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A Full-NT Step Infeasible Interior-Point Algorithm for Mixed Symmetric Cone LCPs

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Abstract. An infeasible interior-point algorithm for mixed symmetric cone linear complementarity problems is proposed. Using the machinery of Euclidean Jordan algebras and Nesterov-Todd search direction, the convergence analysis of the algorithm is shown and proved. Moreover, we obtain a polynomial time complexity bound which matches the currently best known iteration bound for infeasible interior-point methods.

1. Introduction

Mixed symmetric cone linear complementarity problems (MSLCP) are a general class of complementarity problems which contains some well-known and well-studied mathematical and optimization problems such as symmetric optimization (SO) problems, convex quadratic symmetric cone optimization (CQSCO) problems, semidefinite optimization (SDO) problems, linear complementarity problems (LCP), and symmetric cone LCP (SCLCP) .

Let $(\mathcal{U},\circ)$ and $(\mathcal{V},\circ)$ be Euclidean Jordan algebras (EJAs) with dimensions m and n and ranks r_1 and r_2, equipped with the standard inner product $\langle x,s\rangle = Tr(x\circ s)$ and $\mathcal{K}$ be the symmetric cone corresponding with $(\mathcal{V},\circ)$. Furthermore, let $\mathcal{J}=\mathcal{U}\times\mathcal{V}$ be the cartesian EJA with dimension $m+n$ and rank $r=r_1+r_2$.

Let

$$M=\begin{bmatrix} M_{11} & M_{12}\\ M_{21} & M_{22}\end{bmatrix},$$

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be an $(m+n)$-matrix written as a 2×2 block matrix where M_{11} is a $n \times n$ symmetric matrix, $M_{21} = M_{12}^T$, M_{22} is a $m \times m$ positive definite symmetric matrix and so, M_{12} is a $n \times m$ matrix. The MSLCP is the problem of computation a vector triple $(x, s, y) \in \mathcal{K} \times \mathcal{K} \times \mathcal{U}$ such that

$$\begin{pmatrix} s \\ 0 \end{pmatrix} = M \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} q_1 \\ q_2 \end{pmatrix}, \quad x \circ s = 0,$$

where $\binom{q_1}{q_2} \in \mathcal{J}$ and M is the Cartesian symmetric positive semidefinite matrix. That is, for all vectors $u \in \mathcal{J}$, $\langle u, Mu \rangle \geq 0$. Various approaches have been proposed for solving MSLCPs. Among them, interior-point methods (IPMs) attract more attention due to polynomial complexity and efficient implementation.

The study of the polynomial complexity and numerical implementation for a short-step primal-dual IPM for LCPs has been presented by Achache [1]. Mansouri et al. [8, 9] proposed some interior-point algorithms for LCPs. Lin et al. [7] suggested a homogeneous model for solving MSLCPs. Achache et al. [2] suggested a full-Newton step feasible weighted primal-dual path-following interior-point algorithm for monotone LCPs. Motivated by Achache et al. [2], a weighted-path-following interior-point algorithm has been presented by Wang et al. [13] for MSLCPs. Mansouri et al. [10], using the machinery of Euclidean Jordan algebra and Nesterov-Todd (NT) search direction, suggested a feasible path-following interior-point algorithm for MSLCPs.

In the above mentioned citations, the proposed algorithms are feasible. That is, the algorithms need to a feasible starting point and generate a set of feasible iterations during their implementations. However, finding an initial feasible solution of MSLCPs is the main difficulty of feasible interior-point algorithms for solving this class of mathematical problems.

To remedy, in this paper, we propose an infeasible interior-point algorithm for MSLCPs. Infeasible interior-point methods (IIPMs) start with an arbitrary positive point and feasibility is reached as optimality is approached. We prove the convergence analysis of the algorithm and show that the algorithm will terminate after at most $O(r_2 L)$ iterations.

The paper is organized as follows. In Section 2, we list some concepts and results on EJAs and symmetric cones which are required in our analysis. Section 3 is devoted to describe an infeasible interior-point algorithm for MSLCPs in more detail. The main part of this paper, the convergence analysis of the proposed algorithm, is presented in Section 4. Finally, the paper ends with some conclusions and remarks follow in Section 5.

2. Preliminaries

In this section, we outline a minimal of the foundation of the theory of EJAs which will be used in continue of paper.

The classical EJA $(\mathcal{V}, \circ)$ is a finite dimensional vector space over $\mathbb{R}$ equipped with the bilinear map $\circ : (x, y) \to x \circ y \in \mathcal{V}$ and the standard inner product $\langle x, s\rangle = Tr(x \circ s)$, while the Cartesian EJA is a Cartesian product of a finite number (such as N) of classical EJAs with the canonical inner product

$$\langle x, s\rangle = \sum_{i=1}^{N} \langle x^i, s^i\rangle.$$

The related cone of squares corresponding with $(\mathcal{V}, \circ)$ is called the classical symmetric cone $\mathcal{K}$. For each $x \in \mathcal{V}$, $L(x)y = x \circ y$ and $P(x) = 2L(x)^2 - L(x^2)$, where $L(x)^2 = L(x)L(x)$, denote the linear and quadratic representation of $\mathcal{V}$, respectively.

