \in \mathbb{N}.
\end{aligned}
$$

Suppose now that $\delta_{2m-1} < \delta_{2m}$ for some $m \in \mathbb{N}$. Using (2.7), $\varphi \in \Phi_1$ and by properties of H, we conclude that

$$\delta_{2m} = H\left(\delta_{2m}, \varphi(\delta_{2m})\right).$$

Hence it follows that $\delta_{2m} = 0$, and this leads to a contradiction. Hence

$$\delta_{2n} \leq \delta_{2n-1} = M_1(x_{2n}, x_{2n-1}), \quad \text{for all } n \in \mathbb{N}.$$

Similarly we conclude that

$$\delta_{2n+1} \leq \delta_{2n} = M_1(x_{2n}, x_{2n+1}), \quad \text{for all } n \in \mathbb{N}, \tag{2.8}$$

which together with (2.8) ensures

$$\delta_{n+1} \leq \delta_n, \quad \text{for all } n \in \mathbb{N}.$$

This means that the sequence $\{\delta_n\}_{n\in\mathbb{N}}$ is nonincreasing and bounded. Consequently there exists $r \geq 0$ with $\lim_{n\to\infty} \delta_n = r \geq 0$. It follows from

(2.7) and (2.8) that

$$\begin{aligned} r &= \lim_{n\to\infty} \delta_{2n} \\ &\leq \lim_{n\to\infty} H\left(M_1\left(x_{2n}, x_{2n-1}\right), \varphi\left(M_1(x_{2n}, x_{2n-1})\right)\right) \\ &= H\left(\lim_{n\to\infty} \delta_{2n-1}, \lim_{n\to\infty} \varphi(\delta_{2n-1})\right) \\ &= H\left(r, \varphi(r)\right). \end{aligned}$$

Therefore, $r = 0$ or $\varphi(r) = 0$. Hence $r = 0$ and so (2.6) holds.

Next we will prove that $\{y_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence. Because of (2.6), it is sufficient to verify that $\{y_{2n}\}_{n\in\mathbb{N}}$ is a Cauchy sequence. Suppose that $\{y_{2n}\}_{n\in\mathbb{N}}$ is not a Cauchy sequence. Then, by Lemma 1.5 there exist $\varepsilon > 0$ and two subsequences $\{y_{2m(k)}\}_{k\in\mathbb{N}}$ and $\{y_{2n(k)}\}_{k\in\mathbb{N}}$ of $\{y_{2n}\}_{n\in\mathbb{N}}$, with $2n(k) > 2m(k) > 2k$ such that $d\left(y_{2m(k)}, y_{2n(k)}\right) \geq \varepsilon$, $d\left(y_{2m(k)}, y_{2n(k)-2}\right) < \varepsilon$, and the following four sequences

(2.9)
$$d\left(y_{2n(k)}, y_{2m(k)}\right), d(y_{2m(k)}, y_{2n(k)-1}), d(y_{2m(k)+1}, y_{2n(k)}), (y_{2m(k)+1}, y_{2n(k)-1}),$$

tend to ε, when $k \to \infty$.

