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The Existence Theorem for Contractive Mappings on wt-distance in b-metric Spaces Endowed with a Graph and its Application

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Abstract. In this paper, we study the existence and uniqueness of fixed points for mappings with respect to a wt-distance in b-metric spaces endowed with a graph. Our results are significant, since we replace the condition of continuity of mapping with the condition of orbitally G-continuity of mapping and we consider b-metric spaces with graph instead of b-metric spaces, under which can be generalized, improved, enriched and unified a number of recently announced results in the existing literature. Additionally, we elicit all of our main results by a non-trivial example and pose an interesting two open problems for the enthusiastic readers.

1. Introduction and Preliminaries

The symmetric space, as metric-like spaces lacking the triangle inequality is introduced by Wilson [23]. Thereinafter, b-metric spaces are defined by Bakhtin [3] and Czerwik [11].

Definition 1.1. Let X be a nonempty set and $s \geq 1$ be a real number. Suppose that the mapping $d : X \times X \to [0, \infty)$ satisfies

(d_1) $d(x, y) = 0$ if and only if $x = y$;
(d_2) $d(x, y) = d(y, x)$ for all $x, y \in X$;
(d_3) $d(x, z) \leq s[d(x, y) + d(y, z)]$ for all $x, y, z \in X$.

Then d is called a b-metric and (X, d) is called a b-metric space (or a metric type space).

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Obviously, for $s = 1$, a b-metric space is a metric space. Also, for notions such as convergent and Cauchy sequences, completeness, continuity and etc. in b-metric spaces, we refer to [1, 7, 20].

In 1996, Kada et al. [18] introduced the concept of w-distance in metric spaces, where non-convex minimization problems were treated. In 2014, Hussain et al. [16] defined a wt-distance on b-metric spaces and proved some fixed point theorems under wt-distance in a partially ordered b-metric space.

Definition 1.2 ([16])**.** Let (X, d) be a b-metric space and $s \geq 1$ be a given real number. A function $\rho : X \times X \to [0, +\infty)$ is called a wt-distance on X if the following properties are satisfied:

- (ρ_1) $\rho(x, z) \leq s[\rho(x, y) + \rho(y, z)]$ for all $x, y, z \in X$;
- (ρ_2) ρ is b-lower semi-continuous in its second variable, i.e. if $x \in X$ and $y_n \to y$ in X then $\rho(x, y) \leq s \liminf_n \rho(x, y_n)$;
- (ρ_3) for each $\varepsilon > 0$ there exists $\delta > 0$ such that $\rho(z, x) \leq \delta$ and $\rho(z, y) \leq \delta$ imply $d(x, y) \leq \varepsilon$.

Obviously, for $s = 1$, every wt-distance is a w-distance. But, a w-distance is not necessary a wt-distance. Thus, each wt-distance is a generalization of w-distance.

Lemma 1.3 ([16])**.** *Let (X, d) be a b-metric space with parameter $s \geq 1$ and ρ be a wt-distance on X. Also, let $\{x_n\}$ and $\{y_n\}$ be sequences in X, let $\{\alpha_n\}$ and $\{\beta_n\}$ be a sequences in $[0, +\infty)$ converging to zero and $x, y, z \in X$. Then the following conditions hold:*

- (i) *if $\rho(x_n, y) \leq \alpha_n$ and $\rho(x_n, z) \leq \beta_n$ for all $n \in \mathbb{N}$, then $y = z$. In particular, if $\rho(x, y) = 0$ and $\rho(x, z) = 0$, then $y = z$;*
- (ii) *if $\rho(x_n, y_n) \leq \alpha_n$ and $\rho(x_n, z) \leq \beta_n$ for $n \in \mathbb{N}$, then $\{y_n\}$ converges to z;*
- (iii) *if $\rho(x_n, x_m) \leq \alpha_n$ for all $m, n \in \mathbb{N}$ with $m > n$, then $\{x_n\}$ is a Cauchy sequence in X;*
- (iv) *if $\rho(y, x_n) \leq \alpha_n$ for all $n \in \mathbb{N}$, then $\{x_n\}$ is a Cauchy sequence in X.*

The most important graph theory approach to metric fixed point theory introduced so far is attributed to Jachymski [17]. In this approach, the underlying metric space is equipped with a directed graph and the Banach contraction is formulated in a graph language (also, see [5, 12, 14, 21]).

The purpose of this paper is to prove the existence and uniqueness of fixed points for mappings under a wt-distance in b-metric spaces endowed with a graph. Our results are generalizations of some fixed point theorems given in terms of a wt-distance on b-metric spaces to wt-distance from b-metric spaces equipped with a graph G. As an application, we develop our results in the framework of a generalized c-distance on cone b-metric spaces.

