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Inverse Problem for Interior Spectral Data of the Dirac Operator with Discontinuous Conditions

Mohammad Shahriari[1 *], Reza Akbari[2], and Mostafa Fallahi[3]

Abstract. In this paper, we study the inverse problem for Dirac differential operators with discontinuity conditions in a compact interval. It is shown that the potential functions can be uniquely determined by the value of the potential on some interval and parts of two sets of eigenvalues. Also, it is shown that the potential function can be uniquely determined by a part of a set of values of eigenfunctions at an interior point and parts of one or two sets of eigenvalues.

1. Introduction

In the seminal paper, direct and inverse problems for Dirac operators with discontinuities inside an interval were investigated by Amirov in [2]. Furthermore, direct or inverse spectral problems for Dirac operators were extensively studied in [4, 5, 11–14, 16, 18], and the references therein.

In the seminal paper, Hald motivated by the inverse problem for the torsional modes of the earth, investigated Sturm–Liouville problems with a discontinuity at an interior point [6]. Hald proved a Hochstadt–Liebermann result [7] in the case of one transmission condition which was later on extended to two transmission conditions by Willis [17]. More recently, Shahriari and et al. [15] investigated the case with finite number of transmission conditions in Robin and eigenparameter dependent boundary conditions. Moreover, Kobayashi [8] proved a similar result in the case for problems with a reflection symmetry.

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* Corresponding author.

Boundary-value problems often appear in mathematics, mechanics, physics, geophysics, and other branches of natural sciences. The inverse problem of reconstructing the material properties of a medium from data collected outside the medium is of major importance in disciplines ranging from engineering to geosciences. [3, 6, 19] are well-known works about discontinuous inverse eigenvalue problems. Direct and inverse problems for Dirac operators with discontinuities inside an interval were investigated in [2].

In this manuscript, we study the inverse problem for Dirac differential operators with discontinuity conditions. It is shown that the potential functions can be uniquely determined by the value of the potential on some interval and parts of two sets of eigenvalues.

2. Preliminaries

Let us consider the system of differential equation

$$\ell[y(x)] := By'(x) + \Omega(x)y(x) = \lambda y(x), \quad x \in I := [0,d) \cup (d,\pi], \tag{2.1}$$

with

$$B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \Omega(x) = \begin{pmatrix} p(x) & q(x) \\ q(x) & -p(x) \end{pmatrix},$$

and $y(x) = (y_1(x), y_2(x))^T$ subject to the boundary conditions

$$\begin{aligned} U(y) &:= y_1(0)\cos\alpha + y_2(0)\sin\alpha = 0, \\ V(y) &:= y_1(\pi)\cos\beta + y_2(\pi)\sin\beta = 0, \end{aligned} \tag{2.2}$$

and the jump conditions

$$C(y) := y(d+0) - Ay(d-0) = 0, \tag{2.3}$$

with $A = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$. Throughout this paper $p(x)$ and $q(x)$ are real valued functions in $L_2(0,\pi)$, $a,\ b \in \mathbb{R} - \{0\}$, $ab > 0$, $\alpha,\ \beta \in [0,\pi)$ and λ is the spectral parameter. For simplicity we use the notation $L = L(\Omega(x); \alpha; \beta; a; b; d)$ for the above system of differential equation. To obtain a self-adjoint operator we introduce the following weight function

$$w(x) = \begin{cases} 1, & 0 \le x < d, \\ \frac{1}{ab}, & d < x \le \pi. \end{cases} \tag{2.4}$$

