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The Uniqueness Theorem for the Solutions of Dual Equations of Sturm-Liouville Problems with Singular Points and Turning Points

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Abstract. In this paper, linear second-order differential equations of Sturm-Liouville type having a finite number of singularities and turning points in a finite interval are investigated. First, we obtain the dual equations associated with the Sturm-Liouville equation. Then, we prove the uniqueness theorem for the solutions of dual initial value problems.

1. Introduction

This paper focuses on the boundary value problem $L := L\left(q\left(t\right), \phi^2\left(t\right), s\right)$ generated by

$$-y'' + q\left(t\right) y = \lambda \phi^2\left(t\right) y, \quad t \in [0,1], \tag{1.1}$$

and the boundary conditions

$$y\left(0,\lambda\right) = 1, \qquad y'\left(0,\lambda\right) = 0 = y(s,\lambda) = 0, \tag{1.2}$$

where λ is a complex spectral-parameter, the potential q and the weight function ϕ^2 are real, and $s \in (0,1)$. (We write the weight function as squared to simplify the formulas in the next sections.) Moreover, let for sufficiently small fixed $\varepsilon > 0$, $I_{1,\varepsilon} := [0, t_2 - \varepsilon]$ and $I_{2,\varepsilon} := [t_1 + \varepsilon, 1]$, we assume that for $\nu = 1, 2$, the functions

$$\phi_{\nu,0} : \ I_{\nu,\varepsilon} \to R, \qquad \phi_{\nu,0}\left(t\right) := \left(t - t_\nu\right)^{-1} \phi^2\left(t\right),$$

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are not vanishing and realanalytic, and for $t \in I_{\nu,\varepsilon}$, $t \neq t_\nu$, the function $q(t)$ has the form

$$q(t) = A_\nu (t - t_\nu)^{-1} + q_0(t),$$

where A_ν is positive constant, and $q_0(t)$ is a bounded real-analytic function.

There are several methods for studying the inverse spectral problems. Most methods lead to a proof of the uniqueness of the potential function. For example, the *transformation operator method* which was first used for the interval $(-\infty, +\infty)$ or an finite interval, was referred by Marchenco (for more details see [10]). Using this method, Gelfand and Levitan provided necessary and sufficient conditions to determine the Sturm-Liouville operator [6]. Further, recently, in [11] we used this method for investigating an inverse Sturm-Liouville problem on finite intervals. Poschel and Trubowitz studied the inverse spectral theory in [14]. Freiling and Yurko [4, 5] used the *Weyl function method*, and solved the most of the inverse problems of recovering the operator. Using the concept of the Weyl function and its generalizations, they formulated and studied inverse problems for various classes of operators (for other examples, see also [13, 17–19, 22, 23]). Recently, some authors used the *nodal points* of boundary value problems to study the inverse problems. Some applications of this method were given in [2, 3, 7, 8, 16, 20, 21]. In [1], Barcilon presented an other approach to obtain the solution of the inverse problem associated with the equation (1.1) with $q(t) = 0$ on the interval $[0, L]$, under the boundary conditions $y(0) = 0 = y(L)$ for the vibrating string, and investigated the density of $\phi^2(t)$ given the length L and two spectra $\{\lambda_n\}$ and $\{\mu_n\}$ which satisfying the property $0 < \lambda_1 < \mu_1 < \lambda_2 < \mu_2 < \cdots$. The spectrum $\{\lambda_n\}$ is associated with the boundary value problem consisting of the equation (1.1) and the boundary condition $y(0) = 0 = y(L)$, and $\{\mu_n\}$ is associated with (1.1) and the boundary condition $y'(0) = 0 = y(L)$. He obtained the spectrum $\{\lambda_n(t)\}$ and $\{\mu_n(t)\}$ on the interval $[t, L]$ for $t \in [0, L)$, and by using the factorization of the solution in terms of these spectral information, the following dual equations were derived:

$$\frac{d\lambda_n^2}{dt} = -\frac{\lambda_n^2(t) \prod_{i\geq 1} \left(1 - \frac{\lambda_n^2(t)}{\mu_i^2(t)}\right)}{(L-t) \prod_{i\geq 1, i\neq n} \left(1 - \frac{\lambda_n^2(t)}{\lambda_i^2(t)}\right)},$$

$$\frac{d\mu_n^2}{dt} = \mu_n^4(t)\, \phi^2(t)\, (L-t) \frac{\prod_{i\geq 1} \left(1 - \frac{\mu_n^2(t)}{\lambda_i^2(t)}\right)}{\prod_{i\geq 1, i\neq n} \left(1 - \frac{\mu_n^2(t)}{\mu_i^2(t)}\right)}.$$

Then he proved the uniqueness theorem for the solution of the dual equation by a special technique. For another example, in the case when equation (1.1) has one turning point of order $4m+1$, see [8].

In studying inverse spectral problems with singularities and turning points it is needed to investigate the uniqueness theoerm. In this paper, first we present the asymptotic form of the eigenvalues of (1.1)-(1.2), and prove some properties of them. Then, using the canonical form of the solution of L, we obtain the dual equations associated with equation (1.1), and prove the uniqueness theorem for the solution of the dual problems.