Note that (1.3) and (2.9) yield

(2.10)
$$\begin{aligned} M_1(x_{2m(k)}, x_{2n(k)-1}) = \max\Big\{ & d(Ax_{2m(k)}, Bx_{2n(k)-1}), d(Ax_{2m(k)}, Tx_{2m(k)}), \\ & d(Bx_{2n(k)-1}, Sx_{2n(k)-1}), \\ & \frac{1}{2}[d(Ax_{2m(k)}, Sx_{2n(k)-1}) \\ & + d(Tx_{2m(k)}, Bx_{2n(k)-1})], \\ & \frac{d(Ax_{2m(k)}, Sx_{2n(k)-1}) d(Tx_{2m(k)}, Bx_{2n(k)-1})}{1 + d(Ax_{2m(k)}, Bx_{2n(k)-1})}, \\ & \frac{d(Ax_{2m(k)}, Tx_{2m(k)}) d(Bx_{2n(k)-1}, Sx_{2n(k)-1})}{1 + d(Ax_{2m(k)}, Bx_{2n(k)-1})}, \\ & \frac{1 + d(Ax_{2m(k)}, Sx_{2n(k)-1}) + d(Tx_{2m(k)}, Bx_{2n(k)-1})}{1 + d(Ax_{2m(k)}, Tx_{2m(k)}) + d(Bx_{2n(k)-1}, Sx_{2n(k)-1})} \\ & \times d(Ax_{2m(k)}, Tx_{2m(k)})\Big\} \\ = \max\Big\{ & d(y_{2m(k)}, y_{2n(k)-1}), d(y_{2m(k)}, y_{2m(k)+1}), \\ & d(y_{2n(k)-1}, y_{2n(k)}), \frac{1}{2}[d(y_{2m(k)}, y_{2n(k)}) \\ & + d(y_{2m(k)+1}, y_{2n(k)-1})], \end{aligned}$$

$$\frac{d(y_{2m(k)},y_{2n(k)})d(y_{2m(k)+1},y_{2n(k)-1})}{1+d(y_{2m(k)},y_{2n(k)-1})},$$
$$\frac{d(y_{2m(k)},y_{2m(k)+1})d(y_{2n(k)-1},y_{2n(k)})}{1+d(y_{2m(k)},y_{2n(k)-1})},$$
$$\frac{1+d(y_{2m(k)},y_{2n(k)})+d(y_{2m(k)+1},y_{2n(k)-1})}{1+d(y_{2m(k)},y_{2m(k)+1})+d(y_{2n(k)-1},y_{2n(k)})}$$
$$\left.\times\, d(y_{2m(k)},y_{2m(k)+1})\right\}$$
$$\to \quad \max\left\{\varepsilon,0,0,\frac{1}{2}(\varepsilon+\varepsilon),\frac{\varepsilon^2}{1+\varepsilon},0,0\right\}=\varepsilon \text{ as } k\to\infty.$$

Then,

$$d\left(Tx_{2m_k},Sx_{2n_k-1}\right)=d\left(y_{2m_k+1},y_{2n_k}\right)\to\varepsilon,$$

and therefore

$$\varepsilon\le H\left(\varepsilon,\varphi\left(\varepsilon\right)\right)\le\varepsilon,$$

so $H\left(\varepsilon,\varphi\left(\varepsilon\right)\right)=\varepsilon$ i.e., $\varepsilon=0$ or $\varphi\left(\varepsilon\right)=0$.

In both cases this is a contradiction. Hence $\{y_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence.

Assume that $A(X)$ is complete. Observe that $\{y_{2n}\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $A(X)$. Consequently there exists $(z,v)\in A(X)\times X$ with $\lim_{n\to\infty}y_{2n}=z=Av$. It is easy to see

(2.11)
$$z=\lim_{n\to\infty}y_n=\lim_{n\to\infty}Tx_{2n}=\lim_{n\to\infty}Bx_{2n+1}=\lim_{n\to\infty}Sx_{2n-1}=\lim_{n\to\infty}Ax_{2n}.$$

Suppose that $Tv\neq z$. Note that (1.3) and (2.6) imply

$$M_1(v,x_{2n+1})=\max\Big\{d(Av,Bx_{2n+1}),d(Av,Tv),d(Bx_{2n+1},Sx_{2n+1}),$$
$$\frac{1}{2}[d(Av,Sx_{2n+1})+d(Tv,Bx_{2n+1})],$$
$$\frac{d(Av,Sx_{2n+1})d(Tv,Bx_{2n+1})}{1+d(Av,Bx_{2n+1})},$$
$$\frac{d(Av,Tv)d(Bx_{2n+1},Sx_{2n+1})}{1+d(Av,Bx_{2n+1})},$$
$$\left.\frac{1+d(Av,Sx_{2n+1})+d(Tv,Bx_{2n+1})}{1+d(Av,Tv)+d(Bx_{2n+1},Sx_{2n+1})}d(Av,Tv)\right\}$$
$$\to\quad\max\Big\{d(Av,z),d(Av,Tv),d(z,z),\frac{1}{2}[d(Av,z)+d(Tv,z)],$$
$$\frac{d(Av,z)d(Tv,z)}{1+d(Av,z)},\frac{d(Av,Tv)d(z,z)}{1+d(Av,z)},$$

