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Generalized Regular Fuzzy Irresolute Mappings and Their Applications

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Abstract. In this paper, the notion of generalized regular fuzzy irresolute, generalized regular fuzzy irresolute open and generalized regular fuzzy irresolute closed maps in fuzzy topological spaces are introduced and studied. Moreover, some separation axioms and r-GRF-separated sets are established. Also, the relations between generalized regular fuzzy continuous maps and generalized regular fuzzy irresolute maps are investigated. As a natural follow-up of the study of r-generalized regular fuzzy open sets, the concept of r-generalized regular fuzzy connectedness of a fuzzy set is introduced and studied.

1. Introduction

Kubiak [10] and Šostak [16] introduced the fundamental concept of a fuzzy topological structure, as an extension of both crisp topology and fuzzy topology [2], in the sense that not only the objects are fuzzified, but also the axiomatics. In [15, 17], Šostak gave some rules and showed how such an extension can be realized. Chattopadhyay et al., [3] have redefined the same concept under the name gradation of openness. A general approach to the study of topological type structures on fuzzy power sets was developed in [6–8, 10, 11]. Balasubramanian and Sundaram [1] gave the concept of generalized fuzzy closed sets in Chang's fuzzy topology as an extension of generalized closed sets of Levine [13] in topological spaces. Kim and Ko [9] introduced the concept of r-generalized fuzzy closed sets in Sostak's fuzzy topological spaces. Recently, Vadivel and

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Elavarasan [18] introduced r-generalized regular fuzzy closed sets and generalized regular fuzzy continuous function in Sostak's fuzzy topological space.

In this paper, the notion of generalized regular fuzzy irresolute, generalized regular fuzzy irresolute open and generalized regular fuzzy irresolute closed maps in fuzzy topological spaces are introduced and studied. Moreover, some properties of generalized regular fuzzy irresolute maps and r-FRCO-T_1, r-FRCO-T_2, r-GRF-T_1, r-GRF-T_2, r-FRCO-regular, r-FRCO-normal, strongly GRF-regular, strongly GRF-normal, r-GRF-separated sets are established. Also, the relations between generalized regular fuzzy continuous maps and generalized regular fuzzy irresolute maps are investigated. As a natural follow-up of the study of r-generalized regular fuzzy open sets, the concept of r-generalized regular fuzzy connectedness of a fuzzy set is introduced and studied.

2. Preliminaries

Throughout this paper, let X be a nonempty set, $I = [0,\ 1]$ and $I_0 = (0,\ 1]$. For $\lambda \in I^X$, $\overline{\lambda}(x) = \lambda$ for all $x \in X$. For $x \in X$ and $t \in I_0$, a fuzzy point x_t is defined by

$$x_t(y) = \begin{cases} t & \text{if } y = x, \\ 0 & \text{if } y \neq x. \end{cases}$$

Let $Pt(X)$ be the family of all fuzzy points in X. A fuzzy point $x_t \in \lambda$ iff $t < \lambda(x)$. All other notations and definitions are standard, for all in the fuzzy set theory.

Definition 2.1 ([16])**.** A function $\tau : I^X \to I$ is called a fuzzy topology on X if it satisfies the following conditions:

(O1) $\tau(\overline{0}) = \tau(\overline{1}) = 1$,
(O2) $\tau(\bigvee_{i\in\Gamma} \mu_i) \geq \bigwedge_{i\in\Gamma} \tau(\mu_i)$, for any $\{\mu_i\}_{i\in\Gamma} \subset I^X$,
(O3) $\tau(\mu_1 \wedge \mu_2) \geq \tau(\mu_1) \wedge \tau(\mu_2)$, for any $\mu_1,\ \mu_2 \in I^X$.

The pair $(X,\ \tau)$ is called a fuzzy topological space (for short, fts). A fuzzy set λ is called an r-fuzzy open (r-FO, for short) if $\tau(\lambda) \geq r$. A fuzzy set λ is called an r-fuzzy closed (r-FC, for short) set iff $\overline{1} - \lambda$ is an r-FO set.

Theorem 2.2 ([4])**.** *Let (X, τ) be a fts. Then for each $\lambda \in I^X$ and $r \in I_0$, we define an operator $C_\tau : I^X \times I_0 \to I^X$ as follows: $C_\tau(\lambda,\ r) = \bigwedge\{\mu \in I^X : \lambda \leq \mu,\ \tau(\overline{1} - \mu) \geq r\}$. For $\lambda,\ \mu \in I^X$ and $r, s \in I_0$, the operator C_τ satisfies the following statements:*

(C1) $C_\tau(\overline{0}, r) = \overline{0}$,
(C2) $\lambda \leq C_\tau(\lambda,\ r)$,

(C3) $C_\tau(\lambda,\ r) \vee C_\tau(\mu,\ r) = C_\tau(\lambda \vee \mu,\ r)$,
(C4) $C_\tau(\lambda,\ r) \leq C_\tau(\lambda,\ s)$ *if* $r \leq s$,
(C5) $C_\tau(C_\tau(\lambda,\ r),\ r) = C_\tau(\lambda,\ r)$.

Theorem 2.3 ([14])**.** *Let* (X, τ) *be a fts. Then for each* $\lambda \in I^X$ *and* $r \in I_0$*, we define an operator* $I_\tau : I^X \times I_0 \to I^X$ *as follows:* $I_\tau(\lambda,\ r) = \bigvee\{\mu \in I^X : \mu \leq \lambda,\ \tau(\mu) \geq r\}$*. For* $\lambda,\ \mu \in I^X$ *and* $r, s \in I_0$*, the operator* I_τ *satisfies the following statements:*

(I1) $I_\tau(\overline{1}, r) = \overline{1}$,
(I2) $I_\tau(\lambda,\ r) \leq \lambda$,
(I3) $I_\tau(\lambda,\ r) \wedge I_\tau(\mu,\ r) = I_\tau(\lambda \wedge \mu,\ r)$,
(I4) $I_\tau(\lambda,\ r) \leq I_\tau(\lambda,\ s)$ *if* $s \leq r$,
(I5) $I_\tau(I_\tau(\lambda,\ r),\ r) = I_\tau(\lambda,\ r)$.
(I6) $I_\tau(\overline{1} - \lambda, r) = \overline{1} - C_\tau(\lambda, r)$ *and* $C_\tau(\overline{1} - \lambda, r) = \overline{1} - I_\tau(\lambda,\ r)$.

