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A Class of Convergent Series with Golden Ratio Based on Fibonacci Sequence

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ABSTRACT. In this article, a class of convergent series based on Fibonacci sequence is introduced for which there is a golden ratio (i.e. $\frac{1+\sqrt{5}}{2}$), with respect to convergence analysis. A class of sequences are at first built using two consecutive numbers of Fibonacci sequence and, therefore, new sequences have been used in order to introduce a new class of series. All properties of the sequences and related series are illustrated in the work by providing the details including sequences formula, related theorems, proofs and convergence analysis of the series.

1. Introduction

Developing a sequence of numbers (later called the Fibonacci sequence) in which the first two numbers are one, then they are added to get 2, 2 is added to the prior number of 1 to get 3, 3 is added to the prior number of 2 to get 5, 5 is added to the prior number of 3 to get 8, and so on. Hence, the recursive formula of Fibonacci sequence begins as

$$f_1 = f_2 = 1, \qquad f_{n+1} = f_n + f_{n-1}, \quad n \geq 2.$$

Recall that this sequence is a recursive sequence which depends on the values of the previous two entries to obtain the next entry. For the geometric interpretation of Fibonacci sequence, one can increasingly use larger squares of unit dimension; Consider the first number of sequence, i.e. one and add it to second one (look in the center of the Figure 1), the result is two -which is the side length of the square annotated with

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a 2. If squares 1 and 2 are added together, the result is square 3 (side length of 3). The same procedure applies for square 5: square 3 + square 2 (actually squares 1 plus 1). Hence, the Fibonacci sequence can be illustrated by adding squares as described in this paper.

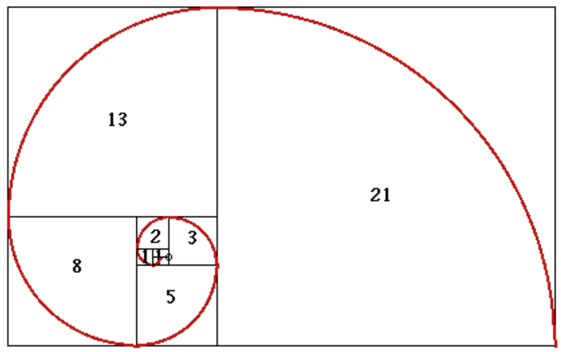

FIGURE 1. Geometric interpretation of Fibonacci numbers

There are a lot of research works done on the Fibonacci sequence. As an example, one can see the relation between Fibonacci numbers and Lucas numbers in graphs which is illustrated in [5]. Also, in [4], authors discussed on a family of Fibonacci means: In fact, a variety of "mean" $M(x, y)$ with $x > 0$, $y > 0$ has been illustrated as follows:

$$M(x, y) = \frac{a(x+y) + 2bxy}{2a + b(x+y)},$$

whereas for $y = x$,

$$\begin{aligned} M(x, y) &= M(x, x) \\ &= \frac{2ax + 2bx^2}{2a + 2bx} \\ &= x, \end{aligned}$$

provided the condition $2a + 2bx \neq 0$ for any $x \in R(x > 0)$. Particular cases are $a > 0$, $b = 0$, $M(x, y) = \frac{x+y}{2}$, the average (arithmetic mean), $a = 0, b > 0, M(x, y) = \frac{2xy}{x+y}$, the harmonic mean, and if $q = \frac{1+\sqrt{5}}{2}, M_q(x, y) = \frac{q(x+y)+2xy}{2q+(x+y)}$, is the golden section mean. Hence the harmonic mean, the arithmetic mean and golden section mean are special cases of the Fibonacci mean. There is a mapping $D : M \to DM$ which gives us the relation between Fibonacci, harmonic and arithmetic mean. On the other hand, $\lim_{n\to\infty} D^n M$ is the golden section mean which is of interest [4]. In this work, we are going to introduce a new applications of the golden ratio which is arising from the Fibonacci numbers.

Remark 1.1. For any $n+1$ entries of Fibonacci sequence, then

$$\begin{aligned}\sum_{i=1}^{n+1} f_i^2 &= f_1^2 + f_2^2 + f_3^2 + \cdots + f_{n+1}^2 \\ &= f_{n+1} f_{n+2},\end{aligned}$$

where $f_1 = f_2 = 1, f_{n+1} = f_n + f_{n-1}; n \geq 2$.