$$
\begin{aligned}
&\qquad\qquad \left.\frac{1+d(Av,z)+d(Tv,z)}{1+d(Av,Tv)+d(z,z)}d(Av,Tv)\right\}\\
&= \max\left\{0, d(z,Tv), 0, \frac{1}{2}d(Tv,z), 0, 0, d(z,Tv)\right\}\\
&= d(Tv,z) \text{ as } n\to\infty,
\end{aligned}
$$

which together with (2.4) give

$$
\begin{aligned}
d(Tv,z) &= \lim_{n\to\infty} d(Tv, y_{2n+2})\\
&= \lim_{n\to\infty} d(Tv, Sx_{2n+1})\\
&\le \lim_{n\to\infty} H(M_1(v,x_{2n+1}), \varphi(M_1(v,x_{2n+1})))\\
&= H(\lim_{n\to\infty} M_1(v,x_{2n+1}), \lim_{n\to\infty} \varphi(M_1(v,x_{2n+1})))\\
&\le H(d(Tv,z), \varphi(d(Tv,z))),
\end{aligned}
$$

so, $d(Tv,z) = 0$ or $\varphi(d(Tv,z)) = 0$. Hence we get a contradiction. Hence $Tv = z$. It follows from (2.2) that there exists a point $w \in X$ with $z = Bw = Tv$. Suppose that $Sw \neq z$. In the light of (1.3) and (2.11), we deduce

$$
\begin{aligned}
M_1(x_{2n}, w) = \max\Bigg\{ & d(Ax_{2n}, Bw), d(Ax_{2n}, Tx_{2n}), d(Bw, Sw),\\
& \frac{1}{2}[d(Ax_{2n}, Sw) + d(Tx_{2n}, Bw)],\\
& \frac{d(Ax_{2n}, Sw)d(Tx_{2n}, Bw)}{1+d(Ax_{2n}, Bw)},\\
& \frac{d(Ax_{2n}, Tx_{2n})d(Bw, Sw)}{1+d(Ax_{2n}, Bw)},\\
& \frac{1+d(Ax_{2n}, Sw)+d(Tx_{2n}, Bw)}{1+d(Ax_{2n}, Tx_{2n})+d(Bw,Sw)}d(Ax_{2n}, Tx_{2n})\Bigg\}\\
\to \quad \max\Bigg\{ & d(z,Bw), d(z,z), d(Bw,Sw), \frac{1}{2}[d(z,Sw)+d(z,Bw)],\\
& \frac{d(z,Sw)d(z,Bw)}{1+d(z,Bw)}, \frac{d(z,z)d(Bw,Sw)}{1+d(z,Bw)},\\
& \frac{1+d(z,Sw)+d(z,Bw)}{1+d(z,z)+d(Bw,Sw)}d(z,z)\Bigg\}\\
= \max & \left\{0, 0, d(z,Sw), \frac{1}{2}d(z,Sw), 0, 0, 0\right\}\\
= d(z&,Sw) \text{ as } n\to\infty,
\end{aligned}
$$

which together with (2.4) yield

$$\begin{aligned} d(z,Sw) &= \lim_{n\to\infty} d(y_{2n+1},Sw) \\ &= \lim_{n\to\infty} d(Tx_{2n},Sw) \\ &\le \lim_{n\to\infty} H(M_1(x_{2n},w),\varphi(M_1(x_{2n},w))) \\ &= H(\lim_{n\to\infty} M_1(x_{2n},w),\lim_{n\to\infty}\varphi(M_1(x_{2n},w))) \\ &\le H(d(z,Sw),\varphi(d(z,Sw))), \end{aligned}$$

