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On Polar Cones and Differentiability in Reflexive Banach Spaces

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Abstract. Let X be a Banach space, $C \subset X$ be a closed convex set included in a well-based cone K, and also let σ_C be the support function which is defined on C. In this note, we first study the existence of a bounded base for the cone K, then using the obtained results, we find some geometric conditions for the set C, so that $\mathrm{int}(\mathrm{dom}\sigma_C) \neq \emptyset$. The latter is a primary condition for subdifferentiability of the support function σ_C. Eventually, we study Gateaux differentiability of support function σ_C on two sets, the polar cone of K and $\mathrm{int}(\mathrm{dom}\sigma_C)$.

1. Introduction

1.1. **A short survay on convex cones.** The study of convex cones and the geometric structure of their bases in a Banach space has many applications in the theory of optimization, economics and engineering which motivate us to study the subject. We recall [3, 6, 5, 11] and the long list of their references for more details.

Throughout this paper, $(X, ||.||)$ is a Banach space (unless it is specified) whose dual X^* is endowed with the dual norm, denoted also by $||.||$. As usual, having a nonempty subset C of X, define:

$$C^+ := \{x^* \in X^* : x^*(x) \geq 0, \forall x \in C\},$$
$$C^- := -C^+,$$
$$C^\perp := C^+ \cap C^-.$$

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Let K be a cone in a Banach space X. We say that K is pointed if $K \cap -K = \{0\}$ and it is solid if the interior of K, say int$K$, is nonempty. A convex subset B of X is called a base for K if 0 is not included in the closure of B, say clB, and $coneB := \{tx : x \in B, t \geq 0\} = K$. Also, a proper convex cone K is called well-based if K has a bounded base. It is known that a cone with a base is necessarily convex and pointed. Also, if X is a real separable normed space, every nontrivial closed pointed convex cone has a base [8, Corollary 3.39]. Note that the separability assumption is essential and cannot be dropped. Krein-Rutman [8], gave an interesting example which shows that the assertion fails in a nonseparable space.

Let K be a cone. The polar of K is defined by

$$K^{\#} := \{x^* \in X^* : x^*(x) > 0, \forall x \in K \backslash \{0\}\}.$$

There is a direct connection between existence of a base for a cone K and the structure of $K^{\#}$. In fact, a convex cone K has a base if and only if $K^{\#} \neq \emptyset$. Indeed, let $K^{\#} \neq \emptyset$. For every $x^* \in K^{\#}$, the set $B_{x^*} := \{x \in K : x^*(x) = 1\}$ defines a base on the cone K. Conversely, from the Hahn-Banach theorem, we could separate B from 0 by $x^* \in K^{\#}$.

Note that the polar cone $K^{\#}$ could be empty. For example, we consider the space $B([a, b])$ of all functions on the real interval $[a, b]$ endowed with the usual " sup " norm and the standard positive cone:

$$K = \{f \in B[a, b] : f(t) \geq 0, \forall t \in [a, b]\},$$

then $K^{\#}$ is empty. See [8] and references therein for more details.

Remark 1.1. If a closed convex cone K is pointed, then int$K^+ =$ int$K^{\#}$ and we have the followings:

- (a_1) The polar cone $K^{\#}$ need not necessarily be the interior of K^+. For example, if K is the nonnegative orthant of the sequence space l_p, $1 < p < \infty$, then int$K^+ = \emptyset$ but $K^{\#}$ is nonempty.
- (a_2) The interior of K^+ could be nonempty. Let $\alpha \in (0, 1)$ and $a^\alpha = (\alpha, \alpha^2, \ldots) \in l_2$ with

$$||a^\alpha||^2 = \frac{\alpha^2}{(1 - \alpha^2)}.$$

For any $0 < \varepsilon < (1 - \alpha^2)^{\frac{1}{2}}$, set

$$K := \{z \in l_2 : a^\alpha z \geq \varepsilon ||a^\alpha||.||z||\}.$$

Then int$K^+ \neq \emptyset$.

Part (a_2) in Remark 1.1 is equivalent to the existence of a bounded base for the cone K as stated in the following theorem.