Proof. In order to see the details of proof the reader is referred to [1]. □

Remark 1.2. Explicit depending formula of Fibonacci sequence is

$$f_n = \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2} \right)^n - \frac{1}{\sqrt{5}} \left(\frac{1-\sqrt{5}}{2} \right)^n, \quad (n \geq 1).$$

Proof. The recursive formula of Fibonacci sequence gives us the following characteristic equation:

$$q^{n+1} = q^n + q^{n-1} \quad \Rightarrow \quad q^{n-1}\left(q^2 - q - 1\right) = 0, \tag{1.1}$$

where we supposed that $f_n = q^n, q \neq 0, n \geq 1$. Therefore, we have two roots of the equation $\left(q^2 - q - 1\right) = 0$:

$$q_1 = \frac{1+\sqrt{5}}{2}, \qquad q_2 = \frac{1-\sqrt{5}}{2}. \tag{1.2}$$

It is easy to show that $G_{n+1} = G_n + G_{n-1}$ where $G_n = c_1 q_1^n + c_2 q_2^n$. Thus, in order to make $G_n = f_n$, we have to find out two coefficients c_1 and c_2 providing:

$$\begin{aligned} G_1 = f_1 \quad &\Rightarrow \quad c_1\left(1+\sqrt{5}\right) + c_2\left(1-\sqrt{5}\right) = 2, \\ G_2 = f_2 \quad &\Rightarrow \quad c_1\left(3+\sqrt{5}\right) + c_2\left(3-\sqrt{5}\right) = 2, \end{aligned}$$

which gives us $c_1 = \frac{1}{\sqrt{5}}, c_2 = \frac{-1}{\sqrt{5}}$. Hence, the proof is completed. □

Note that in the following parts of the paper, we will show that the two roots q_1 and q_2 of the equation (1.2) by ϕ and ϵ, respectively. Now more explorations with Fibonacci numbers can be considered as follows:

$$\begin{aligned}\lim_{n\to\infty} \frac{f_{n+1}}{f_n} &= \lim_{n\to\infty} \frac{\frac{1}{\sqrt{5}}\left(\phi^{n+1} - \epsilon^{n+1}\right)}{\frac{1}{\sqrt{5}}\left(\phi^n - \epsilon^n\right)} \\ &= \lim_{n\to\infty} \frac{1 - \left(\frac{\epsilon}{\phi}\right)^{n+1}}{\frac{1}{\phi}\left(1 - \left(\frac{\epsilon}{\phi}\right)^n\right)} = \phi, \quad \frac{\epsilon}{\phi} < 1\end{aligned} \tag{1.3}$$

$$\lim_{n\to\infty}\frac{f_{n+2}}{f_n}=\lim_{n\to\infty}\frac{\frac{1}{\sqrt{5}}\left(\phi^{n+2}-\epsilon^{n+2}\right)}{\frac{1}{\sqrt{5}}\left(\phi^{n}-\epsilon^{n}\right)}
=\lim_{n\to\infty}\frac{1-\left(\frac{\epsilon}{\phi}\right)^{n+2}}{\frac{1}{\phi^2}\left(1-\left(\frac{\epsilon}{\phi}\right)^{n}\right)}=\phi^2,\quad \frac{\epsilon}{\phi}<1. \tag{1.4}$$

The golden ratio ($\phi=\frac{1+\sqrt{5}}{2}$) could be used to verify the convergence order of secant iteration method

$$x_{n+1}=x_n-\frac{(x_n-x_{n-1})\,f(x_n)}{f(x_n)-f(x_{n-1})},$$

by which one can solve nonlinear equations ($f(x)=0$) instead of Newton iteration scheme when the first derivative of the considered function is not so easy to evaluate in use. It could be noted that the convergence order of secant iteration method is $\phi=\frac{1+\sqrt{5}}{2}$ whereas for the Newton iteration scheme is 2. For more details we refer the reader to [3].