2. The Eigenvalues and the Canonical Solution

We know from [12] that in the sector

$$S := \left\{ \sqrt{\lambda} \middle| -\frac{\pi}{4} \leq \arg(\sqrt{\lambda}) \leq 0 \right\},$$

for $0 < t < t_1$, the boundary value problem L has a countable set of the eigenvalues $\{\lambda_n(t)\}_{n=1}^{\infty}$ of the form

$$\lambda_n(t) = \frac{n^2\pi^2}{f_0^2(t)} - \frac{n\pi^2}{f_0^2(t)} + O(1),$$

where $f_0(t) = \int_0^t |\phi(x)|dx$, and for $t \in \omega_j$, $j = 1, 2$, the spectrum of the boundary value problem L consists of two sequences of negative and positive eigenvalues $\{\mu_{n,j}(t)\}_{n=1}^{\infty}$, $\{\eta_{n,j}(t)\}_{n=1}^{\infty}$ as

$$\begin{cases} \mu_{n,j}(t) = -\frac{n^2\pi^2}{f_j^2(t)} + \frac{n\pi^2}{2f_j^2(t)} + O(1), \\ \eta_{n,j}(t) = \frac{n^2\pi^2}{g_j^2(t)} - \frac{n\pi^2}{2g_j^2(t)} + O(1), \end{cases} \tag{2.1}$$

where $\omega_1 = (t_1, t_2)$, $\omega_2 = (t_2, 1)$,

$$f_1(t) = \int_{t_1}^{t} |\phi(x)|dx,$$
$$f_2(t) = \int_{t_1}^{t_2} |\phi(x)|dx,$$
$$g_1(t) = \int_{0}^{t_1} |\phi(x)|dx,$$
$$g_2(t) = \int_{t_2}^{t} |\phi(x)|dx.$$

Furthermore, since for any fixed $t \in (0, t_1)$, $y(t,\lambda)$ is an entire function of the parameter λ of order $\frac{1}{2}$, it can be represented in the form

$$y(t,\lambda) = f(t) \prod_{n \geq 1} \left(1 - \frac{\lambda}{\lambda_n(t)}\right), \tag{2.2}$$

where $f(t)$ is a function independent of λ but may depend on t, and $\lambda_n(t)$, $n \geq 1$, are the eigenvalues of L on $[0,t]$ for $0 < t < t_1$. Moreover,

$$y(t,\lambda) = \frac{1}{2} |\phi(0)|^{\frac{1}{2}} |\phi(t)|^{-\frac{1}{2}} \prod_{n \geq 1} \frac{(\lambda_n(t) - \lambda) R_+^2(t)}{\zeta_n^2}, \tag{2.3}$$

where

$$R_+(t) = \int_0^t \sqrt{\max\{0, \phi^2(x)\}}\, dx,$$

and $\zeta_n,\ n = 1, 2, \ldots$, are the positive zeros of $J'_{\frac{1}{2}}$, derivative of the Bessel function of order $\frac{1}{2}$. Similarly, for $t \in \omega_j$, $j = 1, 2$, $y(t,\lambda)$ is an entire function of the parameter λ of order $\frac{1}{2}$. Thus, for $t \in \omega_j$, $y(t,\lambda)$ is the form

$$y(t,\lambda) = h_j(t) \prod_{n \geq 1} \left(1 - \frac{\lambda}{\mu_{n,j}(t)}\right) \prod_{n \geq 1} \left(1 - \frac{\lambda}{\eta_{n,j}(t)}\right), \tag{2.4}$$

where $h_j(t)$ is a function independent of λ but may depend on t. Furthermore, we have

$$\begin{aligned} y(t,\lambda) = {} & \frac{\pi}{8} \left\{ |\phi(0)| |\phi(t)|^{-1} R_-(t) R_+(t) \right\}^{\frac{1}{2}} \\ & \times \prod_{n \geq 1} \frac{(\lambda - \mu_{n,j}(t)) R_-^2(s_j)}{\widetilde{j}_n^2} \prod_{n \geq 1} \frac{(\eta_{n,j}(t) - \lambda) R_+^2(z_j)}{\widetilde{j}_n^2}, \end{aligned} \tag{2.5}$$

where

$$R_-(t) = \int_0^t \sqrt{\max\{0, -\phi^2(x)\}}\, dx,$$

$s_1 = z_2 = t$, $z_2 = t_1$, $s_2 = t_2$, and $\widetilde{j}_n,\ n = 1, 2, \ldots$, are the positive zeros of J'_1. The representations (2.3) and (2.5) are called the *canonical form* of the solution $y(t,\lambda)$.

In the following lemma, some properties of the eigenvalues of L are proved.

Lemma 2.1. *Let $\{\eta_{n,j}(t)\}_{n=1}^{\infty}$, $\{\mu_{n,j}(t)\}_{n=1}^{\infty}$, $j = 1, 2$, be the positive and negative eigenvalues of the boundary value problem L on $(t_1, t_2) \cup (t_2, 1)$, respectively. Then, for $j = 1, 2$,*

$$\eta_{n,j}(t) \sum_{k \neq n, k \geq 1} (\eta_{k,j}(t) - \eta_{n,j}(t))^{-1} = O(1), \tag{2.6}$$

$$\mu_{n,j}(t)\sum_{k\neq n,k\geq 1}\left(\mu_{k,j}(t)-\mu_{n,j}(t)\right)^{-1}=O(1), \tag{2.7}$$

$$\begin{cases} \eta_{n,j}(t)\sum\limits_{k\neq n,k\geq 1}\eta'_{k,j}(t)\left\{\eta^2_{k,j}(t)-\eta_{k,j}(t)\,\eta_{n,j}(t)\right\}^{-1}=O(1), \\ \eta_{n,j}(t)\sum\limits_{k\geq 1}\mu'_{k,j}(t)\left\{\mu^2_{k,j}(t)-\mu_{k,j}(t)\,\eta_{n,j}(t)\right\}^{-1}=O(1). \end{cases} \tag{2.8}$$