A Jordan algebra has an identity element, if there exists a unique element $e \in \mathcal{V}$ such that $x\ o\ e = e\ o\ x = x$ for all $x \in \mathcal{V}$. An element $c \in \mathcal{J}$ is said to be idempotent if $c^2 = c$. An idempotent c is primitive if it is nonzero and can not be expressed by sum of two other nonzero idempotents. A set consists of idempotents $\{c_1, c_2, \ldots, c_k\}$ is called a Jordan frame if $c_i\ o\ c_j = 0$ for any $i \neq j$, and $\sum_{i=1}^{k} c_i = e$. For any $x \in \mathcal{V}$, let r be the smallest positive integer such that $\{e, x, x^2, \ldots, x^r\}$ is linearly dependent, r is called the degree of x and is denoted by $\deg(x)$. The rank of $\mathcal{V}$, denoted by rank $(\mathcal{V})$, is defined as the maximum of $\deg(x)$ over all $x \in \mathcal{V}$.

The spectral decomposition theorem (Theorem III.1.2 of [3]) of an EJA $\mathcal{V}$ states that for any $x \in \mathcal{V}$ there exists a Jordan frame $\{c_{1,2}, \ldots, c_r\}$ and real numbers $\{\lambda_1, \lambda_2, \ldots, \lambda_r\}$ (the eigenvalues of x) such that $x = \sum_{i=1}^{r} \lambda_i c_i$. For any $x \in \mathcal{V}$, the norm induced by the standard inner product is named as the Frobenius norm, which is given by $\|x\|_F := \sqrt{\langle x, x\rangle}$. Some other norms related to absolute value of eigenvalues of x, namely norm 1 and norm infinity, are defined as

$$\|x\|_1 = \sum_{i=1}^{r} |\lambda_i(x)|,$$

and $\|x\|_\infty = \max_i |\lambda_i(x)|$.

Here we list some key lemmas which are required in our analysis.

Lemma 2.1 (Lemma 3.2, [4]). *Let* $\mathrm{int}\mathcal{K}$ *be the interior of* $\mathcal{K}$. *For* $x, s \in \mathrm{int}\mathcal{K}$ *there exists a unique* $w \in \mathrm{int}\mathcal{K}$ *such that*

$$x = P(w)s.$$

Moreover,

$$w = P\left(x^{\frac{1}{2}}\right)\left(P(x^{\frac{1}{2}})s\right)^{\frac{-1}{2}} = P\left(s^{\frac{-1}{2}}\right)\left(P\left(s^{\frac{1}{2}}\right)x\right)^{\frac{1}{2}}.$$

The point w is called the NT-scaling point of x and s.

Lemma 2.2 (Lemma 28, [11])**.** *Let $u \in \mathrm{int}\mathcal{K}$. Then*

$$x \circ s = \mu e \quad \Leftrightarrow \quad P(u)\, x \circ P(u)^{-1} s = \mu e.$$

Lemma 2.3 (Lemma 30, [11])**.** *Let $x, s \in \mathrm{int}\mathcal{K}$. Then*

$$\left\| P(x)^{\frac{1}{2}} s - e \right\|_F \leq \| x \circ s - e \|_F .$$

Lemma 2.4 (Theorem 4, [12])**.** *Let $x, s \in \mathrm{int}\mathcal{K}$. Then*

$$\lambda_{\min}\left(P(x)^{\frac{1}{2}} s\right) \geq \lambda_{\min}(x \circ s).$$

3. Infeasible Interior-point Algorithm for MSLCPs

In this section we present an infeasible full-NT step interior-point algorithm for MSLCPs. In contrast with feasible algorithm, in infeasible one, the initial starting point does not belong to the feasible set of the MSLCP. As usual for IIPMs, we suppose that an optimal solution of MSLCP exists and assume that the algorithm starts with the following initial infeasible point

$$\left(x^0, y^0, s^0\right) = (\rho_p e, 0, \rho_d e),$$

where ρ_p and ρ_d are some positive scaler values such that

$$Tr\left(x^0 \circ s^0\right) = r_2 \mu^0, \qquad \mu^0 = \rho_p \rho_d.$$

The main idea of infeasible algorithms is based on a sufficiently proper enough perturbation of the original problem MSLCP and construction of a new problem, namely, the perturbed problem P_ν. While in each iterate of infeasible algorithm the perturbed problem P_ν will be tended to the original problem, the generated ε-solutions of the algorithm converge to the ε-optimal solution of the original problem.

To begin the discussion, first of all, let $r^0 = \binom{r_1^0}{r_2^0}$ denotes the initial value of the residual vector as follows:

$$\begin{pmatrix} r_1^0 \\ r_2^0 \end{pmatrix} = \begin{pmatrix} s^0 \\ 0 \end{pmatrix} - \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \begin{pmatrix} x^0 \\ y^0 \end{pmatrix} - \begin{pmatrix} q_1 \\ q_2 \end{pmatrix}. \tag{3.1}$$

For any ν with $\nu \in (0, 1]$, we consider the perturbed problem P_ν as follows:

$$\begin{pmatrix} s \\ 0 \end{pmatrix} - \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \nu \begin{pmatrix} r_1^0 \\ r_2^0 \end{pmatrix}, \tag{3.2}$$

where $(x,s) \in \operatorname{int}\mathcal{K} \times \operatorname{int}\mathcal{K}$ and $y \in \mathcal{U}$. Clearly, the initial point $\left(x^0, y^0, s^0\right)$ is a strictly feasible solution of the perturbed problem P_ν. This means that the perturbed problem P_ν satisfies the interior-point condition (IPC). The following lemma completely describes the relation between the original problem MSLCP and the perturbed problem P_ν.