Theorem 1.2 ([3] Theorem 2.2)**.** *Let K be a nontrivial closed convex cone in X. Then K is well-based if and only if $K^{\#}$ is solid.*

In fact, B_{x^*} is a bounded base for the closed convex cone K if and only if $x^* \in \text{int}K^{\#}$.

Definition 1.3. Let X be a Banach space, X^* be its dual and let K be a cone in X (see [6] and references therein).

(b_1) K is said to be acute if there is an open half space
$$L_{x^*} = \{x \in X : x^*(x) > 0\},$$
with $x^* \in X^*$, $x^* \neq 0$, such that $clK \subset L_{x^*} \cup \{0\}$.

(b_2) For $x^* \in K^-$ and $\delta > 0$, set $v(x^*, \delta) := \{x \in K : x^*(x) \geq \delta\}$. Recall that the cone K satisfies property (π) (weak property (π)), if there exists $x^* \in K^-$ such that for all $\delta > 0$ the set $v(x^*, \delta)$ is relatively weakly compact (bounded).

(b_3) A closed convex cone K satisfies angle property if there exist $x^* \in X^* \backslash \{0\}$ and $0 < \varepsilon \leq 1$ such that
$$K \subset \{x \in K : x^*(x) \leq \varepsilon ||x^*||.||x||\}.$$

(b_4) A closed convex cone K is said to be a locally weakly compact cone, if for every bounded set A in K, A is relatively weakly compact.

A cone satisfying the (weak) property (π) is pointed and one can replace: 'for all $\delta > 0$' by 'there exists $\delta > 0$'. Cesari and Suryanarrayana showed that there exist infinite dimensional spaces including cones, satisfying angle property. It is of interest to know that in a Banach space X, a closed convex cone with angle property is acute and hence pointed. Also, a closed convex cone K with property (π) is acute (and hence pointed). Furthermore, when X is a reflexive Banach space, angle property implies property (π). However, Cesari and Suryanarayana showed that in the Hilbert space l_2, we can find acute cones which niether have property (π) nor the angle property. See [6, 5, 11] and references therein, for more details.

Theorem 1.4. *Let X be a Banach space and K be a closed convex cone. Then,*

$$K \text{ has angle property} \quad \Rightarrow \quad K \text{ is acute} \quad \Rightarrow \quad K \text{ is pointed.}$$

Also, when X is reflexive

$$K \text{ satisfies property } (\pi) \quad \Rightarrow \quad K \text{ is acute} \quad \Rightarrow \quad K \text{ is pointed.}$$

In 1978, Cesari and Suryanarrayana illustrated an example to show that acuteness (hence either angel property or property (π)) is not satisfied for a half-space. In 1994, Han investigated the relations between cones satisfying angle property and solid cones. The investigation showed that the two classes of cones are dual in some senses. Also, they found a relation between solid cones, acute cones, cones satisfying (weak) property (π) and cones having bounded bases. See, [6, 5] and references therein.

Theorem 1.5. *Let K be a nontrivial closed convex cone in a Banach space X. Then:*

- (c_1) *K has angle property if and only if K has a closed bounded base if and only if K^- is solid if and only if K satisfies the weak property (π).*
- (c_2) *K^- (K) is solid if and only if K (K^-) is well-based.*
- (c_3) *K is acute if and only if K has a closed convex base.*
- (c_4) *K has property (π) if and only if K is relatively weakly compact with weak property (π).*

From (b_4) of Definition 1.3, every convex cone K in a reflexive Banach space is relatively weakly compact. This remark together with (c_1) and (c_4) of Theorem 1.5, imply that in reflexive Banach spaces, a closed convex cone K has angle property if and only if it has property (π). Hence, in reflexive Banach spaces, K^- is solid if and only if K satisfies angle property [6, Theorem 1.2].

1.2. **Convex functions.** Let $U \subset X$ be an open subset of a Banach space X and $f : U \to \mathbb{R}$ be a real valued function. We say that f is Gateaux differentiable at $x \in U$, if for every $h \in X$,

$$f^{'}(x)(h) = \lim_{t \to 0} \frac{f(x+th) - f(x)}{t},$$

exists in $\mathbb{R}$ and is a linear continuous function in h (i.e $f^{'}(x) \in X^*$). The functional $f^{'}(x)$ is then called the Gateaux derivative or Gateaux differential of f at x.