Proof. From (2.1), we get for $t\in(t_1,t_2)\cup(t_2,1)$,

$$\eta_{n,j}(t)=\frac{n^2\pi^2}{g_j^2(t)}-\frac{n\pi^2}{2g_j^2(t)}+O(1),\quad j=1,2.$$

Thus,

$$\begin{aligned}\frac{g_j^2(t)}{\pi^2}\left(\eta_{k,j}(t)-\eta_{n,j}(t)\right)&=\left(k^2-n^2\right)-\left(\frac{k-n}{2}\right)+O(1)\\&=\left(k^2-n^2\right)\left\{1-\frac{k-n+O(1)}{2\left(k^2-n^2\right)}\right\}.\end{aligned}$$

So, we can write

$$\begin{aligned}\frac{\pi^2}{g_j^2(t)}\left(\eta_{k,j}(t)-\eta_{n,j}(t)\right)^{-1}=&\frac{2(k+n)+1}{2\left(k^2-n^2\right)(k+n)}\\&+\left\{\frac{\pi\left(k-n+O(1)\right)}{2g_j(t)\left(k^2-n^2\right)}\right\}^2\left(\eta_{k,j}(t)-\eta_{n,j}(t)\right)^{-1}\\&+\frac{O(1)}{2\left(k^2-n^2\right)^2}.\end{aligned}$$

Hence,

$$\begin{aligned}\sum_{k\neq n,k\geq 1}&\pi^2\left\{g_j^2(t)\left(\eta_{k,j}(t)-\eta_{n,j}(t)\right)\right\}^{-1}\\&=\sum_{k\neq n,k\geq 1}\frac{2(k+n)+1}{2\left(k^2-n^2\right)(k+n)}\\&\quad+\sum_{k\neq n,k\geq 1}\left\{\frac{\pi\left(k-n+O(1)\right)}{2g_j(t)\left(k^2-n^2\right)}\right\}^2\left(\eta_{k,j}(t)-\eta_{n,j}(t)\right)^{-1}\\&\quad+\sum_{k\neq n,k\geq 1}\frac{O(1)}{2\left(k^2-n^2\right)^2}.\end{aligned} \tag{2.9}$$

On the other hand, we know from [15] that

$$\sum_{k\neq n,k\geq 1}\frac{1}{k^2-n^2}=\frac{3}{4n^2},\qquad \sum_{k\neq n,k\geq 1}\frac{1}{\left(k^2-n^2\right)^2}=O\left(\frac{1}{n^2}\right). \tag{2.10}$$

Therefore, we derive

$$\left\{\begin{array}{l}\sum\limits_{k\neq n,k\geq 1}\frac{O(1)}{2(k^2-n^2)^2}=O\left(n^{-2}\right),\\ \sum\limits_{k\neq n,k\geq 1}\left\{\frac{\pi(k-n+O(1))}{2g_j(t)(k^2-n^2)}\right\}^2=O\left(n^{-3}\right).\end{array}\right. \tag{2.11}$$

Moreover, it follows from

$$1\leq\frac{2\left(k+n\right)+1}{k+n}\leq\frac{5}{2},$$

that

$$\begin{aligned}\sum_{k\neq n,k\geq 1}\frac{1}{k^2-n^2} &\leq \sum_{k\neq n,k\geq 1}\frac{2\left(k+n\right)+1}{2\left(k^2-n^2\right)\left(k+n\right)}\\ &\leq \sum_{k\neq n,k\geq 1}\frac{5}{4\left(k^2-n^2\right)}.\end{aligned} \tag{2.12}$$

Thus, according to (2.10) and (2.12) we obtain

$$\sum_{k\neq n,k\geq 1}\frac{2\left(k+n\right)+1}{2\left(k^2-n^2\right)\left(k+n\right)}=O\left(n^{-2}\right). \tag{2.13}$$

The relation (2.13) together with (2.9) and (2.11) yields

$$\sum_{k\neq n,k\geq 1}\left(\eta_{k,j}-\eta_{n,j}\right)^{-1}=O\left(n^{-2}\right).$$

Thus, since $\eta_{n,j}=O\left(n^2\right)$, we conclude that

$$\eta_{n,j}\left(t\right)\sum_{k\neq n,k\geq 1}\left(\eta_{k,j}\left(t\right)-\eta_{n,j}\left(t\right)\right)^{-1}=O\left(1\right),\quad j=1,2.$$

Therefore, (2.6) is proved. By a similar method, we can prove (2.7).