Let us take a look at Figure 1 again. It shows that there is a sequence of squares' areas as follows:

$$s_1=s_2=f_1^2=1,\qquad s_n=f_n^2,\quad ,n=3,4,\dots.$$

From this equations and together with Remark 1.1, we obtain the following result which is the sum of n squares' areas based on Fibonacci numbers:

$$S_n=\sum_{i=1}^{n}s_i^2=f_1^2+f_2^2+\cdots+f_n^2=f_nf_{n+1},\quad (n\geq 1). \tag{1.5}$$

Theorem 1.3. *Let the sequence $\{S_n\}_1^n$ be the sum of n squares' areas as explained in equation (1.5), then:*

(i) *the series $\sum_{n=1}^{\infty}\frac{1}{S_n}$ is convergent, and*
(ii) *the series $\sum_{n=1}^{\infty}S_n$ is divergent.*

Proof. (i) From the equation (1.5), the series $\sum_{n=1}^{\infty}\frac{1}{S_n}$ can be written in the form:

$$\sum_{n=1}^{\infty}\frac{1}{S_n}=\sum_{n=1}^{\infty}\frac{1}{\left(\sum_{j=1}^{n}f_j^2\right)}
=\sum_{n=1}^{\infty}\frac{1}{f_1^2+f_2^2+\cdots+f_n^2}$$

$$= \sum_{n=1}^{\infty} \frac{1}{f_n f_{n+1}},$$

for which we use ratio test of series to verify its convergence. To do that, set $a_n = \frac{1}{f_n f_{n+1}}$ and so we have:

$$\begin{aligned} L &= \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| \\ &= \lim_{n\to\infty} \left| \frac{f_{n+1} f_n}{f_{n+1} f_{n+2}} \right| \\ &= \lim_{n\to\infty} \left| \frac{f_n}{f_{n+2}} \right| \\ &= \frac{1}{\phi^2} \\ &< 1. \end{aligned}$$

Since $L < 1$, hence the series is convergent where its ratio is related to ϕ^2.

(ii) From the equation (1.5), the series $\sum_{n=1}^{\infty} S_n$ can be written in the form:

$$\begin{aligned} \sum_{n=1}^{\infty} S_n &= \sum_{n=1}^{\infty} \left(\sum_{j=1}^{n} f_j^2 \right) \\ &= \sum_{n=1}^{\infty} f_1^2 + f_2^2 + \cdots + f_n^2 \\ &= \sum_{n=1}^{\infty} f_n f_{n+1}, \end{aligned}$$

for which we use ratio test of series to verify its divergence. To do that, set $a_n = f_n f_{n+1}$ and so we have:

$$\begin{aligned} L &= \lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| \\ &= \lim_{n\to\infty} |\frac{f_{n+1} f_{n+2}}{f_{n+1} f_n}| \\ &= \lim_{n\to\infty} |\frac{f_{n+2}}{f_n}| \\ &= \phi^2 \\ &> 1. \end{aligned}$$

This shows that the series is divergent.

□

2. New Class of Sequences Based on Fibonacci Numbers

As explained in the last section, the series $\sum_{n=1}^{\infty} \frac{1}{S_n}$ based on the sum of squares' areas is convergent with a ratio related to the golden ratio ϕ^2. The main sequence was in fact with entries which show the sides or edges with the golden ratio ϕ. Hence, for segment sizes based on Fibonacci numbers, the ratio is the root of squares' areas ration when $n \to \infty$.

2.1. **A sequence of triangles.** Now, we are going to introduce a sequence of triangles where its two sides are f_n and f_{n+1} (two consecutive numbers of Fibonacci sequence) with angle α between them. Therefore, we can gain the inner side of this triangle and so we have a sequence of triangles compared to Fig. 1 which explain a sequence of squares (see Fig. 2 for more explanation). We here denote the inner edge of each triangle with z_n. It is clear that the following relation between the three sides is true:

$$z_n^2 = f_n^2 + f_{n+1}^2 + 2f_nf_{n+1}\cos(\alpha), \quad n = 1, 2, 3, \ldots. \tag{2.1}$$

This suggests a sequence of inner edges numbers of triangle as $\{z_n\}_{n=1}^{\infty}$. Note that in Fig. 1, the sequence of squares sides is $\{f_n\}$.