Lemma 3.1. *If the original problem MSLCP is feasible then the perturbed problem P_ν satisfies the IPC.*

Proof. Let the original problem MSLCP be feasible and $(\bar{x}, \bar{y}, \bar{s})$ be its feasible solution. For any $\nu \in (0,1]$, consider

$$x = (1-\nu)\,\bar{x} + \nu x^0, \qquad y = (1-\nu)\,\bar{y} + \nu y^0, \qquad s = (1-\nu)\,\bar{s} + \nu s^0.$$

Then

$$\begin{aligned}
\begin{pmatrix} (1-\nu)\,\bar{s} + \nu s^0 \\ 0 \end{pmatrix} - M\begin{pmatrix} (1-\nu)\,\bar{x} + \nu x^0 \\ (1-\nu)\,\bar{y} + \nu y^0 \end{pmatrix} - \begin{pmatrix} q_1 \\ q_2 \end{pmatrix}
&= (1-\nu)\left[\begin{pmatrix} \bar{s} \\ 0 \end{pmatrix} - M\begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix}\right] \\
&\quad + \nu\left[\begin{pmatrix} s^0 \\ 0 \end{pmatrix} - M\begin{pmatrix} x^0 \\ y^0 \end{pmatrix}\right] - \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} \\
&= \nu\begin{pmatrix} r_1^0 \\ r_2^0 \end{pmatrix}.
\end{aligned}$$

This shows that (x,y,s) is feasible for perturbed problem. Since $\nu > 0$ and $x, s \in \operatorname{int}\mathcal{K}$, thus proving that P_ν satisfies the IPC. $\square$

The most important result of the above lemma is the existence of the central path for the perturbed problem P_ν. In other words, Lemma 3.1 concludes that for any $\mu > 0$ the following system

$$\begin{aligned}
\begin{pmatrix} s \\ 0 \end{pmatrix} - M\begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \nu\begin{pmatrix} r_1^0 \\ r_2^0 \end{pmatrix}, &\qquad x \in \operatorname{int}\mathcal{K}, \\
x \circ s = \mu e, &\qquad s \in \operatorname{int}\mathcal{K},
\end{aligned}$$

has a unique solution which is denoted by $(x(\mu,\nu), y(\mu,\nu), s(\mu,\nu))$ and it is so-called as a μ-center of the perturbed problem P_ν. Note that since $x^0 \circ s^0 = \mu e$, $\left(x^0, s^0\right)$ is the μ^0-center of the perturbed problem P_1. In the following the parameters μ and ν always satisfy the relation $\mu = \nu\mu^0$.

To simplify, according to $\mu = \nu\mu^0$, we can denote $(x(\mu), y(\mu), s(\mu))$ as a μ-center of P_ν. The set of all μ-centers constructs a guide line, so-called the central path, to an ε-optimal solution of the original problem MSLCP.

In our analysis, we need to measure the closeness of the generated points from the central path. To this end, we use the following proximity measure in our analysis:

$$\delta(v) = \delta(x, s; \mu) = \left\|e - v^2\right\|_F, \tag{3.3}$$

where

$$v := \frac{P(w)^{\frac{1}{2}} s}{\sqrt{\mu}} = \frac{P(w)^{\frac{-1}{2}} x}{\sqrt{\mu}}.$$

The algorithm starts from an initial arbitrary point (x, y, s) with $\delta(x, s; \mu) \leq \tau$. Every main iteration of algorithm consist of a feasibility step, a μ-update and a few centering steps. Feasibility step is designed to find a new feasible point $\left(x^f, y^f, s^f\right)$ in the quadratic convergence neighborhood of the new perturbed problem P_{ν^+} with $\nu^+ = (1-\theta)\nu$. However, we need the feasibility search directions $\left(\Delta^f x, \Delta^f y, \Delta^f s\right)$ to get the new feasible solution of P_{ν^+}. The following system, namely the feasibility search direction system, concludes the following search directions.

$$\begin{aligned}
&M_{11}\Delta^f x + M_{12}\Delta^f y - \Delta^f s = \theta\nu r_1^0,\\
&M_{21}\Delta^f x + M_{22}\Delta^f y = \theta\nu r_2^0,\\
&P(w)^{\frac{-1}{2}} x \circ P(w)^{\frac{1}{2}} \Delta^f s + P(w)^{\frac{1}{2}} s \circ P(w)^{\frac{-1}{2}} \Delta^f x = (1-\theta)\mu e\\
&\quad - P(w)^{\frac{-1}{2}} x \circ P(w)^{\frac{1}{2}} s,
\end{aligned} \tag{3.4}$$

where $\theta \in (0, 1)$ and the third equation is inspired by Lemma 2.2. Since the parameters μ and ν will always be in one-to-one correspondence, then after updating the parameter ν, the parameter μ will be updated to $\mu^+ = (1-\theta)\mu$.

The hard part in our analysis will be to guarantee that the iterate

$$\left(x^f, y^f, s^f\right) = \left(x + \Delta^f x, y + \Delta^f y, s + \Delta^f s\right),$$

is strictly feasible and it satisfies $\delta\left(x^f, s^f; \mu^+\right) \leq \frac{1}{2}$. After the feasibility step, starting from the iterate $(x, y, s) = \left(x^f, y^f, s^f\right)$, a few centering steps are applied to produce a new iterate (x^+, y^+, s^+) such that $\delta(x^+, s^+, \mu^+) \leq \tau$. The centering search direction $(\Delta^c x, \Delta^c y, \Delta^c s)$ is the usual NT-search direction defined by

$$\begin{aligned}
&M_{11}\Delta^c x + M_{12}\Delta^c y - \Delta^c s = 0,\\
&M_{12}\Delta^c x + M_{22}\Delta^c y = 0,\\
&P(w)^{\frac{-1}{2}} x \circ P(w)^{\frac{1}{2}} \Delta^c s + P(w)^{\frac{1}{2}} s \circ P(w)^{\frac{-1}{2}} \Delta^c x = (1-\theta)\mu e\\
&\quad - P(w)^{\frac{-1}{2}} x \circ P(w)^{\frac{1}{2}} s.
\end{aligned} \tag{3.5}$$

A more formal description of the infeasible algorithm can be summarized as below.