Now, since $\eta'_{k,j}\left(t\right)\left(\eta_{k,j}\left(t\right)\right)^{-1}=\frac{3}{t}+O\left(\frac{\ln(k)}{k}\right)$, using (2.6) we get for $j=1,2$,

$$\eta_{n,j}\left(t\right)\sum_{k\neq n,k\geq 1}\eta'_{k,j}\left(t\right)\left\{\eta^2_{k,j}\left(t\right)-\eta_{k,j}\left(t\right)\eta_{n,j}\left(t\right)\right\}^{-1}=O\left(1\right)$$

uniformly on compact sets of $(0,1]$. Similarly, it is indicated that

$$\eta_{n,j}\left(t\right)\sum_{k\geq 1}\mu'_{k,j}\left(t\right)\left\{\mu^2_{k,j}\left(t\right)-\mu_{k,j}\left(t\right)\eta_{n,j}\left(t\right)\right\}^{-1}=O\left(1\right).$$

This completes the proof. $\square$

We can similarly obtain the following formula:

(2.14)

$$\lambda_n(t)\sum_{k\neq n,k\geq 1}\lambda_k'(t)\left\{\lambda_k^2(t)-\lambda_k(t)\lambda_n(t)\right\}^{-1}=O(1),\quad t\in(0,t_1).$$

3. Dual Equations and Uniqueness Theorem

In this section, the system of dual equations associated with the equation (1.1) are presented in Theorems 3.1 and 3.2, and in Theorem 3.3 we prove the associate uniqueness theorem.

Theorem 3.1. *The eigenvalues $\lambda_n(t)$, $n=1,2,3,\ldots$, satisfy*

$$\begin{aligned}(3.1)\qquad &\lambda_n''(t)+\frac{2f'(t)\lambda_n'(t)}{f(t)}\\ &+2\lambda_n(t)\lambda_n'(t)\sum_{k\neq n,k\geq 1}\lambda_k'(t)\left\{\lambda_k^2(t)\left(1-\frac{\lambda_n(t)}{\lambda_k(t)}\right)\right\}^{-1}\\ &-\frac{2(\lambda_n'(t))^2}{\lambda_n(t)}=0,\quad 0<t<t_1,\end{aligned}$$

where

$$f(t)=\frac{1}{2}\left(|\phi(0)||\phi(t)|^{-1}\right)^{\frac{1}{2}}\prod_{n\geq 1}\frac{\lambda_n(t)R_+^2(t)}{\zeta_n^2},$$

and $\lambda_n(t)$, $n\geq 1$, are the eigenvalues of boundary value problem L on $[0,t]$, ζ_n, $n=1,2,\ldots$, are positive zeros of $J'_{\frac{1}{2}}$, and

$$R_+(t)=\int_0^t\sqrt{\max\{0,\phi^2(x)\}}\,dx.$$

Here, $J'_{\frac{1}{2}}$ is the derivative of the Bessel function of order $\frac{1}{2}$.

Proof. For $n=1,2,3,\ldots$, the functions $\lambda_n(t)$ are twice continuously differentiable. Thus, from $y(t,\lambda_n(t))=0$ and by the implicit function theorem, for $t\in(0,t_1)$ we have $\frac{\partial y}{\partial t}+\frac{\partial y}{\partial\lambda_n}.\lambda_n'(t)=0$. Therefore,

$$\frac{\partial^2 y}{\partial t^2}+2\frac{\partial^2 y}{\partial t\partial\lambda_n}\lambda_n'(t)+\frac{\partial^2 y}{\partial\lambda_n^2}\left(\lambda_n'(t)\right)^2+\frac{\partial y}{\partial\lambda_n}\lambda_n''(t)=0.$$

Since $y(t,\lambda_n(t))=0$, from (2.2) we get

$$(3.2)\qquad 2\frac{\partial^2 y}{\partial t\partial\lambda_n}\lambda_n'(t)+\frac{\partial^2 y}{\partial\lambda_n^2}\left(\lambda_n'(t)\right)^2+\frac{\partial y}{\partial\lambda_n}\lambda_n''(t)=0.$$

By virtue of (2.2), the conclusions are

$$\frac{\partial y}{\partial \lambda_n}(t, \lambda_n(t)) = \frac{-f(t)}{\lambda_n(t)} \prod_{k\neq n, k\geq 1} \left(1 - \frac{\lambda_n(t)}{\lambda_k(t)}\right), \tag{3.3}$$

$$\frac{\partial^2 y}{\partial \lambda_n^2}(t, \lambda_n(t)) = \frac{2f(t)}{\lambda_n(t)} \left\{ \sum_{k\neq n, k\geq 1} \frac{1}{\lambda_k(t)} \left(1 - \frac{\lambda_n(t)}{\lambda_k(t)}\right)^{-1} \right\} \times \prod_{k\neq n, k\geq 1} \left(1 - \frac{\lambda_n(t)}{\lambda_k(t)}\right). \tag{3.4}$$

Also, according to (2.14), the series

$$\sum_{k\neq n, k\geq 1} \lambda_n(t)\, \lambda_k'(t) \left\{\lambda_k^2(t) - \lambda_k(t)\, \lambda_n(t)\right\}^{-1},$$

is uniformly convergent on $(0, t_1)$. Thus, it follows from (2.4) that

$$\begin{aligned} \frac{\partial^2 y}{\partial \lambda_n \partial t}(t, \lambda_n(t)) = \Bigg\{ & -\frac{f'(t)}{\lambda_n(t)} + \frac{f(t)}{\lambda_n^2(t)} \\ & - f(t) \sum_{k\neq n, k\geq 1}^{\infty} \frac{\lambda_k'(t)}{\lambda_k^2(t)} \left(1 - \frac{\lambda_n(t)}{\lambda_k(t)}\right)^{-1} \\ & - \frac{f(t)\, \lambda_n'(t)}{\lambda_n(t)} \sum_{k\neq n, k\geq 1} \frac{1}{\lambda_k(t)} \left(1 - \frac{\lambda_n(t)}{\lambda_k(t)}\right)^{-1} \Bigg\} \\ & \times \prod_{k\neq n, k\geq 1} \left(1 - \frac{\lambda_n(t)}{\lambda_k(t)}\right). \end{aligned} \tag{3.5}$$