2. Main Results

Let (X, d) be a b-metric space and G be a directed graph with vertex set $V(G) = X$ such that the edge set $E(G)$ contains all loops; that is, $(x, x) \in E(G)$ for all $x \in X$. Also, let the graph G has no parallel edges. Then the graph G can be easily denoted by the ordered pair $(V(G), E(G))$ and it is said that the b-metric space (X, d) is endowed with the graph G. The b-metric space (X, d) can also be endowed with the graphs G^{-1} and $\widetilde{G}$, where the former is the conversion of G which is obtained from G by reversing the directions of the edges, and the latter is an undirected graph obtained from G by ignoring the directions of the edges. In other words, $V(G^{-1}) = V(\widetilde{G}) = X$, $E(G^{-1}) = \big\{(x, y) : (y, x) \in E(G)\big\}$ and $E(\widetilde{G}) = E(G) \cup E(G^{-1})$. If $x, y \in X$, then a finite sequence $(x_i)_{i=0}^N$ consisting of $N + 1$ vertices is called a path in G from x to y whenever $x_0 = x$, $x_N = y$ and (x_{i-1}, x_i) is an edge of G for $i = 1, \dots, N$. The graph G is called connected if there exists a path in G between each two vertices of G. For more details on graphs, see [6]. In the sequel, let (X, d) be a b-metric space endowed with a graph G with $V(G) = X$ and $\Delta(X) \subseteq E(G)$, where $\Delta(X) = \{(x, x) \in X \times X : x \in X\}$. Also, we denote by $\mathrm{Fix}(T)$ the set of all fixed points of a self-map T on X and we use X_T to denote the set of all points $x \in X$ such that (x, Tx) is an edge of G; that is, $X_T = \{x \in X : (x, Tx) \in E(G)\}$.

Following the idea of Petruşel and Rus [21], we define Picard operators in b-metric spaces.

Definition 2.1. Let (X, d) be a b-metric space. A self-map T on X is called a Picard operator if T has a unique fixed point x_* in X and $T^n x \to x_*$ for all $x \in X$.

We also need a weaker type of continuity in b-metric spaces endowed with a graph. The idea of this definition comes from the definition of orbital continuity considered by Cirić [9] (also, see [2]). Following Jachymski [17], we introduce the concept of orbitally G-continuous for self-map f on b-metric spaces.

Definition 2.2. Let (X, d) be a b-metric space endowed with a graph G. A mapping $T : X \to X$ is called orbitally G-continuous on X if for all $x, y \in X$ and all sequences $\{b_n\}$ of positive integers with $(T^{b_n} x, T^{b_{n+1}} x) \in E(G)$ for all $n \geq 1$, the convergence $T^{b_n} x \to y$ implies $T(T^{b_n} x) \to Ty$.

Trivially, a continuous mapping on a b-metric space is orbitally G-continuous for all graphs G but the converse is not generally true.

Theorem 2.3. *Let (X, d) be a complete b-metric space endowed with the graph G and $s \geq 1$ be a given real number. Also, let ρ be a wt-distance and $T : X \to X$ be an orbitally G-continuous mapping. Suppose that there exist mappings $\alpha, \beta, \gamma : X \to [0, 1)$ such that the following conditions hold:*

- (t_1) *$\alpha(Tx) \leq \alpha(x)$, $\beta(Tx) \leq \beta(x)$ and $\gamma(Tx) \leq \gamma(x)$ for all $x \in X$;*
- (t_2) *$\big(s(\alpha + 2\beta) + (s^2 + s)\gamma\big)(x) < 1$ for all $x \in X$;*
- (t_3) *T preserves the edges of G, that is, $(x, y) \in E(G)$ implies $(Tx, Ty) \in E(G)$ for all $x, y \in X$;*
- (t_4) *for all $x, y \in X$ with $(x, y) \in E(G)$,*

$$\rho(Tx, Ty) \leq \alpha(x)\rho(x, y) + \beta(x)\rho(x, Ty) + \gamma(x)\rho(y, Tx),$$
$$\rho(Ty, Tx) \leq \alpha(x)\rho(y, x) + \beta(x)\rho(Ty, x) + \gamma(x)\rho(Tx, y).$$

Then T has a fixed point if and only if $X_T \neq \emptyset$. Moreover, if $Tx_* = x_*$, then $\rho(x_*, x_*) = 0$. Also, if the subgraph of G with the vertex set $\mathrm{Fix}(T)$ is connected, then the restriction of T to X_T is a Picard operator.

Proof. Because $\mathrm{Fix}(T) \subseteq X_T$, it follows that if T has a fixed point, then X_T is nonempty. Now, let $x_0 \in X_T$. Since T preserves the edges of G, then $(x_n, x_{n+1}) \in E(G)$ for all $n \in \mathbb{N}$, where $x_n = Tx_{n-1} = T^n x_0$. Since $(x_{n-1}, x_n) \in E(G)$, by a simple calculation, we have

$$\begin{aligned}\rho(x_{n+1}, x_n) \leq\ & \alpha(x_0)\rho(x_n, x_{n-1}) + s(\beta + \gamma)(x_0)\rho(x_n, x_{n+1}) \\ & + s[\beta(x_0)\rho(x_{n+1}, x_n) + \gamma(x_0)\rho(x_{n-1}, x_n)].\end{aligned} \tag{2.1}$$

Similarly, we have

$$\begin{aligned}\rho(x_n, x_{n+1}) \leq\ & \alpha(x_0)\rho(x_{n-1}, x_n) + s(\beta + \gamma)(x_0)\rho(x_{n+1}, x_n) \\ & + s[\beta(x_0)\rho(x_n, x_{n+1}) + \gamma(x_0)\rho(x_n, x_{n-1})].\end{aligned} \tag{2.2}$$

Adding up (2.1) and (2.2), we get

$$\begin{aligned}\rho(x_{n+1}, x_n) + \rho(x_n, x_{n+1}) \leq\ & (\alpha + s\gamma)(x_0)[\rho(x_n, x_{n-1}) + \rho(x_{n-1}, x_n)] \\ & + s(2\beta + \gamma)(x_0)[\rho(x_{n+1}, x_n) + \rho(x_n, x_{n+1})].\end{aligned}$$