Primal-Dual Infeasible IPM

Input:
Accuracy parameter $\varepsilon > 0$;
barrier update parameter θ, $0 < \theta < 1$;
threshold parameter $\tau > 0$;

$\left(x^0,\, s^0\right) \in \mathrm{int}\mathcal{K} \times \mathrm{int}\mathcal{K}$ and $\mu^0 > 0$ such that $Tr\left(x^0 \circ s^0\right) = r_2\mu^0$.

begin
$(x, y, \mu) = \left(x^0, y^0, s^0\right) = (\rho_p e, 0, \rho_d e)$;
while $\max\left(r_2\mu,\, \|r\|_F\right) \geq \varepsilon$ **do**
begin
feasibility step:
$(x,\, y,\, s) := (x,\, y,\, s) + \left(\Delta^f x,\, \Delta^f y,\, \Delta^f s\right)$;
μ-update:
$\mu := (1 - \theta)\,\mu$;
centering steps:
while $\delta\,(x,\, s;\; \mu) \geq \tau$ **do**
begin
$(x,\, s) := (x,\, y,\, s) + (\Delta^c x,\, \Delta^c y,\, \Delta^c s)$;
end
end
end

Figure 1. Infeasible full-Newton-step algorithm

4. Analysis of the Method

As we mentioned before, the hard part in our analysis will be to guarantee that the iterate $\left(x^f,\, y^f, s^f\right)$ is strictly feasible and satisfies

$$\delta\left(x^f,\, s^f;\; \mu^+\right) \leq \frac{1}{2}.$$

To simplify, let

$$d_x^f = \frac{P\,(w)^{\frac{-1}{2}}\,\Delta^f x}{\sqrt{\mu}}, \qquad d_s^f = \frac{P\,(w)^{\frac{1}{2}}\,\Delta^f s}{\sqrt{\mu}}. \tag{4.1}$$

It is easy to check that the system (3.4) which defines the search direction $\left(\Delta^f x, \Delta^f y, \Delta^f s\right)$ can be expressed in term of the scaled search directions d_x^f and d_s^f as follows:

$$\begin{aligned}
&\bar{M}_{11} d_x^f + \bar{M}_{12} \frac{\Delta y}{\sqrt{\mu}} - d_s^f = \frac{1}{\sqrt{\mu}} \theta \upsilon p(w)^{\frac{1}{2}} r_1^0,\\
&\bar{M}_{21} d_x^f + M_{22} \frac{\Delta y}{\sqrt{\mu}} = \frac{1}{\sqrt{\mu}} \theta \upsilon r_2^0,\\
&d_x^f + d_s^f = (1-\theta)\, v^{-1} - v,
\end{aligned} \tag{4.2}$$

where

$$\bar{M}_{11} = P(w)^{\frac{1}{2}} M_{11} P(w)^{\frac{1}{2}}, \qquad \bar{M}_{12} = P(w)^{\frac{1}{2}} M_{12}, \qquad \bar{M}_{21} = \bar{M}_{12}^T.$$

Using (4.1), the new iterates after the feasibility step can be calculated as follows:

$$\begin{aligned}
x^f &:= x + \Delta^f x = \sqrt{\mu} P(w)^{\frac{1}{2}} \left(v + d_x^f\right),\\
s^f &:= s + \Delta^f s = \sqrt{\mu} P(w)^{-\frac{1}{2}} \left(v + d_s^f\right).
\end{aligned} \tag{4.3}$$

Since $P(w)^{\frac{1}{2}}$ and $P(w)^{-\frac{1}{2}}$ are automorphisms of $\mathrm{int}\mathcal{K}$, x^f, s^f belong to $\mathrm{int}\mathcal{K}$ if and only if $v + d_x^f$ and $v + d_s^f$ belong to $\mathrm{int}\mathcal{K}$. Using (4.3) and the third equation in (4.2), we have

$$\begin{aligned}
\left(v + d_x^f\right) \circ \left(v + d_s^f\right) &= v^2 + v \circ \left(d_x^f + d_s^f\right) + d_x^f \circ d_s^f\\
&= v^2 + v \circ \left((1-\theta)\, v^{-1} - v\right) + d_x^f \circ d_s^f\\
&= (1-\theta)\, e + d_x^f \circ d_s^f.
\end{aligned} \tag{4.4}$$

The following lemma presents a sufficient condition to have strictly feasible solution.

Lemma 4.1. *The iterates* $\left(x^f, y^f, s^f\right)$ *are strictly feasible if* $(1-\theta)\, e + d_x^f \circ d_s^f \in \mathrm{int}\mathcal{K}$.