Substituting (3.3)-(3.5) into (3.2) we arrive at (3.1). Moreover, from (2.2)-(2.3) we obtain

$$f(t) = \frac{1}{2} \left(|\phi(0)|\, |\phi(t)|^{-1}\right)^{\frac{1}{2}} \prod_{n\geq 1} \frac{\lambda_n(t)\, R_+^2(t)}{\zeta_n^2}, \quad 0 < t < t_1.$$

The proof is complete. □

Theorem 3.2. *For $t \in \omega_j$, $j = 1, 2$, the positive and negative eigenvalues $\eta_{n,j}(t)$, $\mu_{n,j}(t)$, $n \geq 1$, satisfy the equations*

(3.6)

$$\eta_{n,j}''(t) + 2h_j'(t)\,(h_j(t))^{-1}\, \eta_{n,j}'(t) - 2\left(\eta_{n,j}'(t)\right)^2 (\eta_{n,j}(t))^{-1} + 2\eta_{n,j}(t)\, \eta_{n,j}'(t)$$

$$\times\left\{\sum_{k\neq n,k\geq 1}\frac{\eta'_{k,j}(t)}{\eta^2_{k,j}(t)-\eta_{k,j}(t)\,\eta_{n,j}(t)}+\sum_{k\geq 1}\frac{\mu'_{k,j}(t)\,(\mu_{k,j}(t))^{-1}}{\eta_{k,j}(t)-\eta_{n,j}(t)}\right\}=0,$$

(3.7)

$$\mu''_{n,j}(t)+2h'_j(t)\,(h_j(t))^{-1}\,\mu'_{n,j}(t)-2\left(\mu'_{n,j}(t)\right)^2(\mu_{n,j}(t))^{-1}+2\mu_{n,j}(t)\,\eta'_{n,j}(t)$$

$$\times\left\{\sum_{k\neq n,k\geq 1}\frac{\mu'_{k,j}(t)}{\mu^2_{k,j}(t)-\mu_{k,j}(t)\,\mu_{n,j}(t)}+\sum_{k\geq 1}\frac{\eta'_{k,j}(t)\,(\eta_{k,j}(t))^{-1}}{\mu_{k,j}(t)-\mu_{n,j}(t)}\right\}=0,$$

where

(3.8)
$$h_j(t)=\frac{\pi}{8}\{|\phi(0)||\phi(t)|^{-1}R_-(t)\,R_+(t)\}^{\frac{1}{2}}$$
$$\times\prod_{n\geq 1}\frac{-\mu_{n,j}(t)\,R^2_-(s_j)}{\widetilde{j}^2_n}\prod_{n\geq 1}\frac{\eta_{n,j}(t)\,R^2_+(z_j)}{\widetilde{j}^2_n},$$

where $s_1=z_2=t$, $z_2=t_1$, $s_2=t_2$, and $\{\widetilde{j}_n\}_{n\geq 1}$, is the sequence of the positive zeros of J'_1 (here, J'_1 is the derivative of the Bessel function of the first order).

Proof. Since $\eta_{n,j}(t)$, $\mu_{n,j}(t)$, $n\geq 1$, $j=1,2$, are twice continuously differentiable functions, thus for $t\in\omega_j$, $j=1,2$, the conditions

$$y(t,\eta_{n,j}(t))=0,\qquad y(t,\mu_{n,j}(t))=0,$$

give the equations

$$\frac{\partial y}{\partial t}+\frac{\partial y}{\partial\eta_{n,j}}.\eta'_{n,j}(t)=0,\qquad\frac{\partial y}{\partial t}+\frac{\partial y}{\partial\mu_{n,j}}.\mu'_{n,j}(t)=0,$$

and

(3.9)
$$\begin{cases}\frac{\partial y}{\partial\eta_{n,j}}\eta''_{n,j}(t)+2\frac{\partial^2 y}{\partial t\partial\eta_{n,j}}\eta'_{n,j}(t)+\frac{\partial^2 y}{\partial\eta^2_{n,j}}\left(\eta'_{n,j}(t)\right)^2=0,\\ \frac{\partial y}{\partial\mu_{n,j}}\mu''_{n,j}(t)+2\frac{\partial^2 y}{\partial t\partial\mu_{n,j}}\mu'_{n,j}(t)+\frac{\partial^2 y}{\partial\mu^2_{n,j}}\left(\mu'_{n,j}(t)\right)^2=0.\end{cases}$$

Further, for $t\in\omega_j$, $j=1,2$, from (2.4) we have

(3.10)
$$\frac{\partial y}{\partial\lambda}(t,\eta_{n,j}(t))=\frac{-h_j(t)}{\eta_{n,j}(t)}\prod_{k\neq n,k\geq 1}\left(1-\frac{\eta_{n,j}(t)}{\eta_{k,j}(t)}\right)\prod_{k\geq 1}\left(1-\frac{\eta_{n,j}(t)}{\mu_{k,j}(t)}\right),$$