Now our Hilbert space will be $\mathcal{H} := L_2((0,\pi); w)$ associated with the weighted inner product

$$\langle f, g\rangle_{\mathcal{H}} := \int_0^\pi (f_1\bar{g}_1 + f_2\bar{g}_2)\, w, \tag{2.5}$$

where $f = (f_1,\, f_2)^T$ and $g = (g_1,\, g_2)^T \in L^2(0,\pi)$. The corresponding norm will be denoted by $\|f\|_{\mathcal{H}} = \langle f, f\rangle_{\mathcal{H}}^{1/2}$. In this Hilbert space, we construct the operator

$$A : \mathcal{H} \to \mathcal{H}, \tag{2.6}$$

with domain

$$\operatorname{dom}(D) = \left\{ f \in \mathcal{H} \,\middle|\, \begin{array}{l} f, f' \in AC\big((0,d) \cup (d,\pi)\big), \\ \ell f \in L^2(0,\pi),\ C(f) = 0 \end{array} \right\}, \tag{2.7}$$

by $Df = \ell f$ with $f \in \operatorname{dom}(D)$. Throughout this paper $AC\big((0,d) \cup (d,\pi)\big)$ denotes the set of all functions whose restriction to $(0,d)$ or (d,π) is absolutely continuous. In particular, those functions will have limits at the boundary point d. It is easy to see that the operator D is a self-adjoint operator by weighted inner product (2.5). In particular, the eigenvalues of D, and hence of L, are simple and real eigenvalues λ_n, for $n \in \mathbb{Z}$. From the linear differential equations we obtain the modified Wronskian

$$W(u,v) = w(x)\big(u(x)v'(x) - u'(x)v(x)\big), \tag{2.8}$$

is constant on $x \in [0,d) \cup (d,\pi]$ for two solutions $\ell u = \lambda u$, $\ell v = \lambda v$ satisfying the transmission conditions (2.3). We define the characteristic function for the operator L of the form

$$\Delta(\lambda) := w(\pi)V(\varphi(\lambda)). \tag{2.9}$$

The characteristic function $\Delta(\lambda)$ is independent of x.

Let the functions $\varphi(.,\lambda) : I \to \mathbb{R}^2$ be

$$\begin{aligned} &B\varphi'(x) + \Omega(x)\varphi(x) = \lambda\varphi(x), \\ &\varphi_1(0) = \sin\alpha, \qquad \varphi_2(0) = -\cos\alpha, \end{aligned} \tag{2.10}$$

with the jump conditions (2.3) and $\varphi(x,\lambda) = (\varphi_1(x,\lambda), \varphi_2(x,\lambda))^T$. It is shown in [2, 4, 11] and [14] that there exist kernels $K(x,t) = \Big(K_{ij}(x,t)_{i,j=1}^2\Big)$ with entire continuously differentiable on $0 \le t \le x < d$ such that the solution $\varphi(x,\lambda)$ is

$$\varphi(x,\lambda) = \varphi_\circ(x,\lambda) + \int_0^x K(x,t)\varphi_\circ(t,\lambda)dt. \tag{2.11}$$

Here

$$\begin{aligned} \varphi_\circ(x,\lambda) &= (\sin(\lambda x + \alpha), -\cos(\lambda x + \alpha))^T \\ &= (\varphi_{\circ 1}(x,\lambda),\, \varphi_{\circ 2}(x,\lambda))^T . \end{aligned}$$

Using the same calculation of papers [1, 19] and [14], we find that
(2.12)
$$\varphi_{\circ 1}(x,\lambda)=\begin{cases}\sin(\lambda x+\alpha), & 0\le x<d,\\ a^{+}\sin(\lambda x+\alpha)+a^{-}\sin(\lambda(2d-x)+\alpha), & d<x\le\pi,\end{cases}$$

and
(2.13)
$$\varphi_{\circ 2}(x,\lambda)=\begin{cases}-\cos(\lambda x+\alpha), & 0\le x<d,\\ -a^{+}\cos(\lambda x+\alpha)+a^{-}\cos(\lambda(2d-x)+\alpha), & d<x\le\pi,\end{cases}$$

with $a^{\pm}=\frac{1}{2}(a\pm b)$. The characteristic function for $(\varphi_{\circ 1}(x,\lambda),\ \varphi_{\circ 2}(x,\lambda))^T$ is

$$\begin{aligned}\Delta_\circ(\lambda):=&w(\pi)V(\varphi_\circ(\lambda))\\ =&\frac{1}{ab}\left(a^{+}\sin(\lambda\pi+\alpha-\beta)+a^{-}\sin(\lambda(2d-\pi)+\alpha+\beta)\right),\end{aligned}\tag{2.14}$$

the roots λ_n° of the entire function $\Delta_\circ(\lambda)$ are simple and real. The roots of $\Delta_\circ(\lambda)$ are

$$\lambda_n^\circ=n+M_n,$$

where $\sup_n M_n<M<\infty$.