Definition 2.4 ([12])**.** Let $(X,\ \tau)$ be a fts, $\lambda \in I^X$ and $r \in I_0$. Then

(1) a fuzzy set λ is called r-fuzzy regular open (for short, r-fro) if
$$\lambda = I_\tau(C_\tau(\lambda, r), r).$$
(2) a fuzzy set λ is called r-fuzzy regular closed (for short, r-frc) if
$$\lambda = C_\tau(I_\tau(\lambda, r), r).$$
(3) a fuzzy set λ is called r-fuzzy regular clopen (for short, r-frco) set iff λ is r-frc set and r-fro set.

Definition 2.5 ([18])**.** Let $(X,\ \tau)$ be a fts. For $\lambda, \mu \in I^X$ and $r \in I_0$.

(1) The r-fuzzy regular closure of λ, denoted by $RC_\tau(\lambda, r)$, is defined by
$$RC_\tau(\lambda, r) = \bigwedge \left\{\mu \in I^X | \mu \geq \lambda, \mu \text{ is } r\text{-}frc\right\}.$$
(2) The r-fuzzy regular interiror of λ, denoted by $RI_\tau(\lambda, r)$, is defined by
$$RI_\tau(\lambda, r) = \bigvee \left\{\mu \in I^X | \mu \leq \lambda, \mu \text{ is } r\text{-}fro\right\}.$$

Definition 2.6 ([18])**.** Let $f : (X,\ \tau) \to (Y,\ \sigma)$ be a function and $r \in I_0$. Then f is called:

(1) fuzzy regular continuous (for short, FR-continuous) if $f^{-1}(\lambda)$ is an r-fro set in I^X for each $\lambda \in I^Y$ with $\sigma(\lambda) \geq r$.
(2) fuzzy regular open (for short, FR-open) if $f(\lambda)$ is an r-fro set in I^Y for each $\lambda \in I^X$ with $\tau(\lambda) \geq r$.
(3) fuzzy regular closed (for short, FR-closed) if $f(\lambda)$ is an r-frc set in I^Y for each $\lambda \in I^X$ with $\tau(\overline{1} - \lambda) \geq r$.

Definition 2.7 ([9])**.** Let $(X,\ \tau)$ be a fts. For any $\lambda,\ \mu \in I^X$ and $r \in I_0$ a fuzzy set λ is called an r-generalized fuzzy closed (for short, r-gfc) set if $C_\tau(\lambda, r) \leq \mu$, whenever $\lambda \leq \mu$ and $\tau(\mu) \geq r$.

Definition 2.8 ([18])**.** Let $(X,\ \tau)$ be a fts. For any $\lambda,\ \mu \in I^X$ and $r \in I_0$.

(1) A fuzzy set λ is called an r-generalized regular fuzzy closed (for short, r-grfc) set if $RC_\tau(\lambda, r) \leq \mu$, whenever $\lambda \leq \mu$ and $\tau(\mu) \geq r$.
(2) A fuzzy set λ is called an r-generalized regular fuzzy open (for short, r-grfo) set if $\overline{1} - \lambda$ is r-grfc.
(3) A fuzzy set λ is called an r-generalized regular fuzzy clopen (for short, r-grfco) set iff λ is an r-grfc and r-grfo set.

Definition 2.9 ([18])**.** Let $(X,\ \tau)$ be a fts. For $\lambda, \mu \in I^X$ and $r \in I_0$.

(1) The r-generalized regular fuzzy closure of λ, denoted by $GRC_\tau(\lambda, r)$ is defined by
$$GRC_\tau(\lambda, r) = \bigwedge \left\{\mu \in I^X \mid \lambda \leq \mu,\ \mu \text{ is } r\text{-}grfc\right\}$$
(2) The r-generalized regular fuzzy interiror of λ, denoted by $GRI_\tau(\lambda, r)$, and is defined by
$$GRI_\tau(\lambda, r) = \bigvee \left\{\mu \in I^X \mid \lambda \geq \mu,\ \mu \text{ is } r\text{-}grfo\right\}.$$

Definition 2.10 ([9])**.** Let $(X,\ \tau)$ and $(Y,\ \eta)$ be fts's. A function $f : (X,\ \tau) \to (Y,\ \eta)$ is called generalized fuzzy continuous (for short, gf-continuous) if $f^{-1}(\mu)$ is r-grfc for each $\mu \in I^Y,\ r \in I_0$ with $\eta(\overline{1} - \mu) \geq r$.

Definition 2.11 ([18])**.** Let $(X,\ \tau)$ and $(Y,\ \eta)$ be a fts's. Let $f : (X,\ \tau) \to (Y,\ \eta)$ be a function.

(1) f is called generalized regular fuzzy continuous (for short, grf-continuous) iff $f^{-1}(\mu)$ is r-grfc for each $\mu \in I^Y,\ r \in I_0$ with $\eta(\overline{1} - \mu) \geq r$.
(2) f is called generalized regular fuzzy open (for short, grf-open) iff $f(\lambda)$ is r-grfo for each $\lambda \in I^X,\ r \in I_0$ with $\tau(\lambda) \geq r$.
(3) f is called generalized regular fuzzy closed (for short, grf-closed) iff $f(\lambda)$ is r-grfc for each $\lambda \in I^X,\ r \in I_0$ with $\tau(\overline{1} - \lambda) \geq r$.