Let $u_n = \rho(x_{n+1}, x_n) + \rho(x_n, x_{n+1})$. Then

$$u_n \leq (\alpha + s\gamma)(x_0)u_{n-1} + s(2\beta + \gamma)(x_0)u_n.$$

Thus, we have $u_n \leq hu_{n-1}$, where $0 \leq h = \frac{(\alpha+s\gamma)(x_0)}{1-s(2\beta+\gamma)(x_0)} < \frac{1}{s}$ by (t_2). By repeating the procedure, we get $u_n \preceq h^n u_0$ for all $n \in \mathbb{N}$ and hence

$$\rho(x_n, x_{n+1}) \leq u_n \leq h^n[\rho(x_1, x_0) + \rho(x_0, x_1)]. \tag{2.3}$$

Let $m > n$. It follows from (2.3) and $0 \leq sh < 1$ that

$$\begin{aligned}\rho(x_n, x_m) &\leq s[\rho(x_n, x_{n+1}) + \rho(x_{n+1}, x_m)] \\ &\ \vdots \\ &\leq s\rho(x_n, x_{n+1}) + s^2\rho(x_{n+1}, x_{n+2}) + \cdots + s^{m-n}\rho(x_{m-1}, x_m)] \\ &\leq \frac{sh^n}{1-sh}[\rho(x_1, x_0) + \rho(x_0, x_1)].\end{aligned}$$

Since $\frac{sh^n}{1-sh}[\rho(x_1, x_0) + \rho(x_0, x_1)]$ is a sequence in $[0, +\infty)$ converging to 0, $\{x_n\}$ is a Cauchy sequence in X (by using (iii) of Lemma 1.3). Since X is complete, there exists a point $x_* \in X$ such that $x_n = T^n x_0 \to x_*$ as $n \to \infty$. We are going to show that x_* is a fixed point of T. To this end, note that from $x_0 \in X_T$ we have $(T^n x_0, T^{n+1} x_0) \in E(G)$ for all $n \geq 0$. Thus, by orbital G-continuity of T, we get $T^{n+1} x_0 \to T x_*$. Since the limit of a sequence is unique, we conclude $T x_* = x_*$. Thus, x_* is a fixed point of the mapping T. Now, let $T x_* = x_*$ for $x_* \in X$. Then (t_4) implies that

$$\begin{aligned}\rho(x_*, x_*) &= \rho(T x_*, T x_*) \\ &\leq \alpha(x_*)\rho(x_*, x_*) + \beta(x_*)\rho(x_*, T x_*) + \gamma(x_*)\rho(x_*, T x_*) \\ &= (\alpha + \beta + \gamma)(x_*)\rho(x_*, x_*).\end{aligned}$$

Since $0 \leq (\alpha + \beta + \gamma)(x_*) < \big(s(\alpha + 2\beta) + (s^2 + s)\gamma\big)(x_*)$ and $\big(s(\alpha + 2\beta) + (s^2 + s)\gamma\big)(x_*) < 1$ by (t_2), we get that $\rho(x_*, x_*) = 0$.

Next, suppose that the subgraph of G with the vertex set $\mathrm{Fix}(T)$ is connected and $x_{**} \in X$ is a fixed point of T. Then there exists a path $(x_i)_{i=0}^N$ in G from x_* to x_{**} such that $x_1, \ldots, x_{N-1} \in \mathrm{Fix}(T)$; that is, $x_0 = x_*$, $x_N = x_{**}$ and $(x_{i-1}, x_i) \in E(G)$ for $i = 1, \ldots, N$. Therefore, by (t_4), for each $i = 1, 2, \cdots N$, we have

$$\begin{aligned}\rho(x_i, x_{i-1}) &= \rho(T x_i, T x_{i-1}) \\ &\leq \alpha(x_i)\rho(x_i, x_{i-1}) + \beta(x_i)\rho(x_i, T x_{i-1}) + \gamma(x_i)\rho(x_{i-1}, T x_i) \\ &= (\alpha + \beta)(x_i)\rho(x_i, x_{i-1}) + \gamma(x_i)\rho(x_{i-1}, x_i),\end{aligned} \tag{2.4}$$

and

$$\begin{aligned}\rho(x_{i-1}, x_i) &= \rho(T x_{i-1}, T x_i) \\ &\leq \alpha(x_i)\rho(x_{i-1}, x_i) + \beta(x_i)\rho(T x_{i-1}, x_i) + \gamma(x_i)\rho(T x_i, x_{i-1}) \\ &= (\alpha + \beta)(x_i)\rho(x_{i-1}, x_i) + \gamma(x_i)\rho(x_i, x_{i-1}).\end{aligned} \tag{2.5}$$

Hence, by (2.4) and (2.5), we get

$$\rho(x_i, x_{i-1}) + \rho(x_{i-1}, x_i) \leq (\alpha + \beta + \gamma)(x_i)[\rho(x_i, x_{i-1}) + \rho(x_{i-1}, x_i)],$$

which is a contradiction unless $\rho(x_i, x_{i-1}) + \rho(x_{i-1}, x_i) = 0$, since $0 \leq (\alpha+\beta+\gamma)(x_*) < \big(s(\alpha+2\beta)+(s^2+s)\gamma\big)(x_*)$ and $\big(s(\alpha+2\beta)+(s^2+s)\gamma\big)(x_*) < 1$ by (t_2). Hence $\rho(x_i, x_{i-1}) = \rho(x_{i-1}, x_i) = 0$. Now, since $\rho(x_i, x_i) = 0$ and $\rho(x_i, x_{i-1}) = 0$, we have $d(x_i, x_{i-1}) = 0$; that is, $x_i = x_{i-1}$. Thus,

$$x_* = x_0 = x_1 = \cdots = x_{N-1} = x_N = x_{**}.$$

Consequently, the fixed point of T is unique and the restriction of T to X_T is a Picard operator. □