Recall that dom$f$ of a function $f : X \to \bar{\mathbb{R}} := \mathbb{R} \cup \{-\infty, +\infty\}$ is the set $\{x \in X : f(x) < \infty\}$ and f is proper if dom$f \neq \emptyset$ and $f(x) \neq -\infty$ for each $x \in X$. The subdifferential of a proper function f at $x \in$ dom$f$ ($f(x) \neq -\infty$) is defined by

$$\partial f(x) := \{x^* \in X^* : x^*(y-x) \leq f(y) - f(x), \forall y \in X\},$$

and $\partial f(x) = \emptyset$ for $x \in X \backslash \text{dom} f$. Of course, the domain of ∂f is

$$\text{dom}\partial f = \{x \in X : \partial f(x) \neq \emptyset\} (\subset \text{dom} f).$$

Let C be a nonempty subset of the Banach space X. The support function of C is an extended real valued function on X^* defined by

$$\sigma_C : X^* \to \bar{\mathbb{R}}, \qquad \sigma_C(x^*) := \sup_C x^*.$$

It is well-known that for a nonempty subset C of the Banach space X, we have $\sigma_C = \sigma_{\operatorname{conv} C} = \sigma_{clC} = \sigma_{cl(\operatorname{conv} C)}$, where $\operatorname{conv} C$ is the convex hull of C. So we could assume that C is a nonempty, closed and convex set (unless otherwise is specified). Moreover, when the Banach space X is reflexive, we have

$$\partial\sigma_C(x^*) = \{u \in C : x^*(u) = \sigma_C(x^*)\}, \tag{1.1}$$

and $\partial\sigma_C(0) = C$ ([1, 4]).

Rest of the paper is organized as follows. In Section 2, we find some results related to the solidness of polar cones, specially the polar of recession cone of a closed convex set C, and we apply the nonemptiness of $\operatorname{int}(\operatorname{dom}\sigma_C)$. In Section 3, assuming that C is a subset of a closed well-based convex cone K, we study Gateaux differentiability of σ_C on both $K^\#$ and $\operatorname{int}(\operatorname{dom}\sigma_C)$.

2. Conditions in which $\operatorname{int}(\operatorname{dom}\sigma_C) \neq \emptyset$

In Theorem 1.2, it is shown that a nontrivial closed convex cone is well-based if and only if its polar is solid. Here, we show that in reflexive Banach spaces, solidness of the polar cone is equivalent to the existence of a weakly compact base for the cone.

Theorem 2.1. *Let X be a reflexive Banach space and K be a closed convex cone. Then $K^\#$ is solid if and only if K has a weakly compact base B_{x^*}, for some $x^* \in K^\#$.*

Proof. Let $\operatorname{int}K^\# \neq \emptyset$. By Theorem 1.2, for each $x^* \in \operatorname{int}K^\#$, the set B_{x^*} is a bounded base for the cone K and by [10, Theorem 4], $K^\# = \operatorname{int}K^\#$. We show that for each $x^* \in K^\#$, the base B_{x^*} is weakly compact. By the contrary, let $x_0^* \in K^\#$ where $B_{x_0^*}$ is not weakly compact. By James theorem [4, Theorem 3.130], there exists $y^* \in X^*$ such that

$$y^*(x) > \inf\{y^*(b) : b \in B_{x_0^*}\}, \quad \forall x \in B_{x_0^*}.$$

Let $l := \inf\{y^*(b) : b \in B_{x_0^*}\}$. When $l > 0$, define $x_1^* := \frac{y^*}{l} - x_0$ (for $l = 0$, define $x_1^* := y^*$ and for $l < 0$, define $x_1^* := x_0 - \frac{y^*}{l}$). So, we have $x_1^* \in K^\# \backslash \operatorname{int}K^\#$. Indeed, if $x_1^* \in \operatorname{int}K^\#$, there exists $r > 0$ such that $x_1^* + rB_{X^*} \subset K^\#$ (B_{X^*} is the unit ball of X^*). Therefore, for each $k \in K$, we get

$$x_1^*(k) = \frac{y^*(k)}{l} - x_0^*(k) \geq r||k||.$$

From definition of l, there exists $(b_n) \subset B_{x_0^*}$ so that $y^*(b_n) \to l$. So, for each $n \in \mathbb{N}$, $r||b_n|| \geq 0$ and $b_n \to 0$. But this makes a contradition since $0 \notin B_{x_0^*}$. Therefore, $x_1^* \in K^{\#} \backslash \text{int} K^{\#}$ which contradicts $K^{\#} = \text{int} K^{\#}$. So, for each $x^* \in K^{\#}$, the base B_{x^*} is weakly compact. □