Lemma 2.1. *The roots of the function $\Delta_\circ(\lambda)$ are in the following form*

$$\lambda_n^\circ=n-\frac{1}{2}+\frac{\beta-\alpha}{\pi}+\eta_n,$$

where $\eta_n\in(0,1)$, $\alpha,\ \beta\in[0,\pi)$ and $n\in\mathbb{Z}$.

Proof. The zeros of the entire function $\Delta_\circ(\lambda)$ are simple and real. Since $\Delta_\circ(\lambda)$ is type of "sine" [10], the number $\gamma_\delta>0$ exists such that for all n, $|\dot{\Delta}_\circ(\lambda_n^\circ)|\ge\gamma_\delta>0$, where

$$\dot{\Delta}_\circ(\lambda_n^\circ):=\frac{d}{d\lambda}\Delta(\lambda)\Big|_{\lambda=\lambda_n^\circ}.$$

We restrict the domain of $\Delta_\circ(\lambda)$ to the real line. By substituting the points $n-\frac{1}{2}+\frac{\beta-\alpha}{\pi}$ and $n+\frac{1}{2}+\frac{\beta-\alpha}{\pi}$ into $\Delta_\circ(\lambda)$ we see that

$$\Delta_\circ\left(n-\frac{1}{2}+\frac{\beta-\alpha}{\pi}\right)\Delta_\circ\left(n+\frac{1}{2}+\frac{\beta-\alpha}{\pi}\right)<0,\quad\text{for } n\in\mathbb{Z}.$$

According to continuity and differentiability of $\Delta_\circ(\lambda)$ there is a point, say η_n, in the interval $(0,1)$ such that

$$\Delta_\circ\left(n-\frac{1}{2}+\frac{\beta-\alpha}{\pi}+\eta_n\right)=0.$$

We show that there exists exactly one zero in $\left(n-\frac{1}{2}+\frac{\beta-\alpha}{\pi}, n+\frac{1}{2}+\frac{\beta-\alpha}{\pi}\right)$. Suppose that $d=\frac{p}{q}\pi$, where $\frac{p}{q}$ is a rational number in the interval $(0,1)$ and $\omega=e^{i\frac{\lambda\pi}{q}}$. By rewriting Eq. (2.14) in the form

$$\Delta_\circ(\lambda)=\frac{1}{2iab}\Big[b_1\Big(e^{i(\lambda\pi+\alpha-\beta)}-e^{-i(\lambda\pi+\alpha-\beta)}\Big) + b_2\Big(e^{i(\lambda(\frac{2p}{q}-1)\pi+\alpha+\beta)}+e^{-i(\lambda(\frac{2p}{q}-1)\pi+\alpha+\beta)}\Big)\Big], \tag{2.15}$$

and substituting $\omega=e^{i\frac{\lambda\pi}{q}}$ in (2.15), we see that $\Delta_\circ(\lambda)$ is a polynomial of degree $2q$ in term of ω. Since $\Delta_\circ(\lambda)$ is a periodic function with period $T=2q$, there are $2q$ zeros on the interval $\left(\frac{1}{2}+\frac{\beta-\alpha}{\pi},\ 2q+\frac{1}{2}+\frac{\beta-\alpha}{\pi}\right)$ and this shows that for each interval $\left(n-\frac{1}{2}+\frac{\beta-\alpha}{\pi},\ n+\frac{1}{2}+\frac{\beta-\alpha}{\pi}\right)$ there is exactly one zero. □

From (2.9) and (2.11) we get

$$\Delta(\lambda)=\Delta_\circ(\lambda)+O\left(\frac{\exp(|\tau|\pi)}{\lambda}\right), \tag{2.16}$$

where $\tau=\mathrm{Im}\lambda$. The zeros of $\Delta(\lambda)$ are the eigenvalues of L and hence it has only simple and real zeros λ_n. We denote by $y_n(x)=(y_1(x,\lambda_n),y_2(x,\lambda_n))^T$, for $n\in\mathbb{Z}$, the corresponding eigenfunction.