so, $d(z,Sw)=0$ or $\varphi(d(z,Sw))=0$. We get a contradiction and hence $Sw=z$. Thus (2.1) means $Az=ATv=TAv=Tz$ and $Bz=BSw=SBw=Sz$. Suppose that $Tz\ne Sz$. It follows from (2.4) that

$$\begin{aligned} M_1(z,z) = \max\Big\{ & d(Az,Bz),d(Az,Tz),d(Bz,Sz), \\ & \frac{1}{2}[d(Az,Sz)+d(Tz,Bz)], \\ & \frac{d(Az,Sz)d(Tz,Bz)}{1+d(Az,Bz)},\frac{d(Az,Tz)d(Bz,Sz)}{1+d(Az,Bz)}, \\ & \frac{1+d(Az,Sz)+d(Tz,Bz)}{1+d(Az,Tz)+d(Bz,Sz)}d(Az,Tz)\Big\} \\ = \max\Big\{ & d(Tz,Sz),0,0,\frac{1}{2}[d(Tz,Sz)+d(Tz,Sz)], \\ & \frac{d^2(Tz,Sz)}{1+d(Tz,Sz)},0,0\Big\} \\ = d(Tz,Sz) & \end{aligned}$$

and

$$\begin{aligned} d(Tz,Sz) &\le H\left(M_1(z,z),\varphi(M_1(z,z))\right) \\ &= H(d(Tz,Sz),\varphi\left(d(Tz,Sz)\right)), \end{aligned}$$

which is a contradiction and hence $Tz=Sz$.

Suppose that $Tz\ne z$. It follows from (1.3) that

$$\begin{aligned} M_1(z,w) = \max\Big\{ & d(Az,Bw),d(Az,Tz),d(Bw,Sw), \\ & \frac{1}{2}[d(Az,Sw)+d(Tz,Bw)],\frac{d(Az,Sw)d(Tz,Bw)}{1+d(Az,Bw)}, \\ & \frac{d(Az,Tz)d(Bw,Sw)}{1+d(Az,Bw)}, \\ & \frac{1+d(Az,Sw)+d(Tz,Bw)}{1+d(Az,Tz)+d(Bw,Sw)}d(Az,Tz)\Big\} \end{aligned}$$

$$
\begin{aligned}
&= \max\left\{d(Tz,z), 0, 0, \frac{1}{2}[d(Tz,z)+d(Tz,z)], \frac{d^2(Tz,z)}{1+d(Tz,z)}, 0, 0\right\} \\
&= d(Tz,z),
\end{aligned}
$$

which together with (2.4), imply

$$
\begin{aligned}
d(Tz,z) &= d(Tz,Sw) \\
&\leq H(M_1(z,w), \varphi(M_1(z,w))) \\
&= H(d(Tz,z), \varphi(d(Tz,z))),
\end{aligned}
$$

so, $d(Tz,z) = 0$ or $\varphi(d(Tz,z)) = 0$. So, we get a contradiction and hence $Tz = z$. Therefore, z is a common fixed point of A, B, S and T.

Suppose that A, B, S and T have another common fixed point $u \in X \setminus \{z\}$. Using (1.3) and (2.4), we have

$$
\begin{aligned}
M_1(u,z) = \max\Bigg\{ & d(Au,Bz), d(Au,Tu), d(Bz,Sz), \\
& \frac{1}{2}[d(Au,Sz)+d(Tu,Bz)], \frac{d(Au,Sz)d(Tu,Bz)}{1+d(Au,Bz)}, \\
& \frac{d(Au,Tu)d(Bz,Sz)}{1+d(Au,Bz)}, \\
& \frac{1+d(Au,Sz)+d(Tu,Bz)}{1+d(Au,Tu)+d(Bz,Sz)} d(Au,Tu)\Bigg\} \\
= \max & \left\{d(u,z), 0, 0, \frac{1}{2}[d(u,z)+d(u,z)], \frac{d^2(u,z)}{1+d(u,z)}, 0, 0\right\} \\
= d(u&,z),
\end{aligned}
$$

and

$$
\begin{aligned}
d(u,z) &= d(Tu,Sz) \\
&\leq H(M_1(u,z), \varphi(M_1(u,z))) \\
&= H(d(u,z), \varphi(d(u,z))),
\end{aligned}
$$

so, $d(u,z) = 0$ or $\varphi(d(u,z)) = 0$. Hence $d(u,z) = 0$ which is a contradiction. So, z is a unique common fixed point of A, B, S and T in X.