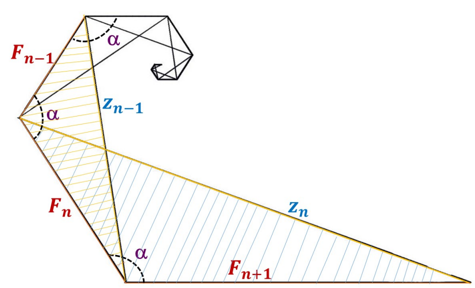

Figure 2. Triangle sequence based on $F_n \& F_{n+1}$

Theorem 2.1. *Let the sequence $\{z_n\}_1^{\infty}$ be the inner side of triangles as mentioned above, then we have:*

$$\lim_{n\to\infty} \frac{z_{n+1}}{z_n} = \phi.$$

Proof. From the definition of z_n and equations (1.4)-(1.5), we have:

$$\lim_{n\to\infty} \frac{z_{n+1}^2}{z_n^2} = \lim_{n\to\infty} \frac{f_{n+1}^2 + f_{n+2}^2 + 2f_{n+1}f_{n+2}\cos(\alpha)}{f_n^2 + f_{n+1}^2 + 2f_nf_{n+1}\cos(\alpha)}$$

$$
\begin{aligned}
&= \lim_{n\to\infty} \frac{1+\frac{f_{n+2}^2}{f_{n+1}^2}+2\frac{f_{n+2}}{f_{n+1}}\cos(\alpha)}{1+\frac{f_n^2}{f_{n+1}^2}+2\frac{f_n}{f_{n+1}}\cos(\alpha)}\\
&= \frac{1+\phi^2+2\phi\cos(\alpha)}{1+\phi^{-2}+2\phi^{-1}\cos(\alpha)}\\
&= \frac{\phi^2\left(1+\phi^2+2\phi\cos(\alpha)\right)}{1+\phi^2+2\phi\cos(\alpha)}\\
&= \phi^2.
\end{aligned}
$$

Thus, the proof will be completed as follows:

$$
\lim_{n\to\infty}\frac{z_{n+1}^2}{z_n^2}=\phi^2 \quad \Rightarrow \quad \lim_{n\to\infty}\frac{z_{n+1}}{z_n}=\phi.
$$

□

Definition 2.1. Assume that f_n and f_{n+1} are two consecutive numbers of Fibonacci sequence and three points

$$
O=(0,0,0), \quad A=(\cos\beta f_n, \sin\beta f_n, 0), \quad B=(\cos\gamma f_{n+1}, \sin\gamma f_{n+1}, 0),
$$

belong to R^3. We define two Fibonacci number based vectors $\overrightarrow{f_n}, \overrightarrow{f_{n+1}} \in R^3$ as follows:

$$
\begin{aligned}
\overrightarrow{f_n} &= \overrightarrow{OA} = (\cos\beta f_n, \sin\beta f_n, 0),\\
\overrightarrow{f_{n+1}} &= \overrightarrow{OB} = (\cos\gamma f_{n+1}, \sin\gamma f_{n+1}, 0),
\end{aligned}
$$

where β is the angle of the vector $\overrightarrow{f_n}$ made by x-axes, γ is the angle of the vector $\overrightarrow{f_{n+1}}$ made by x-axes, and α is the difference of the angles β and γ.

In this work, we assume that $0 \le \beta,\ \gamma \le \pi$ and $\beta < \gamma$. Therefore, $\alpha = \gamma - \beta \ge 0$ and it is the angle between the vectors $\overrightarrow{f_n}$ and $\overrightarrow{f_{n+1}}$. From the definition of the two vectors $\overrightarrow{f_n}$ and $\overrightarrow{f_{n+1}}$ we have:

$$
\begin{aligned}
|\overrightarrow{f_n}| &= \sqrt{(\cos^2\beta+\sin^2\beta)f_n^2} = f_n,\\
|\overrightarrow{f_{n+1}}| &= \sqrt{(\cos^2\gamma+\sin^2\gamma)f_{n+1}^2} = f_{n+1}.
\end{aligned}
$$

2.2. **A Sequence of Parallelepiped.** In this section, we are going to introduce a new sequence of parallelepiped for which the base is the same as illustrated in the first subsection of current section which means that we defined the parallelepiped where its edges length of base is $f_n \& f_{n+1}$ and the angle α is the same between edges with lengths f_n and f_{n+1}.