Theorem 2.12 ([18])**.** *Let $(X,\ \tau)$ be a fts. For each $r \in I_0$ and $\lambda \in I^X$, we define the operators $GRC_\tau, GRI_\tau : I^X \times I_0 \to I^X$ as follows:*
$$GRC_\tau(\lambda, r) = \bigvee \left\{\mu \in I^X \mid \mu \leq \lambda,\ \mu \text{ is } r\text{-}grfc\right\},$$
$$GRI_\tau(\lambda, r) = \bigvee \left\{\mu \in I^X \mid \mu \leq \lambda,\ \mu \text{ is } r\text{-}grfo\right\},$$
Then

(1) $GRI_\tau(\overline{1}-\lambda, r) = \overline{1} - GRC_\tau(\lambda, r)$.
(2) $GRC_\tau(\overline{1}-\lambda, r) = \overline{1} - GRI_\tau(\lambda, r)$.
(3) $RI_\tau(\lambda, r) \le GRI_\tau(\lambda, r) \le \lambda \le RC_\tau(\lambda, r) \le GRC_\tau(\lambda, r)$.

Theorem 2.13 ([18])**.** *Let $(X,\ \tau)$ be a fts, $\lambda \in I^X$ and $r \in I_0$.*
(1) *If λ is an r-frc set, then λ is an r-grfc set.*
(2) *If λ is an r-fro set, then λ is an r-grfo set.*

3. Generalized Regular Fuzzy Irresolute Mappings

Definition 3.1. Let (X, τ) and (Y, η) be fts's. The mapping $f : X \to Y$ is called:
(1) generalized regular fuzzy irresolute (grf-irresolute, for short) if $f^{-1}(\mu)$ is an r-grfc set, for each r-grfc set $\mu \in I^Y,\ r \in I_0$.
(2) generalized regular fuzzy irresolute open (grf-irresolute open, for short) if $f^{-1}(\mu)$ is an r-grfo set, for each r-grfo set $\mu \in I^Y,\ r \in I_0$.
(3) generalized regular fuzzy irresolute closed (grf-irresolute closed, for short) if $f^{-1}(\mu)$ is an r-grfc set, for each r-grfc set $\mu \in I^Y,\ r \in I_0$.
(4) generalized regular fuzzy irresolute homeomorphism (grfi-homeomorphism, for short) iff f is bijective and both f and f^{-1} are grf-irresolute mappings.

Remark 3.2. For a mapping $f : X \to Y$, the following statements are valid:
(1) f is grf-continuous $\Rightarrow$ f is gf-continuous.
(2) f is grf-irresolute $\Rightarrow$ f is grf-continuous.

The converse of Remark 3.2 is not true in general.

Example 3.3. Let $X = Y = \{a,\ b,\ c\}$ and $\gamma \in I^X$ be defined as $\gamma(a) = 0.5,\ \gamma(b) = 0.7,\ \gamma(c) = 0.9$. We define a fuzzy topology $\tau : I^X \to I$ as follows:

$$\tau(\lambda) = \begin{cases} 1 & \text{if } \lambda = \overline{0} \text{ or } \overline{1}, \\ \frac{1}{2} & \text{if } \lambda = \gamma, \\ 0 & \text{otherwise,} \end{cases} \qquad \eta(\lambda) = \begin{cases} 1 & \text{if } \lambda = \overline{0} \text{ or } \overline{1}, \\ \frac{1}{2} & \text{if } \lambda = \gamma, \\ 0 & \text{otherwise,} \end{cases}$$

For $r = 1/3$, $\eta(\overline{1} - (\overline{1} - \gamma)) \ge r$, $\overline{1} - \gamma$ is an r-gfc set in (X, τ) but not r-grfc in (X, τ). Since $C_\tau(\overline{1} - \gamma, r)(= \overline{1} - \gamma) \le \gamma$, $\overline{1} - \gamma \le \gamma$, $\tau(\gamma) \ge r$ but $RC_\tau(\overline{1} - \gamma, r)(= \overline{1}) \not\le \gamma$, $\overline{1} - \gamma \le \gamma$, $\tau(\gamma) \ge r$. Thus the identity function $f : (X, \tau) \to (Y, \eta)$ is gf-continuous but not grf-continuous.

Example 3.4. Let $X = \{a,\ b,\ c\}$, $Y = \{p, q, r\}$ and $\lambda \in I^X, \delta, \mu \in I^Y$ be defined as $\lambda(a) = 0.4,\ \lambda(b) = 0.5,\ \lambda(c) = 0.7$; $\mu(p) = 0.4,\ \mu(q) =$

0.5, $\mu(r) = 0.6$; $\delta(p) = 0.4$, $\delta(q) = 0.5$, $\delta(r) = 0.7$. We define two smooth topologies $\tau, \eta : I^X \to I$ as follows.

$$\tau(\lambda) = \begin{cases} 1 & \text{if } \lambda = \overline{0} \text{ or } \overline{1}, \\ \frac{1}{2} & \text{if } \lambda = \lambda, \\ 0 & \text{otherwise,} \end{cases} \qquad \eta(\lambda) = \begin{cases} 1 & \text{if } \lambda = \overline{0} \text{ or } \overline{1}, \\ \frac{1}{2} & \text{if } \lambda = \mu, \\ 0 & \text{otherwise.} \end{cases}$$

For $r = 1/3$, $\eta(\overline{1} - (\overline{1} - \mu)) \geq r$, $\overline{1} - \mu$ is an r-grfc set in (X, τ). Then the function $f : (X, \tau) \to (Y, \eta)$ is grf-continuous but not grf-irresolute, since the fuzzy set δ is r-grfc set in (Y, η) but not r-grfc set in (X, τ).