(3.11)
$$\frac{\partial^2 y}{\partial\lambda^2}(t,\eta_{n,j}(t))=\frac{2A(t)\,h_j(t)}{\eta_{n,j}(t)}\prod_{k\neq n,k\geq 1}\left(1-\frac{\eta_{n,j}(t)}{\eta_{k,j}(t)}\right)\prod_{k\geq 1}\left(1-\frac{\eta_{n,j}(t)}{\mu_{k,j}(t)}\right),$$

where

$$A(t)=\sum_{k\neq n,k\geq 1}\left(\eta_{k,j}(t)-\eta_{n,j}(t)\right)^{-1}+\sum_{k\geq 1}\left(\mu_{k,j}(t)-\eta_{n,j}(t)\right)^{-1}.$$

On the other hand, it follows from (2.8) that

$$\eta_{n,j}(t)\sum_{k\neq n,k\geq 1}\frac{\eta'_{k,j}(t)}{\eta^2_{k,j}(t)-\eta_{k,j}(t)\,\eta_{n,j}(t)},$$

and

$$\eta_{n,j}(t)\sum_{k\neq n,k\geq 1}\frac{\mu'_{k,j}(t)}{\mu^2_{k,j}(t)-\mu_{k,j}(t)\,\eta_{n,j}(t)},$$

are uniformly convergent on $\omega_1\cup\omega_2$. Therefore, we get by virtue of (3.10) that

(3.12)

$$\begin{aligned}\frac{\partial^2 y}{\partial\lambda\partial t}(t,\eta_{n,j}(t))=&\prod_{k\neq n,k\geq 1}\left(1-\frac{\eta_{n,j}(t)}{\eta_{k,j}(t)}\right)\prod_{k\geq 1}\left(1-\frac{\eta_{n,j}(t)}{\mu_{k,j}(t)}\right)\\&\times\left\{-\frac{h'_j(t)}{\eta_{n,j}(t)}+h_j(t)\left(\frac{\eta'_{n,j}(t)}{\eta^2_{n,j}(t)}-\frac{A(t)\,\eta'_{n,j}(t)}{\eta_{n,j}(t)}-B(t)\right)\right\},\end{aligned}$$

where

$$B(t)=\sum_{k\neq n,k\geq 1}\frac{\eta'_{k,j}(t)}{\eta^2_{k,j}(t)-\eta_{k,j}(t)\,\eta_{n,j}(t)}+\sum_{k\neq n,k\geq 1}\frac{\mu'_{k,j}(t)}{\mu^2_{k,j}(t)-\mu_{k,j}(t)\,\eta_{n,j}(t)}.$$

Substituting (3.10)-(3.12) into (3.9), we arrive at (3.6). Similarly, for $\mu_{n,j}(t)$, $j=1,2$, we get (3.7). Also, using the relations (2.4)-(2.5) we obtain (3.8). □

The system of (3.1), (3.6) and (3.7) is *dual* to the equation (1.1). For $t\in(0,t_1)$, dividing (3.1) by λ'_n and integrating from a fixed number $\alpha\neq 0$ to t, we obtain

$$\lambda'_n(t)=\lambda'_n(\alpha)\left\{\frac{f(\alpha)\,\lambda_n(t)}{f(t)\,\lambda_n(\alpha)}\;\exp\left(-\sum_{k\neq n,k\geq 1}\int_\alpha^t\frac{\lambda'_k(t)\,\lambda_n(t)}{\lambda^2_k(t)-\lambda_k(t)\,\lambda_n(t)}\right)\right\}^2.$$

Similarly, for $t\in\omega_j$, $j=1,2$, dividing (3.6) by $\eta'_{n,j}(t)$ and (3.7) by $\mu'_{n,j}(t)$, and integrating from t to 1, we calculate

$$\eta'_{n,j}(t)=\eta'_{n,j}(1)\left\{\frac{h_j(1)\eta_{n,j}(t)}{h_j(t)\,\eta_{n,j}(1)}M_n(t)\right\}^2, \tag{3.13}$$

$$\mu'_{n,j}(t) = \mu'_{n,j}(1) \left\{ \frac{h_j(1)\mu_{n,j}(t)}{h_j(t)\mu_{n,j}(1)} M_n(t) \right\}^2, \tag{3.14}$$

where

$$M_n(t) = \exp\left\{ \sum_{k\neq n, k\geq 1} \int_t^1 \frac{\eta'_{k,j}(t)\eta_{n,j}(t)}{\eta^2_{k,j}(t) - \eta_{k,j}(t)\eta_{n,j}(t)} d\tau \right.$$
$$\left. + \sum_{k\geq 1} \int_t^1 \frac{\mu'_{k,j}(t)\eta_{n,j}(t)}{\mu^2_{k,j}(t) - \mu_{k,j}(t)\eta_{n,j}(t)} d\tau \right\}.$$

Theorem 3.3. *(Uniqueness theorem) Let L_1 and L_2 be the dual initial value problems consisting of* (3.13) *with initial condition $\eta_{n,2}(1) = \eta_{n,2}$, and* (3.14) *with condition $\mu_{n,2}(1) = \mu_{n,2}$, $n \geq 1$, respectively. Then, the dual problems L_1 and L_2 have unique solutions.*