Theorem 2.2. *The corresponding eigenvalues $\{\lambda_n\}$ of the boundary value problem L admit the following asymptotic form as $n\to\infty$:*

$$\lambda_n=n-\frac{1}{2}+\frac{\beta-\alpha}{\pi}+\eta_n+O\left(\frac{1}{n}\right),$$

where $n\in\mathbb{Z}$, η_n, α and β are defined in Lemma 2.1.

Proof. Let $\lambda_n=\lambda_n^\circ+\epsilon_n$. Using (2.16) we obtain $\epsilon_n=O\left(\frac{1}{n}\right)$. □

We note that, the case $\alpha=0,\ \beta=\pi/2$ and $b=1/a, (a>0)$ the more general proof of Lemma 2.1 and Theorem 2.2 were given in [2].

3. Inverse Problems I

Let us introduce a second Dirac operator $\tilde{L}=\tilde{L}(\tilde{\Omega}(x);\alpha;\beta;d)$ here

$$\tilde{\Omega}(x)=\begin{pmatrix}\tilde{p}(x) & \tilde{q}(x)\\ \tilde{q}(x) & -\tilde{p}(x)\end{pmatrix},$$

with real valued functions $\tilde{p}(x),\ \tilde{q}(x)\in L^2(0,\ \pi)$. The eigenvalues and the corresponding eigenfunctions of $\tilde{L}$ are denoted by $\tilde{\lambda}_n$ and $\tilde{y}_n(x)=(\tilde{y}_{n,1}(x),\tilde{y}_{n,2}(x))^T(n\in\mathbb{Z})$, respectively.

Theorem 3.1. *Let $d \in (0, \frac{\pi}{2}]$ be a jump point and for each $n \in \mathbb{Z}$, $\lambda_n = \tilde{\lambda}_n$, and $\Omega(x) = \tilde{\Omega}(x)$ almost everywhere on $(d, \pi]$. Then $\Omega(x) = \tilde{\Omega}(x)$ almost everywhere on $[0, \pi]$.*

Proof. Let us denote by $\tilde{\varphi}$ the solution of the initial-value problems

$$\begin{aligned} &B\tilde{\varphi}'(x) + \tilde{\Omega}(x)\tilde{\varphi}(x) = \lambda\tilde{\varphi}(x), \\ &\tilde{\varphi}_1(0) = \sin\alpha, \quad \tilde{\varphi}_2(0) = -\cos\alpha, \end{aligned} \tag{3.1}$$

and the jump conditions (2.3). As the same as (2.11), there exist the kernels $\tilde{K}(x,t) = (\tilde{K}_{ij}(x,t)_{i,j=1}^2)$ with entire continuously differentiable on $0 \le t \le x < d$ such that the solution $\tilde{\varphi}(x, \lambda)$ is

$$\tilde{\varphi}(x,\lambda) = \varphi_\circ(x,\lambda) + \int_0^x \tilde{K}(x,t)\varphi_\circ(t,\lambda)dt. \tag{3.2}$$

Here $\varphi_\circ(x, \lambda)$ is defined in (2.12) and (2.13). Multiplying $\tilde{\varphi}^T$ by (2.11), φ^T by (3.2), and subtracting the result and integrating on $[0, d) \cup (d, \pi]$, we obtain

$$\begin{aligned} &\int_0^d \Big[(\Omega(x) - \tilde{\Omega}(x))\varphi(x,\lambda)\Big]^T \tilde{\varphi}(x,\lambda)w(x)dx \\ &\qquad = w(x)(\tilde{\varphi}_2(x,\lambda)\varphi_1(x,\lambda) - \tilde{\varphi}_1(x,\lambda)\varphi_2(x,\lambda))|_0^d|_d^\pi. \end{aligned}$$

Define

$$P(x) = \Omega(x) - \tilde{\Omega}(x), \qquad p_1(x) = p(x) - \tilde{p}(x), \qquad q_1(x) = q(x) - \tilde{q}(x),$$

and

$$H(\lambda) := \int_0^d [P(x)\varphi(x,\lambda)]^T \tilde{\varphi}(x,\lambda)w(x)\,dx. \tag{3.3}$$