Therefore, the height of parallelepiped is h_n as follows:

(2.2)
$$\begin{aligned}\overrightarrow{h_n} =& \overrightarrow{f_n} \times \overrightarrow{f_{n+1}}\\ =&(0,0,(\sin\gamma\cos\beta - \cos\gamma\sin\beta)f_n f_{n+1})\\ =&(0,0,\sin(\gamma-\beta)f_n f_{n+1})\\ =&(0,0,\sin\alpha f_n f_{n+1}),(\alpha=\gamma-\beta),\end{aligned}$$

where α is the angle between two vectors $\overrightarrow{f_n}$ and $\overrightarrow{f_{n+1}}$ for all $n \in \mathbb{N}$ and $\times$ is the cross product of two vectors in R^3 (see Fig. 3).

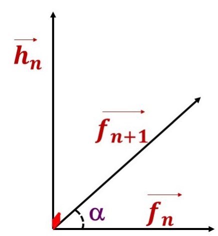

FIGURE 3. Cross product of the new sequences

For example, we insert $n = 1$, $\beta = 0$, $\gamma = \frac{\pi}{2}$, and explain the $\overrightarrow{f_n} \times \overrightarrow{f_{n+1}}$($i.e.\overrightarrow{f_1} \times \overrightarrow{f_2}$) and h_n as follows:

$$\overrightarrow{f_1} = (\cos 0 f_1, \sin 0 f_1, 0) = (f_1, 0, 0) = (1,0,0) \quad \Rightarrow \quad |\overrightarrow{f_1}| = f_1 = 1,$$
$$\overrightarrow{f_2} = (\cos\frac{\pi}{2} f_2, \sin\frac{\pi}{2} f_2, 0) = (0, f_2, 0) = (0,1,0) \quad \Rightarrow \quad |\overrightarrow{f_2}| = f_2 = 1.$$

Hence, the cross product of $\overrightarrow{f_1}$ and $\overrightarrow{f_2}$ is defined as follows:

$$\overrightarrow{h_1} = \overrightarrow{f_1} \times \overrightarrow{f_2} \quad \Rightarrow \quad \overrightarrow{h_1} = (1,0,0) \times (0,1,0) = (0,0,1),$$

this gives us: $h_1 = |\overrightarrow{h_1}| = |\overrightarrow{f_1}||\overrightarrow{f_2}| \sin\alpha = f_1 f_2 \sin\frac{\pi}{2} = f_1 f_2$.

Theorem 2.2. *Suppose that the* $\{h_n\}_{n=1}^{\infty}$ *(defined in equation* (2.2)*) is a sequence of heights related to parallelepiped. Then:*

$$\lim_{n\to\infty} \frac{h_{n+1}}{h_n} = \phi^2.$$

Proof. From the definition of $\{h_n\}_{n=1}^{\infty}$ and the properties of Fibonacci sequence explained in section 1 we have

$$\lim_{n\to\infty} \frac{h_{n+1}}{h_n} = \lim_{n\to\infty} \frac{f_{n+1}f_{n+2}\sin\alpha}{f_n f_{n+1}\sin\alpha} \tag{2.2}$$

$$= \lim_{n\to\infty} \frac{f_{n+2}}{f_n}$$
$$= \phi^2.$$

Hence, the proof is complete. □

3. New Class of Convergent Series

3.1. **Series Based on** $\cos(\alpha)$**:** In this section, we are going to obtain some series based on sequences illustrated in the last section. First we consider the sequence $\{c_n\}_{n=1}^{\infty}$:
Suppose that $c_n^2 = f_n^2 + f_{n+1}^2 + 2f_nf_{n+1}\cos\alpha$ (for $n \geq 1$) as illustrated in (2.1). inserting $\alpha = \frac{\pi}{2}$ gives us a sequence of the form:

(3.1)
$$c_n^2 = f_n^2 + f_{n+1}^2 + 2f_nf_{n+1}\cos(\frac{\pi}{2}) \quad \Rightarrow \quad c_n^2 = f_n^2 + f_{n+1}^2 \quad (n \geq 1).$$