Theorem 3.5. *Let (X, τ) and (Y, η) be fts's. Let $f : X \to Y$ be a mapping. Then the following statements are equivalent:*

(1) *f is grf-irresolute.*
(2) *For each r-grfc set $\mu \in I^Y$, $f^{-1}(\mu)$ is an r-grfc set.*
(3) *$f(GRC_\tau(\lambda, r)) \leq GRC_\eta(f(\lambda), r)$, for each $\lambda \in I^X$ and $r \in I_0$.*
(4) *$GRC_\tau(f^{-1}(\mu), r) \leq f^{-1}(GRC_\eta(\mu, r))$, for each $\mu \in I^Y$ and $r \in I_0$.*
(5) *$f^{-1}(GRI_\eta(\mu, r)) \leq GRI_\tau(f^{-1}(\mu), r)$, for each $\mu \in I^Y$ and $r \in I_0$.*

Proof. (1)⇒(2) Let μ be an r-grfc set in I^Y and $r \in I_0$. Then $\overline{1} - \mu$ is an r-grfo set. Since f is grf-irresolute, $f^{-1}(\overline{1} - \mu) = \overline{1} - f^{-1}(\mu)$ is an r-grfo set in X. Therefore, $f^{-1}(\mu)$ is an r-grfc set in X.

(2)⇒(3) Suppose there exist $\lambda \in I^X$ and $r \in I_0$ such that

$$f(GRC_\tau(\lambda, r)) \nleq GRC_\eta(f(\lambda), r).$$

There exist $y \in Y$ and $t \in I_0$ such that

$$f(GRC_\tau(\lambda, r))(y) > t > GRC_\eta(f(\lambda), r)(y).$$

If $f^{-1}(\{y\}) = \phi$, it is a contradiction because

$$f(GRC_\tau(\lambda, r))(y) = 0.$$

If $f^{-1}(\{y\}) \neq \phi$, there exists $x \in f^{-1}(\{y\})$ such that

$$f(GRC_\tau(\lambda, r))(y) \geq GRC_\tau(\lambda, r)(x) > t > GRC_\eta(f(\lambda), r)(f(x)). \tag{3.1}$$

Since $GRC_\eta(f(\lambda), r)(f(x)) < t$, there exists an r-grfc set $\mu \in I^Y$ with $f(\lambda) \leq \mu$ such that

$$GRC_\eta(f(\lambda), r)(f(x)) \leq \mu(f(x)) < t.$$

Moreover, $f(\lambda) \leq \mu$ implies $\lambda \leq f^{-1}(\mu)$. From (2), $f^{-1}(\mu)$ is r-grfc. Thus, $GRC_\tau(\lambda, r)(x) \leq f^{-1}(\mu)(x) = \mu(f(x)) < t$. It is a contradiction to (3.1).

(3)⇒(4) For all $\mu \in I^Y$ and $r \in I_0$, put $\lambda = f^{-1}(\mu)$. From (3), we have

$$f(GRC_\tau(f^{-1}(\mu), r)) \leq GRC_\eta(f(f^{-1}(\mu)), r) \leq GRC_\eta(\mu, r).$$

It implies

$$GRC_\tau(f^{-1}(\mu), r) \leq f^{-1}(f(GRC_\tau(f^{-1}(\mu), r))) \leq f^{-1}(GRC_\eta(\mu, r)).$$

(4)⇒(5) It can be easily seen from Theorem 2.12 (1).

(5)⇒(1) Let μ be an r-grfo set in Y. From Theorem 2.12 (3), $\mu = GRI_\eta(\mu, r)$. By (5),

$$f^{-1}(\mu) \leq GRI_\tau(f^{-1}(\mu), r).$$

On the other hand, by Theorem 2.12(2),

$$f^{-1}(\mu) \geq GRI_\tau(f^{-1}(\mu), r).$$

Thus, $f^{-1}(\mu) = GRI_\tau(f^{-1}(\mu), r)$, that is, $f^{-1}(\mu)$ is an r-grfo set.

□

Theorem 3.6. *Let (X, τ) and (Y, η) be fts's. Let $f : X \to Y$ be a bijective mapping. Then, the following statements are equivalent.*

(1) *f is grf-irresolute.*
(2) *$GRC_\eta(f(\lambda), r) \leq f(GRC_\tau(\lambda, r))$, for each $\lambda \in I^X$ and $r \in I_0$.*

Proof. (1)⇒(2) Let f be a grf-irresolute mapping, $\lambda \in I^X$ and $r \in I_0$. Then $f^{-1}(GRC_\eta(f(\lambda), r))$ is an r-grfo set in X. By Theorem 3.5 and the fact that f is one-to-one we have

$$f^{-1}(GRC_\eta(f(\lambda), r)) \leq GRC_\tau(f^{-1}(f(\lambda), r)) = GRC_\tau(\lambda, r),$$

Again since f is onto we have

$$GRI_\eta(f(\lambda), r) = ff^{-1}(GRI_\eta(f(\lambda), r)) \leq f(GRI_\tau(\lambda, r)).$$

(2)⇒(1) Let μ be an r-grfo set in Y. Then by Theorem 2.12(3), $\mu = GRI_\eta(\mu, r)$. By (2)

$$f(GRI_\tau(f^{-1}(\mu), r)) \geq GRI_\eta(ff^{-1}(\mu), r) = GRI_\eta(\mu, r) = \mu.$$

It implies

$$GRI_\tau(f^{-1}(\mu), r) = f^{-1}f(GRI_\tau(f^{-1}(\mu), r)) \geq f^{-1}(\mu).$$

Hence, $f^{-1}(\mu) = GRI_\tau(f^{-1}(\mu), r)$, that is, $f^{-1}(\mu)$ is an r-grfo set in X.