From the conditions of this theorem, it follows from the assumptions that

$$H(\lambda_n) = 0, \quad n \in \mathbb{Z}.$$

We can show from (2.11) and (3.2) that

(3.4)

$$\begin{aligned} H(\lambda) = \int_0^d p_1(x)\Bigg[&-\cos 2(\lambda x + \alpha) + \int_0^x R_1(x,t)\exp(2i\lambda t)dt \\ &+ \int_0^x R_2(x,t)\exp(-2i\lambda t)dt\Bigg]dx + \int_0^d q_1(x)\Bigg[-\sin 2(\lambda x + \alpha) \\ &+ \int_0^x R_3(x,t)\exp(2i\lambda t)dt + \int_0^x R_4(x,t)\exp(-2i\lambda t)dt\Bigg]dx, \end{aligned}$$

where $R_i(x,t)$, $i = 1,\ldots,4$, are piecewise-continuously differentiable on $0 \le t \le x \le d$. Therefore it follows that $H(\lambda)$ is an entire function of order not greater than 1. We now claim that

$$H(\lambda) = 0, \tag{3.5}$$

on the whole λ-plane. Using (3.4) and the following inequality

$$|\cos(2\lambda x)| \le \exp(2x|\tau|), \tag{3.6}$$

we see that

$$|H(\lambda)| \le C\exp(2d|\tau|) \tag{3.7}$$

for $|\lambda|$ large enough and some positive constant C. Fix $\delta > 0$ and define $G_\delta := \{\lambda :\ |\lambda - \lambda_n| \ge \delta\}$. Then (see [2, 3])

$$|\Delta(\lambda)| \ge C_1 \exp(|\tau|\pi), \quad \lambda \in G_\delta, \tag{3.8}$$

for some constant $C_1 > 0$. Define

$$\phi(\lambda) := \frac{H(\lambda)^{2p}}{\Delta(\lambda)^q},$$

where p and q are defined in Lemma 2.1. The definition of $\Delta(\lambda)$ and $H(\lambda)$ implies $\phi(\lambda)$ is an entire function of order not greater than 1. It follows from (3.7) and (3.8) that $\phi(\lambda)$ is bounded for all λ-plan. Then it follows from the Phragmen-Lindelöf's and Liouville's Theorem that $\phi(\lambda) = M$ is constant on the whole λ-plan. We can rewrite the equation $H(\lambda)^{2p} = M\Delta(\lambda)^q$ in the form

$$\begin{aligned}
&\Bigg[\int_0^d p_1(x)\Bigg(-\cos 2(\lambda x+\alpha) + \int_0^x R_1(x,t)\exp(2i\lambda t)dt \\
&\qquad + \int_0^x R_2(x,t)\exp(-2i\lambda t)dt\Bigg)dx + \int_0^d q_1(x)\Bigg(-\sin 2(\lambda x+\alpha) \\
&\qquad + \int_0^x R_3(x,t)\exp(2i\lambda t)dt + \int_0^x R_4(x,t)\exp(-2i\lambda t)dt\Bigg)dx\Bigg]^{2p} \\
&\quad = M\Bigg[a^+\sin(\lambda\pi+\alpha-\beta) + a^-\sin(\lambda(2d-\pi)+\alpha+\beta) \\
&\qquad\quad + O\left(\frac{\exp(|\tau|\pi)}{\lambda}\right)\Bigg]^q.
\end{aligned}$$