Example 2.4. Let $X = [0,1]$ and define a mapping $d : X \times X \to \mathbb{R}$ by $d(x,y) = (x-y)^2$ for all $x, y \in X$. Then (X,d) is a complete b-metric space with $s = 2$. Also, let a mapping $T : X \to X$ be defined by $T(1) = 1$ and $Tx = \frac{x^2}{4}$ for all $x \in X - \{1\}$. Obviously, T is not continuous at $x = 1$, and in particular, on the whole X. Now assume that X is endowed with a graph $G = (V(G), E(G))$, where $V(G) = X$ and $E(G) = \{(x,x) : x \in X\} \cup \{(0, \frac{1}{2}), (\frac{1}{2}, 0)\}$. If $x, y \in X$ and $\{b_n\}$ is a sequence of positive integers with $(T^{b_n}x, T^{b_{n+1}}x) \in E(G)$ for all $n \geq 1$ such that $T^{b_n}x \to y$, then $\{T^{b_n}x\}$ is necessarily a constant sequence. Thus, $T^{b_n}x = y$ for all $n \geq 1$ and $T(T^{b_n}x) \to Ty$. Hence, T is orbitally G-continuous on X. Now, consider $\rho : X \times X \to [0,\infty)$ defined by $\rho(x,y) = d(x,y)$ for all $x, y \in X$. Then ρ is a wt-distance. Define the mappings $\alpha(x) = \frac{(x+1)^2}{9}$ and $\beta(x) = \gamma(x) = 0$ for all $x \in X$. Now, we have

(i) $\alpha(Tx) = \frac{1}{9}\big(\frac{x^2}{4}+1\big)^2 \leq \frac{1}{9}\big(x^2+1\big)^2 \leq \frac{(x+1)^2}{9} = \alpha(x)$ for all $x \in X - \{1\}$ and $\alpha(T1) = \alpha(1) = \dfrac{4}{9}$;
(ii) $\beta(Tx) = 0 \leq 0 = \beta(x)$ and $\gamma(Tx) = 0 \leq 0 = \gamma(x)$ for all $x \in X$;
(iii) $\big(2(\alpha+2\beta)+(2^2+2)\gamma\big)(x) = 2\frac{(x+1)^2}{9} < 1$ for all $x \in X$;
(iv) let $x \in X$. Then

$$\rho(Tx,Tx) = 0 = \alpha(x)\rho(x,x) + \beta(x)\rho(x,Tx) + \gamma(x)\rho(x,Tx),$$
$$\rho(Tx,Tx) = 0 = \alpha(x)\rho(x,x) + \beta(x)\rho(Tx,x) + \gamma(x)\rho(Tx,x).$$

or

$$\rho(T\frac{1}{2}, T0) = \frac{1}{256} \leq \alpha(\frac{1}{2})\rho(\frac{1}{2},0) + \beta(\frac{1}{2})\rho(\frac{1}{2},T0) + \gamma(\frac{1}{2})\rho(0,T\frac{1}{2}),$$
$$\rho(T0, T\frac{1}{2}) = \frac{1}{256} \leq \alpha(\frac{1}{2})\rho(0,\frac{1}{2}) + \beta(\frac{1}{2})\rho(T0,\frac{1}{2}) + \gamma(\frac{1}{2})\rho(T\frac{1}{2},0).$$

Therefore, the conditions of Theorem 2.3 are satisfied and hence T has a fixed point $x = 0$ with $\rho(0,0) = 0$.

In Example 2.4, we consider $\rho = d$ (because each b-metric is a wt-distance). One can consider non-trivial examples of wt-distance and check the validity of Theorem 2.3 (for example, see [16, 22]). An immediate consequence of Theorem 2.3 can be stated in the form of the following theorem.

Theorem 2.5. *Let (X, d) be a complete b-metric space endowed with the graph G and $s \geq 1$ be a given real number. Also, let ρ be a wt-distance and $T : X \to X$ be an orbitally G-continuous mapping. Suppose that there exist $\alpha, \beta, \gamma > 0$ with $s(\alpha + 2\beta) + (s^2 + s)\gamma < 1$ such that the following conditions hold:*

- (t_1) *T preserves the edges of G, that is, $(x, y) \in E(G)$ implies $(Tx, Ty) \in E(G)$ for all $x, y \in X$;*
- (t_2) *for all $x, y \in X$ with $(x, y) \in E(G)$,*

$$\rho(Tx, Ty) \leq \alpha\rho(x, y) + \beta\rho(x, Ty) + \gamma\rho(y, Tx),$$
$$\rho(Ty, Tx) \leq \alpha\rho(y, x) + \beta\rho(Ty, x) + \gamma\rho(Tx, y).$$

Then T has a fixed point if and only if $X_T \neq \emptyset$. Moreover, if $Tx_* = x_*$, then $\rho(x_*, x_*) = 0$. Also, if the subgraph of G with the vertex set $\mathrm{Fix}(T)$ *is connected, then the restriction of T to X_T is a Picard operator.*

Proof. We can prove this result by applying Theorem 2.3 with $\alpha(x) = \alpha$, $\beta(x) = \beta$ and $\gamma(x) = \gamma$. □

Several consequences of Theorem 2.3 follow now for particular choices of the graph. For example, consider b-metric (X, d) endowed with the complete graph G_0 whose vertex set coincides with X; that is, $V(G_0) = X$ and $E(G_0) = X \times X$. If we set $G = G_0$ in Theorem 2.3, then it is clear that the set X_T related to any self-map T on X coincides with the whole set X. Then we get the following corollary.