Therefore, we can obtain the following sum related to $\{c_n\}_{n=1}^{\infty}$:

$$c_1^2 = f_1^2 + f_2^2, \quad c_2^2 = f_2^2 + f_3^2, \quad c_3^2 = f_3^2 + f_4^2, \quad \ldots \quad c_n^2 = f_n^2 + f_{n+1}^2,$$

then

$$s_n^2 = 2(f_1^2 + f_2^2 + \cdots + f_n^2 + f_{n+1}^2) - f_1^2 - f_{n+1}^2.$$

using equation (1.5), we obtain

$$\begin{aligned} s_n^2 =& 2f_{n+1}f_{n+2} - 1 - 2f_{n+1}^2 + f_{n+1}^2 \\ =& 2f_{n+1}(f_{n+2} - f_{n+1}) + f_{n+1}^2 - 1 \\ =& 2f_{n+1}f_n + f_{n+1}^2 - 1 \\ =& 2f_{n+1}(2f_n + f_{n+1}) - 1 \\ =& f_{n+1}f_{n+2} + f_{n+1}f_n - 1. \end{aligned}$$

By inserting $C_n = 1 + s_n^2$, we can gain a series of the form:

$$\sum_{n=1}^{\infty} \frac{1}{C_n}.$$

For testing the convergance of this series we use the ratio test as follows:

$$\begin{aligned} L =& \lim_{n\to\infty} \left| \frac{\frac{1}{C_{n+1}}}{\frac{1}{C_n}} \right| \\ =& \lim_{n\to\infty} \left| \frac{\frac{1}{f_{n+2}f_{n+3}+f_{n+2}f_{n+1}}}{\frac{1}{f_{n+1}f_{n+2}+f_nf_{n+1}}} \right| \\ =& \lim_{n\to\infty} \left| \frac{f_{n+1}f_{n+2} + f_nf_{n+1}}{f_{n+2}f_{n+3} + f_{n+2}f_{n+1}} \right| \end{aligned}$$

$$= \lim_{n\to\infty}\left|\frac{\frac{f_n}{f_{n+2}}+1}{\frac{f_{n+3}}{f_{n+1}}+1}\right|$$
$$= \frac{\frac{1}{\phi^2}+1}{\phi^2+1}$$
$$= \frac{1}{\phi^2}\left(\frac{\phi^2+1}{\phi^2+1}\right)$$
$$= \frac{1}{\phi^2}.$$

Since $L < 1$, it leads to the following theorem:

Theorem 3.1. *Let* $\{c_n\}_{n=1}^{\infty}$ *be as in* (3.1) *and suppose that*

$$C_n = 1 + s_n^2 = 1 + \sum_{i=1}^{n} C_i^2, \quad (C^2 = c_1^2 + c_2^2 + c_3^2 + \cdots + c_n^2).$$

then the series $\sum_{i=1}^{n}\left(\frac{1}{C_i}\right)$ *is convergent.*

Let us consider the sequence $\{c_n\}_{n=1}^{\infty}$ again by inserting $\alpha = 0$.

$$\begin{aligned} & c_n^2 = f_n^2 + f_{n+1}^2 + 2f_nf_{n+1}\cos(0) \\ \Rightarrow\quad & c_n^2 = (f_n + f_{n+1})^2 \\ \Rightarrow\quad & c_n = f_n + f_{n+1}. \end{aligned}$$

It means that for any $n \geq 1$, we have $c_n = f_{n+2}$. Defining

$$\begin{aligned} s_n &= \sum_{i=1}^{n} C_i^2 \\ &= c_1^2 + c_2^2 + c_3^2 + \cdots + c_n^2 \\ &= \sum_{i=1}^{n} f_{i+2}^2 = s_1 + s_2 + \cdots + s_n, \end{aligned}$$

gives us roughly the same series based on Fibonacci number squares illustrated in the first section for which the ratio test has been used to verity its convergence property. We remark here that the upper and lower bounds for C_n are $k_1 = f_n+f_{n+1} = f_{n+2}$ and $k_2 = f_{n+1}-f_n = f_{n-1}$ respectively, where $n \geq 2$.