Using the Rimann–Lebesque Lemma, the left side of the above equality tends to 0 az $\lambda \to \infty$, $\lambda \in \mathbb{R}$. Thus we obtain that $M = 0$, so $H(\lambda) = 0$

for all λ. We are going to show that $P(x)=0$ a.e. on $[0,d)$. From definition of $H(\lambda)$ we have

$$\begin{aligned}H(\lambda)=&\int_0^d f_1(x)\Big[\exp(2i\lambda x)+\int_0^x S_{11}(x,t)\exp(2i\lambda t)dt\\&+\int_0^x S_{12}(x,t)\exp(-2i\lambda t)dt\Big]dx+\int_0^d f_2(x)\Big[\exp(-2\lambda ix)\\&+\int_0^x S_{21}(x,t)\exp(2i\lambda t)dt+\int_0^x S_{22}(x,t)\exp(-2i\lambda t)dt\Big]\,dx\\=&\,0,\end{aligned}$$

where

$$f_1(x)=-\frac{\exp(2i\alpha)}{2i}(q_1(x)+ip_1(x)),\quad f_2(x)=\frac{\exp(-2i\alpha)}{2i}(q_1(x)-ip_1(x)),$$

and $S(x,t)=(S_{ij}(x,t))$, $i,j=1,2$, is a matrix which all its entries are piecewise–continuously differentiable on $0\le t\le x\le d$. This can be rewritten as

$$\begin{aligned}&\int_0^d \exp(2i\lambda s)\left[f_1(s)+\int_s^d (f_1(x)S_{11}(x,s)+f_2(x)S_{21}(x,s))dx\right]ds\\&\quad+\int_0^d \exp(-2i\lambda s)\left[f_2(s)+\int_s^d (f_1(x)S_{12}(x,s)+f_2(x)S_{22}(x,s)dx\right]ds\\&=0,\end{aligned}$$

or

$$\int_0^d e_0(\lambda s)^T\left[f(s)+\int_s^d S(x,s)f(x)dx\right]ds=0.$$

Here $e_0(x)=(\exp(2ix),\,\exp(-2ix))^T$ and $f(x)=(f_1(x),\,f_2(x))^T$. From the completeness of $e_0(\lambda s)$ in $\{L^2(0,d)\}^2$, it follows that

$$f(s)+\int_s^d S(x,s)f(x)dx=0,\quad 0<s<b.$$

But this equation is a homogeneous Volterra integral equation and therefore it has only the zero solution. Thus $f(x)=(f_1(x),\,f_2(x))^T=0$ on $0<x<d$, that is, $p_1(x)=q_1(x)=0$ a.e. on $[0,d]$. □

Let $l(n)$ be a subsequence of natural numbers such that

$$l(n)=\frac{n}{\sigma}(1+\epsilon_n),\quad 0<\sigma\le 1,\quad \epsilon_n\to 0,$$

and let μ_n be the eigenvalues of the problem (2.10) and (3.9) and $\tilde{\mu}_n$ be the eigenvalues of the problem (3.1) and (3.9) with the jump conditions (2.3) such that

$$y_1(\pi)\cos\gamma+y_2(\pi)\sin\gamma=0, \tag{3.9}$$

where $\gamma \in [0,\pi)$, $\beta-\gamma \neq k\pi$, and $k \in \mathbb{Z}$.

Theorem 3.2. *Let $d \in (\frac{\pi}{2},\pi]$ be a jump point and $\sigma > \frac{2d}{\pi}-1$. Let $\lambda_n = \tilde{\lambda}_n$ and $\mu_{l(n)} = \tilde{\mu}_{l(n)}$ for each $n \in \mathbb{Z}$, $\Omega(x) = \tilde{\Omega}(x)$ a.e. on $(d,\pi]$. Then $\Omega(x) = \tilde{\Omega}(x)$ a.e. on $[0,\pi]$.*

Proof. From (3.3) and assumptions we get

$$H(\lambda_n) = 0, \qquad H(\mu_{l(n)}) = 0.$$

Now, we show that $H(\lambda) = 0$, for all $\lambda \in \mathbb{C}$. From (3.3) and (3.7) we see that the entire function $H(\lambda)$ is a function of exponential type and

$$|H(\lambda)| \leq Me^{2dr|\sin\theta|}, \tag{3.10}$$

where M is a positive number and $\lambda = re^{i\theta}$. Define the indicator of function $H(\lambda)$ by

$$h(\theta) = \limsup_{\lambda\to+\infty} \frac{\ln|H(re^{i\theta})|}{r}. \tag{3.11}$$