Proof. The proof is similar to the proof of Lemma 4.9 in [10] and is therefor omitted. ☐

Corollary 4.2. *The new iterates* $\left(x^f, y^f, s^f\right)$ *are strictly feasible if*

$$\left\| d_x^f \circ d_s^f \right\|_\infty < 1 - \theta.$$

Due to the elementary properties of norms in EJA, it is easy to verify that

$$|\langle d_x^f, d_s^f \rangle| \le \left\| d_x^f \right\|_F \left\| d_s^f \right\|_F \le \frac{1}{2} \left(\left\| d_x^f \right\|_F^2 + \left\| d_s^f \right\|_F^2 \right), \tag{4.5}$$

$$\left\|d_x^f \circ d_s^f\right\|_\infty \le \left\|d_x^f \circ d_s^f\right\|_F \le \frac{1}{2}\left(\left\|d_x^f\right\|_F^2 + \left\|d_s^f\right\|_F^2\right). \tag{4.6}$$

The following lemma is a direct result of Corollary 4.2 and (4.6).

Lemma 4.3. *If*

$$\left\|d_x^f\right\|_F^2 + \left\|d_s^f\right\|_F^2 < 2\left(1-\theta\right),$$

then the iterates $\left(x^f,\, y^f,\, s^f\right)$ *are strictly feasible.*

In the sequel we proceed by deriving an upper bound for $\delta\left(v^f\right) = \delta\left(x^f, s^f, \mu^+\right)$, where

$$\delta\left(v^f\right) := \left\|e - \left(v^f\right)^2\right\|_F, \qquad v^f := \frac{P\left(w^f\right)^{\frac{1}{2}} s^f}{\sqrt{\mu^+}} = \frac{P\left(w^f\right)^{\frac{-1}{2}} x^f}{\sqrt{\mu^+}}.$$

Lemma 4.4 (Lemma 4.3, [14])**.** *One has*

$$\sqrt{(1-\theta)}v^f \sim \left[P\left(v + d_x^f\right)^{\frac{1}{2}}\left(v + d_s^f\right)\right]^{\frac{1}{2}}.$$

Lemma 4.5. *Assuming* $(1-\theta)\,e + d_x^f \circ d_s^f \in \mathrm{int}\mathcal{K}$, *one has*

$$\delta\left(v^f\right) \le \frac{\left\|d_x^f\right\|_F^2 + \left\|d_s^f\right\|_F^2}{2\left(1-\theta\right)}.$$

Proof. Let w^f be the scaling point related to x^f and s^f. Lemma 4.4 directly results

$$\left(v^f\right)^2 \sim P\left(\frac{v + d_x^f}{\sqrt{1-\theta}}\right)^{\frac{1}{2}}\left(\frac{v + d_x^f}{\sqrt{1-\theta}}\right),$$

hence, using Lemma 2.3 and (4.4) we have

$$\begin{aligned}
\delta\left(v^f\right) = \left\|e - \left(v^f\right)^2\right\|_F &= \left\|e - P\left(\frac{v + d_x^f}{\sqrt{1-\theta}}\right)^{\frac{1}{2}}\left(\frac{v + d_x^f}{\sqrt{1-\theta}}\right)\right\|_F \\
&\le \left\|e - \left(\frac{v + d_x^f}{\sqrt{1-\theta}}\right) \circ \left(\frac{v + d_s^f}{\sqrt{1-\theta}}\right)\right\|_F \\
&\le \frac{1}{1-\theta}\left\|(1-\theta)e - \left(v + d_x^f\right) \circ \left(v + d_s^f\right)\right\|_F \\
&= \frac{1}{1-\theta}\left\|d_x^f \circ d_s^f\right\|_F \\
&\le \frac{\left\|d_x^f\right\|_F^2 + \left\|d_s^f\right\|_F^2}{2\left(1-\theta\right)},
\end{aligned}$$

where the last inequality follows by (4.6). This completes the proof. $\square$

Since in our analysis, we need to have strictly feasible iterates in quadratic convergence region, i.e, $\delta\left(v^f\right) \leq \frac{1}{2}$, this, due to Lemma 4.5, holds only if

$$\left\|d_x^f\right\|_F^2 + \left\|d_s^f\right\|_F^2 \leq 1-\theta. \tag{4.7}$$

4.1. **Upper Bound for $\left\|d_x^f\right\|_F^2 + \left\|d_s^f\right\|_F^2$.** Obtaining an upper bound for the term $\left\|d_x^f\right\|_F^2 + \left\|d_s^f\right\|_F^2$ is the main goal of this section. In the sequel this will enable us to find a default value for the update parameter θ.

The Schur complement theorem gives a characterization for the positive semi-definiteness (definiteness) of a matrix via the positive semi-definiteness (definiteness) of the Schur-complement with respect to a block partitioning of the matrix, which is stated as below. For more detail, we refer the reader to [5].

Lemma 4.6. *(Schur Complement Theorem) Let $A \in \mathbb{R}^{n\times n}$ be a symmetric matrix, $C \in \mathbb{R}^{m\times m}$ be a symmetric positive definite matrix and $B \in \mathbb{R}^{n\times m}$. Then*

$$\begin{bmatrix} A & B \\ B^T & C \end{bmatrix} \succeq 0 \quad \Leftrightarrow \quad A - BC^{-1}B^T \succeq 0,$$

and

$$\begin{bmatrix} A & B \\ B^T & C \end{bmatrix} \succ 0 \quad \Leftrightarrow \quad A - BC^{-1}B^T \succ 0.$$