Recall that recession cone of a subset C of a Banach space X is defined by

$$C_\infty := \{w \in X : x + tw \in C, \forall t \geq 0, \forall x \in C\}.$$

It is clear that if C is closed, then C_∞ is closed, and if C is closed and convex, we have $C_\infty = \bigcap_{t \geq 0} t(C - a)$, where $a \in C$. Note that in finite dimensional Banach spaces, the closed convex set C is bounded if and only if $C_\infty = \{0\}$. But, the latter is not correct for infinite case in general. For example, let $X = l_2$ and $C = \{x \in l_2 : |x_i| \leq 1 \forall i \in \mathbb{N}\}$. It is clear that X is a reflexive Banach space and C is a closed convex unbounded subset of X. We show that $C_\infty = \{0\}$. It is obvious that $0 \in C_\infty$. Let $v \in C_\infty$ where $v \neq 0$. Then $c + tv \in C$ for each $t \geq 0$. Hence, for a sufficiently large t, we may have $|c_i + tv_i| > 1$ for some $i \in \mathbb{N}$ which is a contradicts with definition of C.

Theorem 2.2. *Let X be a reflexive Banach space and K be a closed convex cone. Then the following statements are established.*

(h_1) $\text{int}(\text{dom}\sigma_K) \neq \emptyset$ *if and only if $K^{\#}$ is solid.*
(h_2) $\text{int}(\text{dom}\sigma_C) \neq \emptyset$ *where C is a closed convex subset of K with $C_\infty \neq \{0\}$.*

Proof. For (h_1), since $K_\infty = K$, the "if" part is a consequence of (h_1). For the other part, let $K^{\#}$ be solid. Then for each $(x_n) \subset K$ with $||x_n|| \to \infty$ and $\left(||x_n||^{-1} x_n\right) \xrightarrow{w} u$, one has $u \neq 0$. Otherwise, let $u = 0$. Since the polar of K is solid, each sequence of K which weakly converges to zero is norm convergence. Hence, $||x_n||^{-1} x_n$ is norm convergence to zero, which is a contradiction. To prove (h_2), it is sufficies to piont out that when $C \subset K$, we have $\text{dom}\sigma_K \leq \text{dom}\sigma_C$ and $K^{\#} \subset C_\infty$. □

Let C be a nonempty closed subset of the Banach space X including a half-line. Define $P := cl(\text{conv}\, C)$ and the set-valued function $W_C : X^* \to X$ by

$$\begin{aligned} W_C(x^*) : &= \{u \in C : x^*(u) = \sigma_C(x^*)\} \\ &= C \cap \partial \sigma_C(x^*). \end{aligned}$$

The following theorem is holds.

Theorem 2.3. *Let X be a reflexive Banach space and C be a nonempty closed subset of X which is included in a closed convex well-based cone. Then the following assertions hold:*

(i_1) $\text{int}(\text{dom}\sigma_C) = \text{int}P^-$ *and* σ_C *is continuous on* $\text{int}P^-$.

(i_2) $\partial\sigma_C(x^*)$ *is nonempty and weakly compact for every* $x^* \in \text{int}P^-$. *Also,* $\partial\sigma_C(x^*)$ *is singleton if and only if* σ_C *is Gateaux differentiable at* x^*

(i_3) $\text{dom}W_C \subset \text{dom}\sigma_C \subset P^-$. *Also,* $W_C(x^*)$ *is nonempty and w-compact for every* $x^* \in \text{int}P^-$.

(i_4) $\text{int}(\text{dom}\sigma_C) = \text{int}(\text{dom}\sigma_C)\backslash\{0\} = \text{int}(\text{dom}\sigma_C)\backslash(\text{lin}_0 C)^\perp$.