Since $|\mathrm{Im}\lambda| = r|\sin\theta|$ and $\theta = \arg\lambda$, from (3.10) and (3.11), we obtain

$$h(\theta) = 2d|\sin\theta|. \tag{3.12}$$

Let $n(r)$ be the number of zeros of $H(\lambda)$ in the disk $|\lambda| \leq r$. From Lemma 2.1 and Theorem 2.2 we see that there are $1+2r[1+o(1)]$ of λ_n and $1+2r\sigma[1+o(1)]$ of $\mu_{l(n)}$ located inside the disc of radius r. Therefore

$$n(r) = 2 + 2r[1+\sigma+o(1)].$$

Hence

$$\lim_{n\to\infty} \frac{n(r)}{r} = 2(\sigma+1).$$

Using the condition $\sigma > \frac{2d}{\pi}-1$ and from (3.12), we get

$$\lim_{n\to\infty} \frac{n(r)}{r} \geq 2(\sigma+1) > \frac{4d}{\pi} \geq \frac{1}{2\pi}\int_0^{2\pi} h(\theta)d\theta. \tag{3.13}$$

According to [9], for any entire function $H(\lambda)$ of exponential type, not identically zero, we see that the following inequality holds:

$$\liminf_{n\to\infty} \frac{n(r)}{r} \leq \frac{1}{2\pi}\int_0^{2\pi} h(\theta)d\theta. \tag{3.14}$$

From the inequalities (3.13) and (3.14), the relation (3.5) holds. By applying the similar method of the proof of Theorem 3.1, we obtain $\Omega(x) = \tilde{\Omega}(x)$ a.e. on $[0,\pi]$. □

Let $m(n)$ be a subsequence of natural numbers such that

$$m(n) = \frac{n}{\sigma_1}(1+\epsilon_{1n}), \quad 0 < \sigma_1 \leq 1, \quad \epsilon_{1n} \to 0. \tag{3.15}$$

Corollary 3.3. *Let $d \in (0, \frac{\pi}{2}]$ be a jump point and $\sigma_1 > \frac{2d}{\pi}$. Let $\lambda_{m(n)} = \tilde{\lambda}_{m(n)}$ for each $n \in \mathbb{Z}$, $\Omega(x) = \tilde{\Omega}(x)$ almost everywhere on $(d, \pi]$. Then $\Omega(x) = \tilde{\Omega}(x)$ a.e. on $[0, \pi]$.*

Proof. By using Theorems 3.1 and 3.2 we can easily prove this corollary. □

Corollary 3.4. *Let $d \in (0, \pi)$ be a jump point, $\sigma_1 > \frac{2d}{\pi}$, and $\sigma > \frac{2d}{\pi} - 1$. Let $\lambda_{m(n)} = \tilde{\lambda}_{m(n)}$ and $\mu_{l(n)} = \tilde{\mu}_{l(n)}$ for each $n \in \mathbb{N}$, $q(x) = \tilde{q}(x)$ a.e. on $(d, \pi]$. Then $q(x) = \tilde{q}(x)$ a.e. on $[0, \pi]$.*

Proof. Using Theorems 3.1, 3.2 and Corollary 3.3 we can easily prove Corollary 4.7. □

4. Inverse Problem II

In this section, by the similar definition of Section 3, we consider the second Dirac operator $\tilde{L} = \tilde{L}(\tilde{\Omega}(x); \alpha; \beta; d)$. So we have the new inverse problem of the following form:

Theorem 4.1. *If*

$$\lambda_n = \tilde{\lambda}_n, \qquad W\left(y_n, \tilde{y}_n\right)_{d-0} = 0,$$

for any $n \in \mathbb{Z}$ and $d \leq \frac{\pi}{2}$ then $p(x) = \tilde{p}(x)$, $q(x) = \tilde{q}(x)$ a.e. on the $[0, d)$.

Proof. Using the similar proof of Theorem 3.1 we obtain this result. □

Remark 4.2. We can easily obtain if y and z are the solutions of (2.1) and satisfy the jump conditions (2.3) and $W(y, z)_{(d-0)} = 0$ then $W(y, z)_{(d+0)} = 0$.