Proof. The problems L_1 and L_2 can be written as

$$\frac{d\tau}{dt} = p(\tau), \qquad \tau(1) = \tau := \{\eta_{1,2}(1), \eta_{2,2}(1), \eta_{3,2}(1), \ldots\},$$
$$\frac{d\sigma}{dt} = r(\sigma), \qquad \sigma(1) = \sigma := \{\mu_{1,2}(1), \mu_{2,2}(1), \mu_{3,2}(1), \ldots\},$$

respectively, where

$$p_n(\tau) = p_n = \eta'_{n,2}(1) \left\{ \frac{h_2(1)\eta_{n,2}(t)}{h_2(t)\eta_{n,2}(1)} M_n(t) \right\}^2, \tag{3.15}$$

$$r_n(\sigma) = r_n = \mu'_{n,2}(1) \left\{ \frac{h_2(1)\mu_{n,2}(t)}{h_2(t)\mu_{n,2}(1)} M_n(t) \right\}^2. \tag{3.16}$$

From (2.1), (3.8) and $h_2(t) = O(1)$, we have $\eta_{n,2}(t) = O(n^2)$, $\mu_{n,2}(t) = O(n^2)$. So, since $\eta_{n,2}(1) = \eta_{n,2}$, $\mu_{n,2}(1) = \mu_{n,2}$, we get

$$\begin{cases} \eta'_{n,2}(1) \left\{ \frac{h_2(1)\eta_{n,2}(t)}{h_2(t)\eta_{n,2}(1)} \right\}^2 = O(n^4), \\ \mu'_{n,2}(1) \left\{ \frac{h_2(1)\mu_{n,2}(t)}{h_2(t)\mu_{n,2}(1)} \right\}^2 = O(n^4). \end{cases} \tag{3.17}$$

Also, according to (2.8), the series

$$\sum_{k\neq n, k\geq 1} \frac{\eta'_{k,2}(t)\eta_{n,2}(t)}{\eta^2_{k,2}(t) - \eta_{k,2}(t)\eta_{n,2}(t)}, \qquad \sum_{k\geq 1} \frac{\mu'_{k,2}(t)\eta_{n,2}(t)}{\mu^2_{k,2}(t) - \mu_{k,2}(t)\eta_{n,2}(t)},$$

are uniformly convergent on $(t_2, 1]$. Therefore, the integration and summation in (3.15)-(3.16) are interchanged. Thus, from (2.8) it is calculated that

$$\exp\left\{ \sum_{k\neq n, k\geq 1} \int_t^1 \frac{\eta'_{k,j}(t)\eta_{n,j}(t)}{\eta^2_{k,j}(t) - \eta_{k,j}(t)\eta_{n,j}(t)} d\tau \right.$$

$$+\sum_{k\geq 1}\int_t^1 \frac{\mu_{k,j}'(t)\,\eta_{n,j}(t)}{\mu_{k,j}^2(t)-\mu_{k,j}(t)\,\eta_{n,j}(t)}d\tau\Bigg\} = O(1).$$

This together with (3.15)-(3.17) gives $p_n(\tau) = p_n = O\left(n^4\right)$, $r_n(\sigma) = r_n = O\left(n^4\right)$. Now, we define the following normed space

$$\Lambda := \left\{\beta = (\beta_n)_{n=1}^{\infty}\,|\|\beta\| := \sum_{n=1}^{\infty}\frac{|\beta_n|}{n^6} < \infty\right\}.$$

Let $\Lambda^* \subset \Lambda$ be a subset containing nonzero members of Λ of the form

$$\beta_n = \frac{n\pi^2}{K^2}\left(\pm n \mp \frac{1}{4}\right) + O(1), \quad n = 1, 2, 3, \ldots,$$

where K is a fixed number. It is easy to show that Λ^* contains the eigenvalues $\eta_{n,j}$ and $\mu_{n,j}$, $n \geq 1$, $j = 1, 2$. Also, from $p_n(\tau) = O(n^4)$ and $r_n(\sigma) = O(n^4)$, we can conclude that the functions p and r are two maps from Λ^* into Λ. Moreover, since Λ^* is a convex space, so for $0 \leq \delta \leq 1$ and $\gamma_1, \gamma_2 \in \Lambda^*$, the functions $p_n(\delta\gamma_1 + (1-\delta)\gamma_2)$ and $r_n(\delta\gamma_1 + (1-\delta)\gamma_2)$ are entire with respect to δ on $[0, 1]$. Therefore,

$$p_n(1) - p_n(0) = p_n(\gamma_1) - p_n(\gamma_2) = \left.\frac{dp_n}{d\delta}\right|_{\delta=\delta_n},$$

$$r_n(1) - r_n(0) = r_n(\gamma_1) - r_n(\gamma_2) = \left.\frac{dr_n}{d\delta}\right|_{\delta=\widetilde{\delta}_n},$$

for some $\delta_n, \widetilde{\delta}_n \in [0, 1]$. On the other hand, p_n and r_n are the functions of $\eta_{n,2}$ and $\mu_{n,2}$, respectively. So,