Corollary 2.6. *Let (X, d) be a complete b-metric space endowed with the graph G_0 and $s \geq 1$ be a given real number. Also, let ρ be a wt-distance and $T : X \to X$ be continuous. Suppose that there exist mappings $\alpha, \beta, \gamma : X \to [0, 1)$ such that the following conditions hold:*

- (t_1) *$\alpha(Tx) \leq \alpha(x)$, $\beta(Tx) \leq \beta(x)$ and $\gamma(Tx) \leq \gamma(x)$ for all $x \in X$;*
- (t_2) *$\big(s(\alpha + 2\beta) + (s^2 + s)\gamma\big)(x) < 1$ for all $x \in X$;*
- (t_3) *for all $x, y \in X$,*

$$\rho(Tx, Ty) \leq \alpha(x)\rho(x, y) + \beta(x)\rho(x, Ty) + \gamma(x)\rho(y, Tx),$$
$$\rho(Ty, Tx) \leq \alpha(x)\rho(y, x) + \beta(x)\rho(Ty, x) + \gamma(x)\rho(Tx, y).$$

Then T is a Picard operator.

Now, suppose that $(X, \sqsubseteq)$ is a poset. Consider on the poset X the graph G_1 given by $V(G_1) = X$ and $E(G_1) = \big\{(x, y) \in X \times X : x \sqsubseteq y\big\}$. Since $\sqsubseteq$ is reflexive, it follows that $E(G_1)$ contains all loops, too. Let $G = G_1$ in Theorem 2.3. Then we obtain the following fixed point corollary in complete b-metric spaces associated with a wt-distance ρ and endowed with a partial order.

Corollary 2.7. *Let (X,d) be a complete b-metric space endowed with the graph G_1 and $s \geq 1$ be a given real number. Also, let ρ be a wt-distance and $T : X \to X$ be a nondecreasing and orbitally G_1-continuous mapping. Suppose that there exist mappings $\alpha,\beta,\gamma : X \to [0,1)$ such that the following conditions hold:*

- (t_1) *$\alpha(Tx) \leq \alpha(x)$, $\beta(Tx) \leq \beta(x)$ and $\gamma(Tx) \leq \gamma(x)$ for all $x \in X$;*
- (t_2) *$\big(s(\alpha+2\beta)+(s^2+s)\gamma\big)(x) < 1$ for all $x \in X$;*
- (t_3) *for all $x,y \in X$ with $x \sqsubseteq y$,*

$$\rho(Tx,Ty) \leq \alpha(x)\rho(x,y)+\beta(x)\rho(x,Ty)+\gamma(x)\rho(y,Tx),$$
$$\rho(Ty,Tx) \leq \alpha(x)\rho(y,x)+\beta(x)\rho(Ty,x)+\gamma(x)\rho(Tx,y).$$

Then T has a fixed point in X if and only if there exists $x_0 \in X$ such that $x_0 \sqsubseteq Tx_0$. Moreover, if $Tx_* = x_*$, then $\rho(x_*,x_*) = 0$. Also, if the subgraph of G_1 with the vertex set $\mathrm{Fix}(T)$ is connected, then the restriction of T to the set of all points in $x \in X$ such that $x \sqsubseteq Tx$ is a Picard operator.

For our next consequence, consider on the poset X the graph G_2 defined by $V(G_2) = X$ and $E(G_2) = \big\{(x,y) \in X \times X : x \sqsubseteq y \ \vee \ y \sqsubseteq x\big\}$. Then, an ordered pair $(x,y) \in X \times X$ is an edge of G_2 if and only if x and y are comparable elements of $(X,\sqsubseteq)$. If we set $G = G_2$ in Theorem 2.3, then we obtain another fixed point theorem in complete b-metric spaces associated with a wt-distance ρ and endowed with a partial order.

Corollary 2.8. *Let (X,d) be a complete b-metric space endowed with the graph G_2 and $s \geq 1$ be a given real number. Also, let ρ be a wt-distance and $T : X \to X$ be an orbitally G_2-continuous mapping which maps comparable elements of X onto comparable elements. Suppose that there exist mappings $\alpha,\beta,\gamma : X \to [0,1)$ such that the following conditions hold:*

- (t_1) *$\alpha(Tx) \leq \alpha(x)$, $\beta(Tx) \leq \beta(x)$ and $\gamma(Tx) \leq \gamma(x)$ for all $x \in X$;*
- (t_2) *$\big(s(\alpha+2\beta)+(s^2+s)\gamma\big)(x) < 1$ for all $x \in X$;*
- (t_3) *for all $x,y \in X$ where x and y are comparable,*

$$\rho(Tx,Ty) \leq \alpha(x)\rho(x,y)+\beta(x)\rho(x,Ty)+\gamma(x)\rho(y,Tx),$$
$$\rho(Ty,Tx) \leq \alpha(x)\rho(y,x)+\beta(x)\rho(Ty,x)+\gamma(x)\rho(Tx,y).$$

Then T has a fixed point in X if and only if there exists $x_0 \in X$ such that x_0 and Tx_0 are comparable. Moreover, if $Tx_* = x_*$, then $\rho(x_*,x_*) = 0$. Also, if the subgraph of G_2 with the vertex set $\mathrm{Fix}(f)$ is connected, then the restriction of f to the set of all points $x \in X$ such that x and Tx are comparable is a Picard operator.