Proof. (i_1) It is clear that

$$cl(\text{dom}\sigma_C) = (cl(convC))^-_\infty = P^-,$$

(for convex sets, the weak and norm closure coincide). Theorem 2.2 implies that $\text{int}(\text{dom}\sigma_C) \neq \emptyset$. So, from [5, Lemma 12], convexity of $\text{dom}\sigma_C$ implies that

$$\text{int}(\text{dom}\sigma_C) = \text{int}[cl(\text{dom}\sigma_C)].$$

Hence, $\text{int}(\text{dom}\sigma_C) = \text{int}P^-$. Also, from [2, Proposition 4.1.5], the support function σ_C is continuous on $x^* \in \text{dom}\sigma_C$ if and only if $x^* \in \text{int}(\text{dom}\sigma_C)$.

(i_2) The subdifferential of a proper convex function is nonempty, convex and w-compact at any point of continuity from its domain [1, Theorem 7.13]. The second allegation comes from Smulyan lemma [4, Theorem 7.17].

(i_3) Fix $u_0 \in C$ and $x^* \in \text{int}P^-$. Define

$$C_0 := \{u \in C : x^*(u) \geq x^*(u_0)\}.$$

It is clear that C_0 is nonempty, closed and $\sigma_{C_0}(x^*) = \sigma_C(x^*)$. We show that C_0 is bounded. By the contrary, assume that there exists $(x_n) \subset C_0$ with $||x_n|| \to \infty$. We may assume that $x_n||x_n||^{-1}$ weakly converges to v. Note that from (g_1) of Theorem 2.5, we have $v \neq 0$ and $v \in P\backslash\{0\}$. Since $x^*(x_n) \geq x^*(u_0)$ for every $n \in \mathbb{N}$, we get the contradiction $0 > x^*(v) \geq 0$. Hence, C_0 is bounded, and so weakly compact. Now, by James theorem [4, Theorem 3.130], x^* attains its supermum on C_0 at $v \in C_0$ and $\sigma_C(x^*) = x^*(v)$. It follows that $v \in W_C(x^*)$. So, $W_C(x^*)$ is nonempty for each $x^* \in \text{int}P^-$. Moreover, the fact that the intersection of a closed set and a weakly compact set is weakly compact completes the proof.

(i_4) This is a consequence of (i_1) and definitions of $\text{int}P^-$ and $(\text{lin}_0 C)^\perp$. □

Remark 2.4. Note that $\partial\sigma_C(0) = C$ and σ_C is Gateaux differentiable on 0 if and only if C is singleton. Also, $x^* \in X^*$ is constant on C if and only if x^* belongs to $(\text{lin}_0 C)^\perp$. Hence, when C is not singleton, σ_C is not Gateaux differentiable on $(\text{lin}_0 C)^\perp$. But, according to (i_4), under our assumptions, $\text{int}(\text{dom}\sigma_C) \cap (\text{lin}_0 C)^\perp = \emptyset$ and we can speak about differentiability of σ_C on $\text{int}(\text{dom}\sigma_C)$.

Corollary 2.5. *Let X be a reflexive Banach space and C be a closed convex set that $C_\infty \neq \{0\}$. The followings are equivalent:*

- (g_1) $C_\infty^\# \neq \varnothing$ *and for every sequence $(x_n) \in C$ with $||x_n|| \to \infty$ and $||x_n||^{-1} x_n \overset{w}{\to} u$, one has $u \neq 0$.*
- (g_2) $\text{int}(\text{dom}\sigma_C) \neq \emptyset$.
- (g_4) C_∞ *is well-based ($C_\infty^\#$ is solid).*
- (g_5) *there exists $x^* \in K^\#$ such that B_{x^*} is a weakly compact base for the cone K.*
- (g_6) C_∞ *has property (π) (weak-property (π)).*
- (g_7) C_∞ *has angle property.*

3. Differentiability of σ_C

In this section, let X be a reflexive Banach space and C be a nonempty closed subset of X which is included in a closed convex well-based cone K (unless otherwise is stated).