□

Theorem 3.7. *Let (X, τ) and (Y, η) be fts's. Let $f : X \to Y$ be a mapping. Then the following statements are equivalent:*

(1) *f is grf-irresolute open.*
(2) *$f(GRI_\tau(\lambda, r)) \leq GRI_\eta(f(\lambda), r)$, for each $\lambda \in I^X$ and $r \in I_0$.*

(3) $GRI_\tau(f^{-1}(\mu), r) \leq f^{-1}(GRI_\eta(\mu, r))$, *for each* $\mu \in I^Y$ *and* $r \in I_0$.

(4) *For any* $\mu \in I^Y$ *and any* r*-grfc set* $\lambda \in I^X$ *with* $f^{-1}(\mu) \leq \lambda$, *there exists an* r*-grfc set* $\rho \in I^Y$ *with* $\mu \leq \rho$ *such that* $f^{-1}(\rho) \leq \lambda$.

Proof. (1)⇒(2) For each $\lambda \in I^X$, since $GRI_\tau(\lambda, r) \leq \lambda$ from Theorem 2.12(2), we have $f(GRI_\tau(\lambda, r)) \leq f(\lambda)$. From (1), $f(GRI_\tau(\lambda, r))$ is r-grfo. Hence $f(GRI_\tau(\lambda, r) \leq GRI_\eta(f(\lambda), r)$.

(2)⇒(3) Let (2) holds. Take $\lambda = f^{-1}(\mu),\ \lambda \in I^X$ and apply part (2),

$$f(GRI_\tau(f^{-1}(\mu), r)) \leq GRI_\eta(f(f^{-1}(\mu)), r) \leq GRI_\eta(\mu, r).$$

It implies $GRI_\tau(f^{-1}(\mu), r) \leq f^{-1}(GRI_\eta(\mu, r))$.

(3)⇒(4) Let λ be an r-grfc subset of X such that $f^{-1}(\mu) \leq \lambda$. Since $\overline{1} - \lambda \leq f^{-1}(\overline{1} - \mu)$ and $GRI_\tau(\overline{1} - \lambda, r) = \overline{1} - \lambda$,

$$GRI_\tau(\overline{1} - \lambda, r) = \overline{1} - \lambda \leq GRI_\tau(f^{-1}(\overline{1} - \mu), r).$$

From (3),

$$\overline{1} - \lambda \leq GRI_\tau(f^{-1}(\overline{1} - \mu), r) \leq f^{-1}(GRI_\eta(\overline{1} - \mu, r)).$$

It implies

$$\begin{aligned} \lambda &\geq \overline{1} - f^{-1}(GRI_\eta(\overline{1} - \mu, r)) \\ &= f^{-1}(\overline{1} - GRI_\eta(\overline{1} - \mu, r)) \\ &= f^{-1}(GRC_\eta(\mu, r)). \end{aligned}$$

Hence there exists an r-grfc set $GRC_\eta(\mu, r) \in I^Y$ with $\mu \leq GRC_\eta(\mu, r)$ such that $f^{-1}(GRC_\eta(\mu, r)) \leq \lambda$.

(4)⇒(1) Let ω be an r-grfo subset of X. Put $\mu = \overline{1} - f(\omega)$ and $\lambda = \overline{1} - \omega$ such that λ is r-grfc. We obtain

$$f^{-1}(\mu) = f^{-1}(\overline{1} - f(\omega)) = \overline{1} - f^{-1}(f(\omega)) \leq \overline{1} - \omega = \lambda.$$

From (4), there exists an r-grfc set ρ with $\mu \leq \rho$ such that $f^{-1}(\rho) \leq \lambda = \overline{1} - \omega$. It implies $\omega \leq \overline{1} - f^{-1}(\rho) = f^{-1}(\overline{1} - \rho)$. Thus, $f(\omega) \leq f(f^{-1}(\overline{1} - \rho)) \leq \overline{1} - \rho$. On the other hand, since $\mu \leq \rho$,

$$f(\omega) = \overline{1} - \mu \geq \overline{1} - \rho.$$

Hence $f(\omega) = \overline{1} - \rho$, that is, $f(\omega)$ is an r-grfo set.

□

The following theorem can be proved using the same argument as in the proof of Theorem 3.7.

Theorem 3.8. *Let* $(X,\ \tau)$ *and* $(Y,\ \eta)$ *be fts's. Let* $f : X \to Y$ *be a mapping. Then the following statements are equivalent:*

(1) *f is grf-irresolute closed.*
(2) *$f(GRC_\tau(\lambda, r)) \geq GRC_\eta(f(\lambda), r)$, for each $\lambda \in I^X$ and $r \in I_0$.*

Theorem 3.9. *Let $(X,\ \tau)$ and $(Y,\ \eta)$ be fts's. Let $f : X \to Y$ be a bijective mapping.*

(1) *f is grf-irresolute closed map iff for each $\mu \in I^Y$ and $r \in I_0$, $f^{-1}(GRC_\eta(\mu, r)) \leq GRC_\tau(f^{-1}(\mu), r)$.*
(2) *f is a grf-irresolute closed map iff f is grf-irresolute open map.*

Proof. (1) (Necessity): Let f be grf-irresolute closed. From Theorem 3.8, we have:

$$f(GRC_\tau(\lambda, r)) \geq GRC_\eta(f(\lambda), r).$$

Let $\mu \in I^Y$ and put $\lambda = f^{-1}(\mu)$, we have

$$\begin{aligned} f(GRC_\tau(f^{-1}(\mu), r)) &\geq GRC_\eta(f(f^{-1}(\mu)), r) \\ &= GRC_\eta(\mu, r). \end{aligned}$$

This implies

$$\begin{aligned} GRC_\tau(f^{-1}(\mu), r) &= f^{-1}\left(f(GRC_\tau(f^{-1}(\mu), r))\right) \\ &\geq f^{-1}(GRC_\eta(\mu, r)). \end{aligned}$$

(Sufficiency): On the other hand let the condition is satisfied and let $\mu \in I^X$ such that μ is r-grfc. Then $f(\mu) \in I^Y$. Applying the condition we have

$$GRC_\tau(f^{-1}f(\mu, r)) \geq f^{-1}(GRC_\eta(f(\mu), r)).$$

This implies that $GRC_\tau(\mu, r) \geq f^{-1}(GRC_\eta(f(\mu), r))$. Then,

$$f(GRC_\tau(\mu, r)) \geq GRC_\eta(f(\mu), r).$$

So by Theorem 3.8 f is grf-irresolute closed.