3.2. **Series Based on** $\sin(\alpha)$**:** Let us consider the sequence of the heights $\{h_n\}_{n=1}^{\infty}$ built by cross product of two consecutive numbers of Fibonacci sequence. In fact, we have $h_n = f_n f_{n+1} \sin\alpha$. Inserting $\alpha = \frac{\pi}{2}$ gives the spacial case of $\{h_n\}_{n=1}^{\infty}$ where $h_n = f_n f_{n+1}$ for $n \geq 1$. Now, by defining a series of the form

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{1}{h_n} &= \sum_{n=1}^{\infty} \frac{1}{f_n f_{n+1}} \\ &= \sum_{n=1}^{\infty} \frac{1}{f_1^2 + f_2^2 + \cdots + f_n^2},\end{aligned}$$

it can be concluded that this series is convergent too:

$$\begin{aligned}L &= \lim_{n\to\infty} \left| \frac{\frac{1}{h_{n+1}}}{\frac{1}{h_n}} \right| \\ &= \lim_{n\to\infty} \left| \frac{h_n}{h_{n+1}} \right| \\ &= \lim_{n\to\infty} \left| \frac{f_n f_{n+1}}{f_{n+1} f_{n+2}} \right| \\ &= \lim_{n\to\infty} \left| \frac{f_n}{f_{n+2}} \right| \\ &= \frac{1}{\phi^2}.\end{aligned}$$

Since $\frac{1}{\phi^2} < 1$, therefore by using the ratio test on series, one can say that the series is convergent. The volume of parallelepiped based on two secutive numbers of Fibonacci Sequence, where α is the same as used in the sequence of triangles and heights $\{c_n\}_{n=1}^{\infty}$ and $\{h_n\}_{n=1}^{\infty}$ respectively, can be obtained as follows:

$$\begin{aligned}V_n &= (f_n f_{n+1} \cos\alpha)(f_n f_{n+1} \sin\alpha) \\ &= \frac{1}{2}(f_n f_{n+1})^2 \sin(2\alpha).\end{aligned}$$

If we choose $\alpha = \frac{\pi}{4}$, then the sequence of volumes $\{V_n\}_{n=1}^{\infty}$ can be shown with the following from:

$$V_n = \frac{1}{2}(f_n f_{n+1})^2, \quad n = 1, 2, 3, \ldots, \tag{3.2}$$

Now, we define a convergent series of the form:

$$\sum_{n=1}^{\infty} \frac{1}{V_i} = \sum_{i=1}^{\infty} \left(\frac{\sqrt{2}}{f_i f_{i+1}} \right)^2$$

$$= \sum_{i=1}^{\infty} \left(\frac{\sqrt{2}}{f_1^2 + f_2^2 + \cdots + f_i^2} \right)^2 .$$

Using the ratio test again, it can be written that

$$\begin{aligned} L &= \lim_{n\to\infty} \left| \frac{\frac{1}{V_{n+1}}}{\frac{1}{V_n}} \right| \\ &= \lim_{n\to\infty} \left| \frac{V_n}{V_{n+1}} \right| \\ &= \lim_{n\to\infty} \left| \left(\frac{f_n f_{n+1}}{f_{n+1} f_{n+2}} \right)^2 \right| \\ &= \lim_{n\to\infty} \left| \left(\frac{f_n}{f_{n+2}} \right)^2 \right| \\ &= \left(\frac{1}{\phi^2} \right)^2 \\ &= \frac{1}{\phi^4}. \end{aligned}$$

4. Conclusion

In this work, new classes of sequences based on two Fibonacci numbers are given. The first class of sequences peresnted in this paper, is the areas of triangles as shown in Fig. 2. The ratio of the inner edge of this type of triangles is related to golden ratio ϕ. The second class of sequences presented, which are based on cross product of two vector $\overrightarrow{a}$ and $\overrightarrow{b}$ where $|\overrightarrow{a}|$ and $\left|\overrightarrow{b}\right|$, are $f_n \& f_{n+1}$, respectively ($\overrightarrow{h_n} = \overrightarrow{a} \times \overrightarrow{b}$). It is clear that by changing the value of angle α between two vectors $\overrightarrow{a}$ and $\overrightarrow{b}$, a new sequence based on Fibonacci numbers can be obtained. Also, two classes of convergent series based on two above mentioned sequences have been presented with required proofs and analysis. Some theorems and needed proofs have been illustrated in the paper too.

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