Let $\varepsilon > 0$ be a fixed number. Recall that two elements $x, y \in X$ are said to be ε-close if $d(x, y) < \varepsilon$. Define the ε-graph G_3 by $V(G_3) = X$ and $E(G_3) = \{(x, y) \in X \times X : \ d(x, y) < \varepsilon\}$. We see that $E(G_3)$ contains all loops. Finally, if we set $G = G_3$ in Theorem 2.3, then we get the following consequence of our fixed point theorem in complete b-metric spaces associated with a wt-distance ρ.

Corollary 2.9. *Let (X, d) be a complete b-metric space endowed with the graph G_3 and $s \geq 1$ be a given real number. Also, let ρ be a wt-distance and $T : X \to X$ orbitally G_3-continuous mapping which maps e-close elements of X onto e-close elements. Suppose that there exist mappings $\alpha, \beta, \gamma : X \to [0, 1)$ such that the following conditions hold:*

- (t_1) *$\alpha(Tx) \leq \alpha(x)$, $\beta(Tx) \leq \beta(x)$ and $\gamma(Tx) \leq \gamma(x)$ for all $x \in X$;*
- (t_2) *$\big(s(\alpha + 2\beta) + (s^2 + s)\gamma\big)(x) < 1$ for all $x \in X$;*
- (t_3) *for all $x, y \in X$ where x and y are ε-close elements,*

$$\rho(Tx, Ty) \leq \alpha(x)\rho(x, y) + \beta(x)\rho(x, Ty) + \gamma(x)\rho(y, Tx),$$
$$\rho(Ty, Tx) \leq \alpha(x)\rho(y, x) + \beta(x)\rho(Ty, x) + \gamma(x)\rho(Tx, y).$$

Then T has a fixed point in X if and only if there exists $x_0 \in X$ such that x_0 and Tx_0 are ε-close. Moreover, if $Tx_* = x_*$, then $\rho(x_*, x_*) = 0$. Also, if the subgraph of G_3 with the vertex set $\mathrm{Fix}(T)$ *is connected, then the restriction of T to the set of all points $x \in X$ such that x and Tx are ε-close is a Picard operator.*

In Corollaries 2.6, 2.7, 2.8 and 2.9, consider $\alpha(x) = \alpha$, $\beta(x) = \beta$ and $\gamma(x) = \gamma$. Then we have the same result similar to Theorem 2.5.

Remark 2.10. For Banach-type fixed point result with respect to a wt-distance on b-metric spaces with parameter $s \geq 1$, we use the condition

$$\rho(Tx, Ty) \leq \alpha\rho(x, y), \qquad \alpha \in [0, \frac{1}{s}).$$

In Theorem 2.3, set $s = 1$. Then we obtain the following theorem in the framework of a w-distance in metric spaces endowed with a graph.

Theorem 2.11. *Let (X, d) be a complete metric space endowed with the graph G, ρ be a w-distance and $T : X \to X$ be an orbitally G-continuous mapping. Suppose that there exist mappings $\alpha, \beta, \gamma : X \to [0, 1)$ such that the following conditions hold:*

- (t_1) *$\alpha(Tx) \leq \alpha(x)$, $\beta(Tx) \leq \beta(x)$ and $\gamma(Tx) \leq \gamma(x)$ for all $x \in X$;*
- (t_2) *$\big(\alpha + 2\beta + 2\gamma\big)(x) < 1$ for all $x \in X$;*
- (t_3) *T preserves the edges of G, that is, $(x, y) \in E(G)$ implies $(Tx, Ty) \in E(G)$ for all $x, y \in X$;*
- (t_4) *for all $x, y \in X$ with $(x, y) \in E(G)$,*

$$\rho(Tx, Ty) \leq \alpha(x)\rho(x, y) + \beta(x)\rho(x, Ty) + \gamma(x)\rho(y, Tx),$$

$$\rho(Ty,Tx) \leq \alpha(x)\rho(y,x) + \beta(x)\rho(Ty,x) + \gamma(x)\rho(Tx,y).$$

Then T has a fixed point if and only if $X_T \neq \emptyset$. Moreover, if $Tx_* = x_*$, then $\rho(x_*, x_*) = 0$. Also, if the subgraph of G with the vertex set $\mathrm{Fix}(T)$ *is connected, then the restriction of T to X_T is a Picard operator.*

An immediate consequence of Theorems 2.5 and 2.11 can be stated in the form of the following theorem.