(2) Applying Theorem 3.8 and taking the complement we have the required result.

□

From the above theorems we obtain the following theorem

Theorem 3.10. *Let $f : (X,\ \tau) \to (Y,\ \eta)$ be a bijective mapping from a fts $(X,\ \tau)$ into a fts $(Y,\ \eta)$. For each $\lambda \in I^X,\ \mu \in I^Y$ and $r \in I_0$, the following statements are equivalent:*

(1) *f is grfi-homeomorphism.*
(2) *f is grf-irresolute and grf-irresolute open.*
(3) *f is grf-irresolute and grf-irresolute closed.*
(4) *$f(GRI_\tau(\lambda, r)) = GRI_\eta(f(\lambda), r)$.*
(5) *$f(GRC_\tau(\lambda, r)) = GRC_\eta(f(\lambda), r)$.*
(6) *$GRI_\tau(f^{-1}(\mu), r) = f^{-1}(GRI_\eta(\mu, r))$.*

(7) $GRC_\tau(f^{-1}(\mu), r) = f^{-1}(GRC_\eta(\mu, r))$.

Proof. (1)⇒(3) Let λ be an r-grfc set in X. Then $\overline{1} - \lambda$ is an r-grfo set in X. By Definition 3.1 (4), f^{-1} is grf-irresolute,

$$(f^{-1})^{-1}(\overline{1} - \lambda) = f(\overline{1} - \lambda) = \overline{1} - f(\lambda),$$

is an r-grfo set in Y. Then $f(\lambda)$ is an r-grfc set in Y. Thus f is a grf-irresolute closed function.

(3)⇒(5) It can be easily proved by Theorem 3.5 (3) and Theorem 3.8.

(5)⇒(7) From Theorems 3.5 (4) and 3.9 (1), we get the result.

(7)⇒(1) It follows from Theorem 3.9 (1) and Theorem 3.5.

Taking compliment to above implications we get the other implications. □

4. Applications

Definition 4.1. A fts (X, τ) is said to be:

(i) r-FRCO-T_1 if for each pair of distinct fuzzy points x_α and y_β of X there exist r-frco sets λ and μ containing x_α and y_β, respectively such that $x_\alpha \notin \mu$ and $y_\beta \notin \lambda$.

(ii) r-FRCO-T_2 if for each pair of distinct fuzzy points x_α and y_β of X there exist disjoint r-frco sets λ and μ in X such that $x_\alpha \in \lambda$ and $y_\beta \in \mu$.

(iii) r-GRF-T_1 if for each pair of distinct fuzzy points x_α and y_β of X there exist r-grfo sets λ and μ such that $x_\alpha \in \lambda$, $x_\alpha \notin \mu$ and $y_\beta \notin \lambda$, $y_\beta \in \mu$.

(iv) r-GRF-T_2 (r-GRF-Hausdorff) if for each pair of distinct fuzzy points x_α and y_β of X there exist disjoint r-grfo sets λ and μ in X such that $x_\alpha \in \lambda$ and $y_\beta \in \mu$.

Definition 4.2 ([5])**.** The mapping $f : (X, \tau) \to (Y, \eta)$ is called slightly generalized regular fuzzy continuous (for short, sgrf-continuous) if for each $\lambda \in I^X, \mu \in I^Y, r \in I_0$ such that μ is an r-frco set and $f(\lambda) \leq \mu$, there exist an r-grfo set $\nu \in I^X$ such that $\lambda \leq \nu$ and $f(\nu) \leq \mu$.

Definition 4.3. The mapping $f : (X, \tau) \to (Y, \eta)$ is called slightly generalized regular fuzzy continuous (for short, sgrf-continuous) if for each fuzzy point $x_\alpha \in X$ and each r-grfco set $\lambda \in I^Y$ containing $f(x_\alpha)$, there exists an r-grfo set $\mu \in I^X$ containing x_α such that $f(\mu) \leq \lambda$.

Theorem 4.4. *If $f : X \to Y$ is a sgrf-continuous injection and Y is r-FRCO-T_1, then X is r-GRF-T_1.*

Proof. Suppose that Y is r-FRCO-T_1. For any distinct fuzzy points x_α and y_β in X, there exist r-frco sets λ, μ in Y. By Theorem 2.13, clearly λ and μ are r-grfco sets such that $f(x_\alpha) \in \lambda$, $f(y_\beta) \notin \lambda$, $f(x_\alpha) \notin \mu$ and

$f(y_\beta) \in \mu$. Since f is sgrf-continuous, $f^{-1}(\lambda)$ and $f^{-1}(\mu)$ are r-grfo sets in X such that $x_\alpha \in f^{-1}(\lambda)$, $y_\beta \notin f^{-1}(\lambda)$, $x_\alpha \notin f^{-1}(\mu)$ and $y_\beta \in f^{-1}(\mu)$. This shows that X is r-GRF-T_1. □

Theorem 4.5. *If $f : X \to Y$ is a sgrf-continuous injection and Y is r-FRCO-T_2, then X is GRF-T_2.*

Proof. For any pair of distinct fuzzy points x_α and y_β in X, there exist disjoint r-frco sets λ and μ in Y. By Theorem 2.13, clearly λ and μ are r-grfco sets such that $f(x_\alpha) \in \lambda$ and $f(y_\beta) \in \mu$. Since f is sgrf-continuous, $f^{-1}(\lambda)$ and $f^{-1}(\mu)$ are r-grfo sets in X containing x_α and y_β respectively. We have $f^{-1}(\lambda) \wedge f^{-1}(\mu) = \phi$. This shows that X is r-GRF-T_2. □

Definition 4.6. A fuzzy space is called r-FRCO-regular (respectively strongly GRF-regular) if for each r-frco (respectively r-grfc) set η and each fuzzy point $x_\alpha \notin \eta$, there exist disjoint r-FO sets λ and μ such that $\eta \leq \lambda$ and $x_\alpha \in \mu$.