One can easily check that system (3.4), by eliminating $\Delta^f y$ and using (4.1) reduces to

$$\widetilde{M} d_x^f - d_s^f = \frac{1}{\sqrt{\mu}} \theta \nu p\left(w\right)^{\frac{1}{2}} \left(r_1^0 - M_{12}{M_{22}}^{-1} r_2^0\right), \tag{4.8}$$

$$d_x^f + d_s^f = \left(1-\theta\right) v^{-1} - v, \tag{4.9}$$

where $\widetilde{M} = p\left(w\right)^{\frac{1}{2}} \left(M_{11} - M_{12}{M_{22}}^{-1} M_{21}\right) p(w)^{\frac{1}{2}}$. Since the matrix M has the positive semidefinite property, Lemma 4.6 guarantees the matrix $M_{11} - M_{12}{M_{22}}^{-1}M_{21}$ has also the positive semidefinite property. Therefore, due to the following relations, $\widetilde{M}$ also has the Cartesian positive semidefinite property.

$$\begin{aligned} \left\langle u, \widetilde{M} u \right\rangle &= \left\langle u, p\left(w\right)^{\frac{1}{2}} \left(M_{11} - M_{12}{M_{22}}^{-1} M_{21}\right) p\left(w\right)^{\frac{1}{2}} u \right\rangle \\ &= \left\langle p\left(w\right)^{\frac{1}{2}} u, \left(M_{11} - M_{12}{M_{22}}^{-1} M_{21}\right) p\left(w\right)^{\frac{1}{2}} u \right\rangle. \end{aligned}$$

Since $\widetilde{M}$ is positive semidefinite, one may easily verify that the coefficient matrix in the linear system (4.8) is nonsingular. Hence, this system uniquely defines the search directions d_x^f and d_s^f. (see, e.g., [6]).

Setting

$$a := \frac{1}{\sqrt{\mu}} \theta \nu p(w)^{\frac{1}{2}} \left(r_1^0 - M_{12} M_{22}{}^{-1} r_2^0 \right),$$

and $b := (1-\theta) v^{-1} - v$, the following lemma gives an upper bound for the single term $\left\| d_x^f \right\|_F^2 + \left\| d_s^f \right\|_F^2$.

Lemma 4.7. *The solution* $\left(d_x^f, d_s^f \right)$ *of the linear system (4.8) satisfies the following relations:*

$$\left\| d_x^f \right\|_F \leq \|a+b\|_F, \tag{4.10}$$

$$\left\| d_x^f \right\|_F^2 + \left\| d_s^f \right\|_F^2 \leq \|b\|_F^2 + 2\|a\|_F \|a+b\|_F. \tag{4.11}$$

Proof. By adding the two equation of system (4.8), we deduce

$$\left(\widetilde{M} + I \right) d_x^f = a + b.$$

Since $\widetilde{M}$ is a positive semidefinite matrix, then (4.10) is concluded. To prove (4.11) we have

$$\begin{aligned}
\left\| d_x^f \right\|_F^2 + \left\| d_s^f \right\|_F^2 &= \left\| d_x^f + d_s^f \right\|_F^2 - 2\langle d_x^f, d_s^f \rangle = \|b\|_F^2 - 2\langle d_x^f, \widetilde{M} d_x^f - a \rangle \\
&= \|b\|_F^2 - 2\langle d_x^f, \widetilde{M} d_x^f \rangle + 2\langle d_x^f, a \rangle \\
&\leq \|b\|_F^2 + 2\|a\|_F \left\| d_x^f \right\|_F \\
&\leq \|b\|_F^2 + 2\|a\|_F \|a+b\|_F,
\end{aligned}$$

where the last inequality follows from (4.10). This completes the proof. □

To compute an upper bound for the term $\left\| d_x^f \right\|_F^2 + \left\| d_s^f \right\|_F^2$, using Lemma 4.7 and setting $\tilde{r} = \left(r_1^0 - M_{12} M_{22}{}^{-1} r_2^0 \right)$, we have

$$\begin{aligned}
\left\| d_x^f \right\|_F^2 + \left\| d_s^f \right\|_F^2 \leq{} & \left\| (1-\theta) v^{-1} - v \right\|_F^2 \\
& + 2 \left\| \frac{\theta \nu}{\sqrt{\mu}} p(w)^{\frac{1}{2}} \tilde{r} \right\|_F \left\| \frac{\theta \nu}{\sqrt{\mu}} p(w)^{\frac{1}{2}} \tilde{r} + (1-\theta) v^{-1} - v \right\|_F \\
\leq{} & \left\| (1-\theta) v^{-1} - v \right\|_F^2 \\
& + 2 \left\| \frac{\theta \nu}{\sqrt{\mu}} p(w)^{\frac{1}{2}} \tilde{r} \right\|_F \left(\left\| \frac{\theta \nu}{\sqrt{\mu}} p(w)^{\frac{1}{2}} \tilde{r} \right\|_F + \left\| (1-\theta) v^{-1} - v \right\|_F \right).
\end{aligned}$$

(4.12)

In what follows, we proceed to estimate some upper bounds for the terms $\left\|(1-\theta)\,v^{-1}-v\right\|_F$ and $\left\|\frac{1}{\sqrt{\mu}}\theta\nu p\,(w)^{\frac{1}{2}}\,\tilde{r}\right\|_F$ respectively. Using (3.3) and Lemma 4.5 in [10], we have

$$\begin{aligned}\left\|(1-\theta)\,v^{-1}-v\right\|_F &= \left\|v^{-1}\circ\left((1-\theta)\,e-v^2\right)\right\|_F \\ &\leq \frac{1}{\lambda_{\min}(v)}\left\|e-v^2-\theta e\right\|_F \\ &\leq \frac{1}{q(\delta)}\left(\left\|e-v^2\right\|_F+\theta\left\|e\right\|_F\right) \leq \frac{1}{q\,(\delta)}\left(\delta+\theta\sqrt{r_2}\right),\end{aligned} \tag{4.13}$$

where $q\,(\delta)=\sqrt{1-\delta}$.