Corollary 4.3. *Let $d \in (\frac{\pi}{2}, \pi)$ be a jump point. Let $\lambda_n = \tilde{\lambda}_n$, and $W(y_n, \tilde{y}_n)_{(d-0)} = 0$, for each $n \in \mathbb{Z}$. Then $\Omega(x) = \tilde{\Omega}(x)$ a.e. on $(d, \pi]$.*

Proof. To prove that $\Omega(x) = \tilde{\Omega}(x)$ a.e. on $(d, \pi]$, we will consider the supplementary problem $\widehat{L}$ by changing x by $\pi - x$. By using the similar proof of Theorem 4.1 we prove this theorem. □

Remark 4.4. For $d = \frac{\pi}{2}$ from Theorems 4.1 and Corollary 4.3, we get $\Omega(x) = \tilde{\Omega}(x)$ a.e. on $[0, \pi]$.

Theorem 4.5. *Let $d \in (\frac{\pi}{2}, \pi]$ be a jump point and $\sigma > \frac{2a}{\pi} - 1$. Let*

$$\lambda_n = \tilde{\lambda}_n, \qquad \mu_{l(n)} = \tilde{\mu}_{l(n)}, \qquad W(y_n, \tilde{y}_n)_{(d-0)} = 0,$$

for each $n \in \mathbb{Z}$. Then $\Omega(x) = \tilde{\Omega}(x)$ a.e. on $[0, d) \cup (d, \pi]$. Note that $l(n)$ is defined in (4.1).

Proof. By using the similar proof of Theorem 3.2 we obtain this result. ☐

Corollary 4.6. *Let* $d \in (0, \frac{\pi}{2}]$ *be a jump point and* $\sigma_1 > \frac{2d}{\pi}$. *Let* $\lambda_{m(n)} = \tilde{\lambda}_{m(n)}$ $W(y_n, \tilde{y}_n)_{d-0} = 0$, *for each* $n \in \mathbb{Z}$. *Then* $\Omega(x) = \tilde{\Omega}(x)$ *a.e. on* $[0, \pi]$.

Proof. By using Theorems 4.1 and 4.5, we can easily prove this corollary. ☐

Let $r(n)$ be a subsequence of natural numbers such that

$$r(n) = \frac{n}{\sigma_2}(1 + \epsilon_{2n}), \quad 0 < \sigma_2 \leq 1, \epsilon_{2n} \to 0. \tag{4.1}$$

Corollary 4.7. *Let* $a \in (\frac{\pi}{2}, \pi)$ *be a jump point, fix* $\sigma > \frac{2a}{\pi} - 1$ *and* $\sigma_2 > 2 - \frac{2a}{\pi}$. *If for each* $n \in \mathbb{N}$

$$\lambda_n = \tilde{\lambda}_n, \qquad \mu_{l(n)} = \tilde{\mu}_{l(n)}, \qquad W(y_{r(n)}, \tilde{y}_{r(n)})_{(a-0)} = 0,$$

then $\Omega(x) = \tilde{\Omega}(x)$ *a.e. on* $[0, \pi]$.

Proof. Using the similar proof of Theorems 4.1, 4.5, and Corollary 4.6 we obtain easily the result of this corollary. ☐

5. Conclusion

In this paper, the inverse Dirac differential operator with a transmission and Robin boundary conditions was studied. For this purpose, a new Hilbert space by defining a new inner product for obtaining a self–adjoint operator was defined. So, the asymptotic form of solutions, eigenvalues and eigenfunctions of this problem was obtained. Finally, we formulated two types of inverse problems for Dirac operator based on [6, 7, 18].

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[1] Department of Mathematics, Faculty of Science, University of Maragheh, P.O. Box 55136-553, Maragheh, Iran.
E-mail address: `shahriari@maragheh.ac.ir`

[2] Department of Mathematical Sciences, Payame Noor University, Iran.
E-mail address: `r9reza@yahoo.com`

[3] Department of Mathematics, Faculty of Science, University of Maragheh, P.O. Box 55136-553, Maragheh, Iran.
E-mail address: `fallahi1mostafa@yahoo.com`