Definition 4.7. A fuzzy space is called r-FRCO-normal (respectively strongly GRF-normal) if for every pair of disjoint r-frco (respectively r-grfc) set η_1 and η_2 in X, there exist disjoint r-FO sets λ and μ such that $\eta_1 \leq \lambda$ and $\eta_2 \leq \mu$.

Theorem 4.8. *If f is sgrf-continuous injective fr-open function from a strongly GRF-regular space X onto a fuzzy space Y, then Y is r-FRCO-regular.*

Proof. Let η be an r-frco set in Y and be $y_\beta \notin \eta$. Take $y_\beta = f(x_\alpha)$. Since f is sgrf-continuous, $f^{-1}(\eta)$ is an r-grfc set. Take $\gamma = f^{-1}(\eta)$. We have $x_\alpha \notin \gamma$. Since X is strongly GRF-regular, there exist disjoint r-FO sets λ and μ such that $\gamma \leq \lambda$ and $x_\lambda \in \mu$. We obtain that $\eta = f(\gamma) \leq f(\lambda)$ and $y_\beta = f(x_\alpha) \in f(\mu)$ such that $f(\lambda)$ and $f(\mu)$ are disjoint r-fro sets [since f is a FR-open function]. This shows that Y is r-FRCO-regular. □

Theorem 4.9. *If f is sgrf-continuous injective FR-open function from a strongly GRF-normal space X onto a fuzzy space Y, then Y is r-FRCO-normal.*

Proof. Let η_1 and η_2 be disjoint r-frco sets in Y. Since f is sgrf-continuous, $f^{-1}(\eta_1)$ and $f^{-1}(\eta_2)$ are r-grfc sets. Take $\lambda = f^{-1}(\eta_1)$ and $\mu = f^{-1}(\eta_2)$. We have $\lambda \wedge \mu = \overline{0}$. Since X is strongly GRF-normal, there exist disjoint r-FO sets α and β such that $\lambda \leq \alpha$ and $\mu \leq \beta$. We obtain that $\eta_1 = f(\lambda) \leq f(\alpha)$ and $\eta_2 = f(\mu) \leq f(\beta)$ such that $f(\alpha)$ and $f(\beta)$ are disjoint r-fro sets [since f is a FR-open function]. Thus, Y is r-FRCO-normal. □

Definition 4.10. Let (X,τ) be a fts and $\lambda,\mu \in I^X$, $r \in I_0$. The two fuzzy sets λ and μ are said to be r-GRF-separated iff $\lambda \overline{q} GRC_\tau(\mu,r)$ and $\mu \overline{q} GRC_\tau(\lambda,r)$.

Definition 4.11. Let (X,τ) be a fts and $\lambda \in I^X$, $r \in I_0$. A fuzzy set λ is said to be an r-GRF-connected if it cannot be expressed as the union of two r-GRF-separated sets.

Theorem 4.12. *Let (X,τ) be a fts and $\lambda,\mu \in I^X$, $r \in I_0$.*

(1) *If λ,μ are r-GRF-separated and ν,η are non-null fuzzy sets such that $\nu \le \lambda$, $\eta \le \mu$, then ν,η are also r-GRF-separated.*
(2) *If $\lambda \overline{q} \mu$ and either both are r-grfo or both r-grfc, then λ and μ are r-GRF-separated.*
(3) *If λ,μ are either both r-grfo or both r-grfc, then $\lambda \wedge (\overline{1}-\mu)$ and $\mu \wedge (\overline{1}-\lambda)$ are r-GRF-separated.*

Proof. (1) and (2) are obvious.

(3) Let λ and μ be both r-grfo. Since $\lambda \wedge (\overline{1}-\mu) \le \overline{1}-\mu$, $GRC_\tau(\lambda \wedge (\overline{1}-\mu), r) \le \overline{1}-\mu$ and hence $GRC_\tau\left(\lambda \wedge (\overline{1}-\mu), r\right) \overline{q} \mu$. Then

$$GRC_\tau(\lambda \wedge (\overline{1}-\mu), r) \overline{q} (\mu \wedge (\overline{1}-\lambda)).$$

Again, since

$$\mu \wedge (\overline{1}-\lambda) \le \overline{1}-\lambda, \qquad GRC_\tau\left(\mu \wedge (\overline{1}-\lambda), r\right) \le \overline{1}-\lambda,$$

and hence $GRC_\tau(\mu \wedge (\overline{1}-\lambda), r) \overline{q} \lambda$. Then

$$GRC_\tau(\mu \wedge (\overline{1}-\lambda), r) \overline{q} (\lambda \wedge (\overline{1}-\mu)).$$

Thus $\lambda \wedge (\overline{1}-\mu)$ and $\mu \wedge (\overline{1}-\lambda)$ are r-GRF-separated.

Similarly we can prove when λ and μ are r-grfc.

□

Theorem 4.13. *Let (X,τ) be a fts and $r \in I_0$. The two non-null fuzzy sets λ and μ are r-GRF-separated iff there exist two r-grfo sets ν,ω such that $\lambda \le \nu$, $\mu \le \omega$, $\lambda \overline{q} \omega$ and $\mu \overline{q} \nu$.*

Proof. For two r-GRF-separated sets λ and μ, $\mu \le \overline{1} - GRC_\tau(\lambda,r) = \omega$ and $\lambda \le \overline{1} - GRC_\tau(\mu,r) = \nu$(say), where ω and ν are clearly r-grfo, then $\omega \overline{q} GRC_\tau(\lambda,r)$ and $\nu \overline{q} GRC_\tau(\mu,r)$. Thus, $\lambda \overline{q} \omega$ and $\mu \overline{q} \nu$.