Theorem 2.12. *Let (X,d) be a complete metric space endowed with the graph G and ρ be a w-distance and $T : X \to X$ be an orbitally G-continuous mapping. Suppose that there exist $\alpha, \beta, \gamma > 0$ with $\alpha + 2\beta + 2\gamma < 1$ such that the following conditions hold:*

- (t_1) *T preserves the edges of G, that is, $(x,y) \in E(G)$ implies $(Tx,Ty) \in E(G)$ for all $x,y \in X$;*
- (t_2) *for all $x,y \in X$ with $(x,y) \in E(G)$,*

$$\rho(Tx,Ty) \leq \alpha\rho(x,y) + \beta\rho(x,Ty) + \gamma\rho(y,Tx),$$
$$\rho(Ty,Tx) \leq \alpha\rho(y,x) + \beta\rho(Ty,x) + \gamma\rho(Tx,y).$$

Then T has a fixed point if and only if $X_T \neq \emptyset$. Moreover, if $Tx_* = x_*$, then $\rho(x_*, x_*) = 0$. Also, if the subgraph of G with the vertex set $\mathrm{Fix}(T)$ *is connected, then the restriction of T to X_T is a Picard operator.*

In Theorem 2.11, consider G_0, G_1, G_2 and G_3 instead of G. Then we have the same results in Corollaries 2.6, 2.7, 2.8 and 2.9 in the framework of a w-distance in metric spaces endowed with a graph. Also, in Remark 2.10, set $s = 1$. Then for Banach-type fixed point result with respect to a w-distance on metric spaces, we use the condition $\rho(Tx,Ty) \leq \alpha\rho(x,y)$ for all $x,y \in X$, where $\alpha \in [0,1)$.

3. Application to Nonlinear Analysis

In 2011, Cvetković et al. [10] defined cone metric type spaces as an extension of cone metric spaces introduced by Huang and Zhang [15]. On the other hand, Bao et al. [4] defined a generalized c-distance in cone b-metric spaces as a generalization of both wt-distance and c-distance introduced by Hussain et al. [16] and Cho et al. [8] (also, see [13, 19, 22] and references therein).

Let E be a real Banach space. Then a subset P of E is called a cone if and only if

- (a) P is closed, non-empty and $P \neq \{\theta\}$;
- (b) $a,b \in \mathbb{R}, a,b \geq 0, x,y \in P$ imply that $ax + by \in P$;
- (c) if $x, -x \in P$, then $x = \theta$.

Given a cone $P \subset E$, we define a partial ordering $\preceq$ with respect to P by $x \preceq y$ if and only if $y - x \in P$. We shall write $x \prec y$ if $x \preceq y$ and

$x \neq y$. Moreover, we denote $x \ll y$ if and only if $y - x \in intP$ where $intP$ is the interior of P. If $intP \neq \emptyset$, then the cone P is called solid. The cone P is named normal if there is a number $k > 0$ such that for all $x, y \in E$, $\theta \preceq x \preceq y$ implies that $\|x\| \leq k\|y\|$.

Definition 3.1 ([10])**.** Let X be a nonempty set, $s \geq 1$ be a real number, E be a real Banach space with zero element θ, and P be a cone in E. Suppose that $d : X \times X \to P$ is a mapping satisfying the following conditions:

- (d_1) $\theta \preceq d(x, y)$ for all $x, y \in X$ and $d(x, y) = \theta$ if and only if $x = y$;
- (d_2) $d(x, y) = d(y, x)$ for all $x, y \in X$;
- (d_3) $d(x, z) \preceq s[d(x, y) + d(y, z)]$ for all $x, y, z \in X$.

Then the pair (X, d) is called cone metric type space (or cone b-metric space).

For notions such as convergent and Cauchy sequences, completeness, continuity and etc. in cone b-metric spaces, we refer to [10].

Definition 3.2 ([4])**.** Let (X, d) be a cone b-metric space with parameter $s \geq 1$. A function $q : X \times X \to E$ is called a generalized c-distance on X if the following properties are satisfied:

- (q_1) $\theta \preceq q(x, y)$ for all $x, y \in X$;
- (q_2) $q(x, z) \preceq s[q(x, y) + q(y, z)]$ for all $x, y, z \in X$;
- (q_3) for $x \in X$, if $q(x, y_n) \preceq u$ for some $u = u_x$ and all $n \geq 1$, then $q(x, y) \preceq su$ whenever $\{y_n\}$ is a sequence in X converging to a point $y \in X$;
- (q_4) for all $c \in E$ with $\theta \ll c$, there exists $e \in E$ with $\theta \ll e$ such that $q(z, x) \ll e$ and $q(z, y) \ll e$ imply $d(x, y) \ll c$.

Example 3.3. [4, 22] Let $E = C^1([0, 1], \mathbb{R})$ with the norm $\|x\| = \|x\|_\infty + \|x'\|_\infty$ and consider the non-normal cone $P = \{x \in E : x(t) \geq 0 \text{ for all } t \in [0, 1]\}$. Also, let $X = [0, \infty)$ and define a mapping $d : X \times X \to E$ by $d(x, y) = |x - y|^s \psi$ for all $x, y \in X$, where $\psi : [0, 1] \to \mathbb{R}$ is defined by $\psi(t) = 2^t$ for all $t \in [0, 1]$. Then (X, d) is a cone b-metric space with $s \in \{1, 2\}$. Define a mapping $q : X \times X \to E$ by $q(x, y) = y^s \psi$ for all $x, y \in X$ and $s \in \{1, 2\}$. Then q is a generalized c-distance.

Note that for a generalized c-distance in cone b-metric space

- $q(x, y) = q(y, x)$ does not necessarily hold for all $x, y \in X$;
- $q(x, y) = \theta$ is not necessarily equivalent to $x = y$ for all $x, y \in X$.