Substituting (4.13) in (4.12), we have

$$\begin{aligned}\left\|d_x^f\right\|_F^2+\left\|d_s^f\right\|_F^2 \leq \frac{1}{q^2\,(\delta)}\left(\delta+\theta\sqrt{r_2}\right)^2+2\left\|\frac{\theta\nu}{\sqrt{\mu}}p(w)^{\frac{1}{2}}\tilde{r}\right\|_F \\ \times\left(\frac{1}{q\,(\delta)}\left(\delta+\theta\sqrt{r_2}\right)+\left\|\frac{\theta\nu}{\sqrt{\mu}}p\,(w)^{\frac{1}{2}}\,\tilde{r}\right\|_F\right).\end{aligned} \tag{4.14}$$

To proceed, we have to specify our initial iterate $\left(x^0,s^0\right)$. We assume that ρ_p and ρ_d are such that

$$\begin{aligned}&\|x^*\|_\infty \leq \rho_p,\\ &\max\left\{\|s^*\|_\infty,\ \rho_p\left\|M_{11}-M_{12}M_{22}^{-1}M_{21}\right\|_\infty\right\} \leq \rho_d,\end{aligned} \tag{4.15}$$

for some optimal solution $(x^*,\ y^*,\ s^*)$, and as usual we start the algorithm with

$$x^0=\rho_p\,e,\qquad y^0=0,\qquad s^0=\rho_d\,e,\qquad \mu^0=\rho_p\,\rho_d. \tag{4.16}$$

For such starting point, we have clearly

$$\begin{aligned}0\preceq x^0-x^*\preceq \rho_p e,\\ 0\preceq s^0-s^*\preceq \rho_d e.\end{aligned}$$

Let (x^*,y^*,s^*) be an optimal solution of the original problem MSLCP. Then

$$\begin{aligned}\tilde{r} &= \left(r_1^0-M_{12}M_{22}^{-1}r_2^0\right)\\ &= \left(s^0-M_{11}x^0-M_{12}y^0-q_1\right)-M_{12}M_{22}^{-1}\left(-M_{21}x^0-M_{22}y^0-q_2\right)\\ &= \left(s^0-s^*\right)-M_{11}\left(x^0-x^*\right)+M_{12}M_{22}^{-1}M_{21}\left(x^0-x^*\right)\\ &= \left(s^0-s^*\right)-\left(M_{11}-M_{12}M_{22}^{-1}M_{21}\right)\left(x^0-x^*\right).\end{aligned}$$

(4.17)

In order to obtain an upper bound for

$$\left\| \frac{\theta\nu}{\sqrt{\mu}} p(w)^{\frac{1}{2}} \tilde{r} \right\|_F = \frac{\theta\nu}{\sqrt{\mu}} \left\| p(w)^{\frac{1}{2}} \tilde{r} \right\|_F$$

we write, using (4.17),

(4.18)

$$\begin{aligned}
\left\| p(w)^{\frac{1}{2}} \tilde{r} \right\|_F &= \left\| p(w)^{\frac{1}{2}} \left(\left(s^0 - s^*\right) - \left(M_{11} - M_{12} M_{22}^{-1} M_{21}\right) \left(x^0 - x^*\right) \right) \right\|_F \\
&\leq \left\| p(w)^{\frac{1}{2}} \left(s^0 - s^*\right) \right\|_F \\
&\quad + \left\| p(w)^{\frac{1}{2}} \left(M_{11} - M_{12} M_{22}^{-1} M_{21}\right) \left(x^0 - x^*\right) \right\|_F .
\end{aligned}$$

We first consider the term $\left\| p(w)^{\frac{1}{2}} \left(s^0 - s^*\right) \right\|_F$. Using the fact that $p(w)^{\frac{1}{2}}$ is self adjoint with respect to the inner product and $p(w)\, e = w^2$, we have

(4.19)

$$\begin{aligned}
\left\| p(w)^{\frac{1}{2}} \left(s^0 - s^*\right) \right\|_F^2 &= \langle p(w) \left(s^0 - s^*\right), \left(s^0 - s^*\right) \rangle \\
&= \langle p(w) \left(s^0 - s^*\right), \rho_d e \rangle - \langle p(w) \left(s^0 - s^*\right), \rho_d e - \left(s^0 - s^*\right) \rangle \\
&\leq \langle p(w) \left(s^0 - s^*\right), \rho_d e \rangle = \rho_d \langle \left(s^0 - s^*\right), p(w)\, e \rangle \\
&= \rho_d \langle p(w)\, e, \rho_d e \rangle - \rho_d \langle p(w)\, e, \rho_d e - \left(s^0 - s^*\right) \rangle \\
&\leq \rho_d^2 Tr\left(w^2\right) \leq \rho_d^2 Tr\,(w)^2 .
\end{aligned}$$

On the other hand to obtain an upper bound for the term

$$\left\| p(w)^{\frac{1}{2}} \left(M_{11} - M_{12} M_{22}^{-1} M_{21}\right) \left(x^0 - x^*\right) \right\|_F ,$$

by using (4.15) and (4.1), we proceed as follows

(4.20)

$$\begin{aligned}
&\left\| p(w)^{\frac{1}{2}} \left(M_{11} - M_{12} M_{22}^{-1} M_{21}\right) \left(x^0 - x^*\right) \right\|_F \\
&\qquad \leq \left\| p(w)^{\frac{1}{2}} \right\|_\infty \left\| M_{11} - M_{12} M_{22}^{-1} M_{21} \right\|_\infty \left\| x^0 - x^* \right\|_F \\
&\qquad \leq \rho_p \sqrt{r} \left\| p(w)^{\frac{1}{2}} \right\|_\infty \left\| M_{11} - M_{12} M_{22}^{-1} M_{21} \right\|_\infty \\
&\qquad \leq \rho_d \sqrt{r} \left\| p(w)^{\frac{1}{2}} \right\|_\infty .
\end{aligned}$$