Similarly, we conclude that A, B, S and T have a unique common fixed point in X if one of $B(X), S(X)$ or $T(X)$ is complete. This completes the proof. □

Similar to the proof of Theorem 2.1, we have the following results whose proofs are omitted.

Theorem 2.2. *Let A, B, S and T be self mappings in a metric space (X, d) satisfying (2.1)-(2.3) and*

$$d(Tx, Sy) \leq H(M_2(x, y), \varphi(M_2(x, y))), \quad \text{for all } x, y \in X, \tag{2.12}$$

where $H \in \mathcal{C}$, $\varphi \in \Phi_1$ and M_2 is defined by (1.4)*. Then A, B, S and T have a unique common fixed point in X.*

Theorem 2.3. *Let A, B, S and T be self mappings in a metric space (X, d) satisfying (2.1)-(2.3) and*

$$d(Tx, Sy) \leq H(M_3(x, y), \varphi(M_3(x, y))), \quad \text{for all } x, y \in X, \tag{2.13}$$

where $H \in \mathcal{C}$, $\varphi \in \Phi_1$ and M_3 is defined by (1.5)*. Then A, B, S and T have a unique common fixed point in X.*

Remark 2.4. If in Theorem 2.3 put $H(u, v) = \lambda u$, $0 < \lambda < 1$, $T = S$ and $A = B = I_X$ (identity mapping of X), we obtain the Ćirić [4] generalized contraction.

Remark 2.5. Let us to mention also an important recent paper [10] which showed that some of the results involving two functions φ, ψ can be reduced to the case of one function say ψ. It was also shown in [21] that conditions on functions φ and ψ can be weaker.

Remark 2.6. It is worth to mention that in all previously results, we can use also as well as the function $\psi \in \Phi_1$ with both hand. For example, Theorem 2.1 became

- Let A, B, S and T be self mappings in a metric space (X, d) such that $\{A, T\}, \{B, S\}$ are weakly compatible, $T(X) \subseteq B(X)$, $S(X) \subseteq A(X)$ as well as one of $A(X), B(X), S(X)$ and $T(X)$ is complete and for all $x, y \in X$

$$\psi(d(Tx, Sy)) \leq H(\psi(M_1(x, y)), \varphi(M_1(x, y))),$$

where $H \in \mathcal{C}, \psi, \varphi \in \Phi_1$ and M_1 is denoted by (1.3). This new result is according to [10] equivalent to Theorem 2.1.

We finally pose the following problems:

Problem 2.7. Does Theorem 2.1 hold for $\varphi \in \Phi_2$?

Problem 2.8. Does Theorem 2.1 hold if instead of (2.2) we suppose that there exist the sequences $\{x_n\}, \{y_n\} \subset X$ such that (2.5) hold?

Problem 2.9. Does Theorem 2.1 hold for $\varphi \in \Phi_2$ as well as if instead of (2.2) we suppose that there exist the sequences $\{x_n\}, \{y_n\} \subset X$ such that (2.5) hold?

3. Conclusion

Hence, putting any function from the class $\mathcal{C}$ one can obtain some new results from the fixed point theory which generalize several known results from literature. Putting in all previously results $H(u,v) = u - v$, we obtain genuine generalizations of all results in [1], [2], [5], [13], [15], [25]. This namely shows that, using any C-class function H one can (from already known examples) obtain new ones. Therefore, our new approach in this paper is significant and very useful for researchers who involved in the fixed point theory. Also, Lemma 2.1. from [11] give us much shorter as well as nicer proofs than ones in literature. For other details regarding generalizations in the theory of fixed point for contractive conditions see also [17].

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