Theorem 3.1. *The support function σ_C is Gateaux differentiable on* $\text{int}(\text{dom}\sigma_C)$ *if and only if*

$$\forall x, y \in \partial\sigma_C(\text{int}(\text{dom}\sigma_C)), x \neq y, \forall \lambda \in (0,1):$$
$$\lambda x + (1-\lambda) y \notin \partial\sigma_C(\text{int}(\text{dom}\sigma_C)). \tag{3.1}$$

Proof. Let 3.1 hold. Consider $x^* \in \text{int}(\text{dom}\sigma_C)$ where σ_C is not Gateaux differentiable on x^*. From (i_2) of Theorem 2.3, σ_C is subdifferentiable on $\text{int}(\text{dom}\sigma_C)$ and there exist $x, y \in \partial\sigma_C$ with $x \neq y$. Therefore, the convexity of $\partial\sigma_C$ implies that $\lambda x + (1-\lambda)y \in \partial\sigma_C(x^*)$ for all $\lambda \in (0,1)$, which is a contradiction.

Now, let σ_C be Gateaux differentiable on $\text{int}(\text{dom}\sigma_C)$. By the contrary, let $x_0, y_0 \in \partial\sigma_C(\text{int}(\text{dom}\sigma_C))$, where $x_0 \neq y_0$ and $\lambda_0 \in (0,1)$ such that $z := \lambda_0 x_0 + (1-\lambda_0) y_0 \in \partial\sigma_C(\text{int}(\text{dom}\sigma_C))$. So, there exists $x^* \in (\text{int}(\text{dom}\sigma_C))$ such that $z \in \partial\sigma_C(x^*)$ which implies that

$$x^*(z) = \sigma_C(x^*) \geq \lambda_0 x^*(x_0) + (1-\lambda_0) x^*(y_0) = x^*(z).$$

It means that $x^*(x_0) = x^*(y_0) = \sigma_C(x^*)$ and $x_0, y_0 \in \partial\sigma_C(x^*)$. But, by (i_2) of Theorem 2.3, $\partial\sigma_C(x^*)$ is singleton for each $x^* \in \text{int}(\text{dom}\sigma_C)$, which is a contradiction. □

Theorem 3.2. *σ_C is Gateaux differentiable on* $\text{int}(\text{dom}\sigma_C)$ *if*

$$\forall x, y \in C,\ x \neq y, \forall \lambda \in (0,1) : \lambda x + (1-\lambda) y \in H, \tag{3.2}$$

where $H := C + (C_\infty \backslash \{0\})$.

Proof. First, we show that $\partial\sigma_C(\text{int}(\text{dom}\sigma_C))$ is a subset of $C\backslash H$. Letting $y \in \partial\sigma_C(\text{int}(\text{dom}\sigma_C)) \cap H$ (by the contrary), there exist $y^* \in \text{int}(\text{dom}\sigma_C)$, $u \in C$ and $v \in C_\infty\backslash\{0\}$ such that $y \in \partial\sigma_C(y^*)$ and $y = u + v$. So,

$$\sigma_C(y^*) > y^*(u) + y^*(v) = y^*(y) = \sigma_C(y^*),$$

which is not possible. Now, it is easy to show that (3.2) implies the following condition:

$$\forall x, y \in C\backslash H, x \neq y, \forall \lambda \in (0,1) : \lambda x + (1-\lambda) y \notin C\backslash H. \tag{3.3}$$

In fact, the implications (3.2) $\Rightarrow$ (3.3) $\Rightarrow$ (3.1) hold. Therefore, (3.2) and (3.3) together imply that σ_C is Gateaux differentiable on $\text{int}(\text{dom}\sigma_C)$. □

Definition 3.3. Let K be a closed convex pointed cone and C be a nonempty set in X. The set of Pareto minimal points, S-properly points, Borwien properly minimal points, and Henig global properly minimal points of C with respect to K are shown by,

$$Min(C,K) := \{u \in C : C \cap (u+K) = \{u\}\} = C\backslash(C + (K\backslash\{0\})),$$
$$S - PMin(C,K) := \{u \in C : \exists y^* \in K^{\#}, \forall y \in C, y^*(u) \leq y^*(y)\},$$
$$Bo - Min(C,K) := \{u \in C : clcone(C-u) \cap (-K) = \{0\})\},$$

and

$$GHe - PMin(C,K) := \Big\{u \in C : \exists \text{ a proper convex cone } P \text{ with } K\backslash\{0\} \subset \text{int}P \text{ such that } (C-u) \cap (-\text{int}P) = \emptyset\Big\},$$

respectively.