$$p_n(\gamma_1) - p_n(\gamma_2) = \left.\frac{dp_n}{d\delta}\right|_{\delta=\delta_n} = \sum_{k=1}^{\infty}\left.\frac{dp_n}{d\eta_{k,2}}\right|_{\delta_n}(\upsilon_k - \widetilde{\upsilon}_k), \tag{3.18}$$

$$r_n(\gamma_1) - r_n(\gamma_2) = \left.\frac{dr_n}{d\delta}\right|_{\delta=\widetilde{\delta}_n} = \sum_{k=1}^{\infty}\left.\frac{dr_n}{d\mu_{k,2}}\right|_{\widetilde{\delta}_n}(\upsilon_k - \widetilde{\upsilon}_k), \tag{3.19}$$

where $\gamma_1 = (\upsilon_k)_{k=1}^{\infty}$, $\gamma_2 = (\widetilde{\upsilon}_k)_{k=1}^{\infty}$. Computing the coefficients $\frac{dp_n}{d\eta_{k,2}}$, $\frac{dr_n}{d\mu_{k,2}}$ from (3.15)-(3.16), and substituting into (3.18)-(3.19), we conclude that

(3.20)

$$p_n(\gamma_1) - p_n(\gamma_2) = \widetilde{p}_n\sum_{k\neq n, k\geq 1}\left\{(\upsilon_k - \widetilde{\upsilon}_k)\left(-\frac{2}{\widetilde{\eta}_k} - 2\int_t^1 C(\tau)d\tau\right)\right\},$$

where $\widetilde{p}_n = p_n(\gamma_1\delta_n + (1-\delta_n)\gamma_2)$, $\widetilde{\eta}_k = \upsilon_k\delta_n + (1-\delta_n)\widetilde{\upsilon}_k$, and

$$C(t) = \frac{\widetilde{\eta}_n(t)\,\widetilde{\eta}_k''(t)}{(\widetilde{\eta}_k^2(t) - \widetilde{\eta}_k(t)\,\widetilde{\eta}_n(t))\widetilde{\eta}_k'(t)} + \frac{\widetilde{\eta}_k'(t)}{(\widetilde{\eta}_k(t) - \widetilde{\eta}_n(t))^2}$$

$$- \frac{(2\widetilde{\eta}_k(t)\,\widetilde{\eta}_n(t) - \widetilde{\eta}_n^2(t))\widetilde{\eta}_k'(t)}{(\widetilde{\eta}_k(t) - \widetilde{\eta}_n(t))^2\widetilde{\eta}_k^2(t)}.$$

Moreover,

(3.21)

$$r_n(\gamma_1) - r_n(\gamma_2) = \widetilde{r}_n \sum_{k\neq n, k\geq 1} \left\{ (\upsilon_k - \widetilde{\upsilon}_k)\left(-\frac{2}{\widetilde{\mu}_k} - 2\int_t^1 D(\sigma)\, d\sigma \right) \right\},$$

where $\widetilde{r}_n = p_n(\gamma_1\widetilde{\delta}_n + (1-\widetilde{\delta}_n)\gamma_2)$, $\widetilde{\mu}_k = \upsilon_k\widetilde{\delta}_n + (1-\widetilde{\delta}_n)\widetilde{\upsilon}_k$, and

$$D(t) = \frac{\widetilde{\mu}_n(t)\,\widetilde{\mu}_k''(t)}{(\widetilde{\mu}_k^2(t) - \widetilde{\mu}_k(t)\,\widetilde{\mu}_n(t))\widetilde{\mu}_k'(t)} + \frac{\widetilde{\mu}_k'(t)}{(\widetilde{\mu}_k(t) - \widetilde{\mu}_n(t))^2}$$
$$- \frac{(2\widetilde{\mu}_k(t)\,\widetilde{\mu}_n(t) - \widetilde{\mu}_n^2(t))\widetilde{\mu}_k'(t)}{(\widetilde{\mu}_k(t) - \widetilde{\mu}_n(t))^2\widetilde{\mu}_k^2(t)}.$$

Since Λ^* is a convex space, so $\widetilde{\eta}_k$, $\widetilde{\mu}_k \in \Lambda^*$, $\frac{\widetilde{\eta}_n''}{\widetilde{\eta}_n'} = O(1)$, $\frac{\widetilde{\mu}_n''}{\widetilde{\mu}_n'} = O(1)$, and

$$\sum_{k\neq n, k\geq 1} \int_t^1 \frac{\widetilde{\eta}_k'}{(\widetilde{\eta}_k - \widetilde{\eta}_n)^2} = O\left(n^{-4}\right), \qquad \sum_{k\neq n, k\geq 1} \int_t^1 \frac{\widetilde{\mu}_k'}{(\widetilde{\mu}_k - \widetilde{\mu}_n)^2} = O\left(n^{-4}\right).$$

Thus, there exists positive numbers M_1, M_2 such that dividing (3.20)-(3.21) by $(2n)^6$ and summing with respect to n, the following inequalities are obtained:

$$\|p_n(\gamma_1) - p_n(\gamma_2)\| \leq M_1\|\gamma_1 - \gamma_2\|, \quad \|r_n(\gamma_1) - r_n(\gamma_2)\| \leq M_2\|\gamma_1 - \gamma_2\|.$$

Consequently, the functions p and r satisfy the Lipschitz condition. Therefore, the dual initial value problems L_1 and L_2 have unique solutions. □

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