Conversely, let ν and ω be r-grfo sets such that $\lambda \le \nu$, $\mu \le \omega$, $\lambda \overline{q} \omega$ and $\mu \overline{q} \nu$. Then $\lambda \le \overline{1}-\omega$, $\mu \le \overline{1}-\nu$. Hence $GRC_\tau(\lambda,r) \le \overline{1}-\omega$, $GRC_\tau(\mu,r) \le \overline{1}-\nu$, which imply that $GRC_\tau(\lambda,r)\overline{q}\mu$ and $GRC_\tau(\mu,r)\overline{q}\lambda$. Thus λ and μ are r-GRF-separated. □

Theorem 4.14. *Let (X,τ) be a fts, $r \in I_0$ and λ be a non-null r-GRF-connected set. If $\lambda \le \mu \le GRC_\tau(\lambda,r)$ then μ is also r-GRF-connected.*

Proof. Suppose that μ is not r-GRF-connected. Then there exist r-GRF-separated sets ω_1 and ω_2 in X such that $\mu = \omega_1 \vee \omega_2$. Let $\nu = \lambda \wedge \omega_1$ and $\omega = \lambda \wedge \omega_2$. Then $\lambda = \nu \vee \omega$. Since $\nu \leq \omega_1$ and $\omega \leq \omega_2$, by Theorem 4.12 (1), ν and ω are r-GRF-separated, contradicting the r-GRF-connectedness of λ. Thus μ is r-GRF-connected. □

Definition 4.15. A fuzzy set λ in a topological space (X, τ) is said to be r-GRF-connected if λ cannot be expressed as the union of two r-grfo sets.

Equivalently, a fuzzy topological space (X, τ) is said to be r-GRF-connected if $\overline{0}$ and $\overline{1}$ are the only fuzzy sets which are both r-grfo and r-grfc.

Theorem 4.16. *A fuzzy topological space (X, τ) is r-GRF-connected iff X has no non-zero r-grfo sets λ and μ such that $\lambda \vee \mu = \overline{1}$.*

Proof. (Necessity) Suppose (X, τ) is r-GRF-connected. If X has two non-zero r-grfo sets λ and μ such that $\lambda \vee \mu = \overline{1}$, then λ is a proper r-grfo and r-grfc subset of X. Hence, X is not r-GRF-connected, a contradiction.

(Sufficiency) If (X, τ) is not r-GRF-connected then it has a proper fuzzy subset λ of X which is both r-grfo and r-grfc. So $\mu = \overline{1} - \lambda$, is a r-grfo set of X such that $\lambda \vee \mu = \overline{1}$, which is a contradiction. □

Theorem 4.17. *If $f : (X, \tau) \to (Y, \sigma)$ is grf-continuous surjection and (X, τ) is r-GRF-connected, then (Y, σ) is r-fuzzy connected.*

Proof. Let X be a r-GRF-connected space and Y is not r-fuzzy connected. Then, there exists a proper fuzzy set λ of Y such that $\lambda \neq \overline{0}, \lambda \neq \overline{1}$ and λ is both r-FO and r-FC set. Since, f is grf-continuous, $f^{-1}(\lambda)$ is both r-grfo and r-grfc set in X such that $f^{-1}(\lambda) \neq \overline{0}$ and $f^{-1}(\lambda) \neq \overline{1}$. Hence, X is not r-GRF-connected, a contradiction. □

Theorem 4.18. *Let $f : X \to Y$ be a grf-irresolute mapping, $\lambda \in I^X$ and $r \in I_0$. If λ is r-GRF-connected set in X, then so is $f(\lambda)$ in Y.*

Proof. Suppose that $f(\lambda)$ is not r-GRF-connected in Y. Then there exist r-GRF-connected sets μ and ν in Y such that $f(\lambda) = \mu \vee \nu$. Since μ and ν are r-GRF-separated, by Theorem 4.13, there exists two r-grfo sets ω_1 and ω_2 such that $\mu \leq \omega_1$, $\nu \leq \omega_2$, $\mu \overline{q} \omega_2$ and $\nu \overline{q} \omega_1$. Now, since $f$is grf-irresolute, $f^{-1}(\omega_1)$ and $f^{-1}(\omega_2)$ are r-grfo sets in X and

$$\lambda \leq f^{-1} f(\lambda) = f^{-1}(\mu \vee \nu) = f^{-1}(\mu) \vee f^{-1}(\nu).$$

For $\mu \overline{q} \omega_2$ and $\nu \overline{q} \omega_1$, we have $\mu \leq \overline{1} - \omega_2$ and $\nu \leq \overline{1} - \omega_1$ i.e., $f^{-1}(\mu) \leq \overline{1} - f(\omega_2)$ and $f^{-1}(\nu) \leq \overline{1} - f(\omega_1)$. Hence $f^{-1}(\mu) \overline{q} f^{-1}(\omega_2)$ and $f^{-1}(\nu) \overline{q} f^{-1}(\omega_1)$. By Theorem 4.13, $f^{-1}(\mu)$ and $f^{-1}(\nu)$ are r-GRF-separated in X. Since

$\lambda = (\lambda \wedge f^{-1}(\mu)) \vee (\lambda \wedge f^{-1}(\nu))$, and $\lambda \wedge f^{-1}(\mu)$ and $\lambda \wedge f^{-1}(\nu)$ are r-GRF-separated in X, from Theorem 4.12(1), λ is not r-GRF-connected set. It is a contradiction. □

Theorem 4.19. *If $f : (X, \tau) \to (Y, \sigma)$ is grf-irresolute surjection and X is r-GRF-connected, then Y is so.*

Proof. Similar to the proof of the above Theorem 4.18. □

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