Remark 3.4. Let K be a closed convex well-based cone and $C \subset K$ be closed and convex with $C_\infty \neq \{0\}$.

(j_1) $S - PMin(C,K) = \partial\sigma_C(-K^{\#}) = \bigcup_{x^* \in K^{\#}} \partial\sigma_C(-x^*)$.

(j_2) Since $C+K$ is closed, we get the following results (see [9] for more details)

$$\begin{aligned} S-PMin(C,K) &= S-PMin(C+K,K) \\ &= GHe-PMin(C,K) \\ &= Bo-Min(C,K) \\ &\subset Min(C,K). \end{aligned}$$

Theorem 3.5. *σ_C is Gateaux differentiable on $-K^{\#}$ if and only if σ_{C+K} is Gateaux differentiable on* $\mathrm{int}(\mathrm{dom}\sigma_{C+K})$.

Proof. By the assumption, we get $C_\infty \subset K$ and $(C+K)_\infty = K_\infty = K$. Moreover, $\sigma_{C+K} = \sigma_C + \sigma_K = \sigma_C + \iota_K$. Since the space is reflexive, we have

$$-K^{\#} = \mathrm{int}K^- \subset \mathrm{dom}\sigma_{C+K} = K^- \cap \mathrm{dom}\sigma_C \subset K^- \cap C_\infty^- = K^-.$$

By taking the interior of the both sides, we have $\mathrm{int}(\mathrm{dom}\sigma_{C+K}) = -K^{\#}$ and

$$\begin{aligned} \partial\sigma_{C+K}[\mathrm{int}(\mathrm{dom}\sigma_{C+K})] &= \partial\sigma_{C+K}(-K^{\#}) \\ &= S-PMin(C+K,K) \\ &= S-PMin(C,K) \\ &= \partial\sigma_C(-K^{\#}). \end{aligned}$$

So $\partial\sigma_C$ is singleton on $-K^{\#}$ if and only if $\partial\sigma_{C+K}$ is singleton on $\mathrm{int}(\mathrm{dom}\sigma_{C+K})$ which means that σ_C is Gateaux differentiable on $-K^{\#}$ if and only if σ_{C+K} is Gateaux differentiable on $\mathrm{int}(\mathrm{dom}\sigma_{C+K})$. □

Theorem 3.6. *σ_C is Gateaux differentiable on $-K^{\#}$ if and only if*

$$\begin{aligned} &\forall x, y \in S-PMin(C,K),\ x \neq y, \\ &\forall \lambda \in (0,1) : \lambda x + (1-\lambda)y \notin S-PMin(C,K). \end{aligned} \tag{3.4}$$

Proof. By Theorem 3.5, σ_C is Gateaux differentiable on $-K^{\#}$ if and only if σ_{C+K} is Gateaux differentiable on $\mathrm{int}(\mathrm{dom}\sigma_{C+K})$. Now, one obtains the result by using Theorem 3.1 for $\mathrm{int}(\mathrm{dom}\sigma_{C+K})$. □

Corollary 3.7. (h_1) *Since $S-PMin(C,K)$ is a subset of $Min(C,K)$, considering the following condition,*

$$\begin{aligned} &\forall x, y \in Min(C,K),\ x \neq y, \\ &\forall \lambda \in (0,1) : \lambda x + (1-\lambda)y \notin Min(C,K), \end{aligned} \tag{3.5}$$

we have the implication 3.5 ⇒ 3.4. Hence, 3.5 implies that σ_C is differentiable on $-K^{\#}$. Moreover, from the equalities in (j_2) of Remark 3.4, we could replace S-properly minimal points in *condition 3.4, by Borwien properly minimal points and Henig global properly minimal points of C with respect to K.*

(h_2) *By taking $K := C_\infty$ in Theorem 3.6, we get $K^\# = -\mathrm{int}(\mathrm{dom}\sigma_C)$. So, Theorem 3.1 is a consequense of Theorem 3.6.*

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