Lemma 3.4. *Let (X, d) be a cone b-metric space with parameter $s \geq 1$ and q be a generalized c-distance on X. Also, let $\{x_n\}$ and $\{y_n\}$ be sequences in X and $x, y, z \in X$, and $\{u_n\}$ and $\{v_n\}$ be two sequences in P converging to θ. Then the following conditions hold:*

(i) *if* $q(x_n, y) \preceq u_n$ *and* $q(x_n, z) \preceq v_n$ *for* $n \in \mathbb{N}$, *then* $y = z$. *In particular, if* $q(x, y) = \theta$ *and* $q(x, z) = \theta$, *then* $y = z$;
(ii) *if* $q(x_n, y_n) \preceq u_n$ *and* $q(x_n, z) \preceq v_n$ *for* $n \in \mathbb{N}$, *then* $\{y_n\}$ *converges to* z;
(iii) *if* $q(x_n, x_m) \preceq u_n$ *for* $m, n \in \mathbb{N}$ *with* $m > n$, *then* $\{x_n\}$ *is a Cauchy sequence in* X;
(iv) *if* $q(y, x_n) \preceq u_n$ *for* $n \in \mathbb{N}$, *then* $\{x_n\}$ *is a Cauchy sequence in* X.

Proof. The proof is similar to a c-distance and a wt-distance in [8, 16]. □

Note that Definition 2.1, Definition 2.2 and other preliminaries on b-metric spaces can be introduced in the framework of cone b-metric spaces. Now, we obtain the following results in the framework of a generalized c-distance in cone b-metric spaces endowed with a graph. Since the procedure of proofs are similar to b-metric version, we remove them.

Theorem 3.5. *Let* (X, d) *be a complete cone* b*-metric space endowed with the graph* G *and* $s \geq 1$ *be a given real number. Also, let* q *be a generalized* c*-distance and* $T : X \to X$ *be an orbitally* G*-continuous mapping. Suppose that there exist mappings* $\alpha, \beta, \gamma : X \to [0, 1)$ *such that the following conditions hold:*

(t_1) $\alpha(Tx) \leq \alpha(x)$, $\beta(Tx) \leq \beta(x)$ *and* $\gamma(Tx) \leq \gamma(x)$ *for all* $x \in X$;
(t_2) $\big(s(\alpha + 2\beta) + (s^2 + s)\gamma\big)(x) < 1$ *for all* $x \in X$;
(t_3) T *preserves the edges of* G, *that is,* $(x, y) \in E(G)$ *implies* $(Tx, Ty) \in E(G)$ *for all* $x, y \in X$;
(t_4) *for all* $x, y \in X$ *with* $(x, y) \in E(G)$,

$$q(Tx, Ty) \preceq \alpha(x)q(x, y) + \beta(x)q(x, Ty) + \gamma(x)q(y, Tx),$$
$$q(Ty, Tx) \preceq \alpha(x)q(y, x) + \beta(x)q(Ty, x) + \gamma(x)q(Tx, y).$$

Then T *has a fixed point if and only if* $X_T \neq \emptyset$. *Moreover, if* $Tx_* = x_*$, *then* $q(x_*, x_*) = \theta$. *Also, if the subgraph of* G *with the vertex set* $\mathrm{Fix}(T)$ *is connected, then the restriction of* T *to* X_T *is a Picard operator.*

An immediate consequence of Theorem 3.5 can be stated in the form of the following theorem.

Theorem 3.6. *Let* (X, d) *be a complete cone* b*-metric space endowed with the graph* G *and* $s \geq 1$ *be a given real number. Also, let* q *be a generalized* c*-distance and* $T : X \to X$ *be an orbitally* G*-continuous mapping. Suppose that there exist* $\alpha, \beta, \gamma > 0$ *with* $s(\alpha+2\beta)+(s^2+s)\gamma < 1$ *such that the following conditions hold:*

(t_1) T preserves the edges of G, that is, $(x,y) \in E(G)$ implies $(Tx,Ty) \in E(G)$ for all $x,y \in X$;

(t_2) for all $x,y \in X$ with $(x,y) \in E(G)$,

$$q(Tx,Ty) \preceq \alpha q(x,y) + \beta q(x,Ty) + \gamma q(y,Tx),$$
$$q(Ty,Tx) \preceq \alpha q(y,x) + \beta q(Ty,x) + \gamma q(Tx,y).$$

Then T has a fixed point if and only if $X_T \neq \emptyset$. Moreover, if $Tx_* = x_*$, then $q(x_*,x_*) = \theta$. Also, if the subgraph of G with the vertex set $\mathrm{Fix}(T)$ *is connected, then the restriction of T to X_T is a Picard operator.*

In Theorem 3.5, consider G_0, G_1, G_2 and G_3 instead of G. Then we have the same results in Corollaries 2.6, 2.7, 2.8 and 2.9 in the framework of a generalized c-distance in cone b-metric spaces endowed with a graph. Also, for Banach-type fixed point result with respect to a generalized c-distance on cone b-metric spaces with parameter $s \geq 1$, we use the condition $q(Tx,Ty) \preceq \alpha q(x,y)$ with $\alpha \in [0, \frac{1}{s})$. Now, set $s = 1$. Then we get Theorem 3.5 and its consequents with respect to a c-distance in cone metric spaces.

4. Conclusion

In this paper, we replace the condition of continuity of mapping with the condition of orbitally G-continuity of mapping and we consider b-metric spaces endowed with graph instead of b-metric spaces, under which can be unified some theorems of the existing literature. Now, we finish this paper with some questions.

Question 4.1. Can one obtain the same results of this paper by considering some another conditions instead of the continuity of the mapping T?

Question 4.2. Can one prove main theorem and its corollaries by considering one contractive-type relation instead of two contractive-type relations?

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