By the definition of the quadratic representation, we have

$$\begin{aligned}\left\|p\left(w\right)^{\frac{1}{2}}\right\|_\infty &= \left\|2L(w^{\frac{1}{2}})^2 - L(w)\right\|_\infty \\ &\leq 2\left\|L(w^{\frac{1}{2}})\right\|_\infty^2 + \left\|L(w)\right\|_\infty \\ &\leq 3\lambda_{\max}\left(w\right) \leq Tr\left(w\right),\end{aligned} \tag{4.21}$$

where the second inequality follows because of $\|L(w)\|_\infty = \|w\|_\infty$. Substituting (4.21) into (4.20) and then substituting the square root of (4.19) and (4.20) into (4.18), we have

$$\left\|p\left(w\right)^{\frac{1}{2}}\tilde{r}\right\|_F \leq \left(1+3\sqrt{r}\right)\rho_d Tr\left(w\right). \tag{4.22}$$

To continue, we need an upper bound for Tr(w), which we will derive in the following lemma.

Lemma 4.8. *Let $x, s \in \text{int}\mathcal{K}$ and w be the scaling point of x and s. Then*

$$Tr\left(w\right) \leq \frac{\|x\|_1}{\sqrt{\mu}q\left(\delta\right)}. \tag{4.23}$$

Proof. For the moment, let $u = \left(P\left(x^{\frac{1}{2}}\right)s\right)^{\frac{-1}{2}}$. Then, by Lemma 2.1, $w = P\left(x^{\frac{1}{2}}\right)u$. Using that $P\left(x^{\frac{1}{2}}\right)$ is self-adjoint, $P\left(x^{\frac{1}{2}}\right)e = x$ and also Lemma 2.4 in [14], we obtain

$$\begin{aligned}Tr\left(w\right) = \langle P\left(x^{\frac{1}{2}}\right)u, e\rangle &= \langle u, P\left(x^{\frac{1}{2}}\right)e\rangle \\ &\leq \lambda_{\max}(u)Tr(x) \\ &= \lambda_{\max}\left(P\left(x^{\frac{1}{2}}\right)s\right)^{\frac{-1}{2}}Tr\left(x\right).\end{aligned}$$

Due to

$$P\left(s^{\frac{1}{2}}\right)x \sim P\left(x^{\frac{1}{2}}\right)s \sim \left(P\left(w^{\frac{1}{2}}\right)s\right)^2 \sim \left(P\left(w^{\frac{-1}{2}}\right)x\right)^2 = \mu v^2,$$

we have

$$\begin{aligned}Tr\left(w\right) &= \lambda_{\max}\left(P\left(x^{\frac{1}{2}}\right)s\right)^{\frac{-1}{2}}Tr\left(x\right) \\ &= \frac{Tr\left(x\right)}{\lambda_{\min}\left(P\left(x^{\frac{1}{2}}\right)s\right)^{\frac{1}{2}}} \\ &= \frac{Tr\left(x\right)}{\sqrt{\mu}\lambda_{\min}(v)}\end{aligned}$$

$$\leq \frac{\|x\|_1}{\sqrt{\mu}q\left(\delta\right)},$$

where the last inequality follows from Lemma 4.5 in [10]. The result is derived. □

Substituting (4.23) into (4.22) gives

$$\left\|p(w)^{\frac{1}{2}}\tilde{r}\right\|_F \leq (1+3\sqrt{r})\rho_d \frac{\|x\|_1}{\sqrt{\mu}q\left(\delta\right)}. \tag{4.24}$$

To complete our computation, we need to estimate an upper bound for $\|x\|_1$.

Lemma 4.9. *Let $(x,\, y,\, s)$ be feasible for the perturbed problem P_ν and $\left(x^0,\, y^0,\, s^0\right)$ as defined in (4.16). Then for any optimal solution*

$$\left(x^*,\, y^*,\, s^*\right),$$

we have

$$\begin{aligned}\nu\left(\langle x^0, s\rangle + \langle s^0, x\rangle\right) \leq \nu^2\langle x^0, s^0\rangle + \langle x, s\rangle + \nu\left(1-\nu\right)\left(\langle x^0, s^*\rangle + \langle s^0, x^*\rangle\right)\\ -\left(1-\nu\right)\left(\langle s, x^*\rangle + \langle x, s^*\rangle\right).\end{aligned}$$

Proof. From (3.2) and the definition of the perturbed problem P_ν, it is easily seen that

$$\begin{aligned}&\nu\begin{pmatrix}s^0\\0\end{pmatrix} + (1-\nu)\begin{pmatrix}s^*\\0\end{pmatrix} - \begin{pmatrix}s\\0\end{pmatrix}\\
&\quad = \nu\left[\begin{pmatrix}r_1^0\\r_2^0\end{pmatrix} + M\begin{pmatrix}x^0\\y^0\end{pmatrix} + \begin{pmatrix}q_1\\q_2\end{pmatrix}\right] + (1-\nu)\begin{pmatrix}s^*\\0\end{pmatrix}\\
&\quad\quad - \left[\nu\begin{pmatrix}r_1^0\\r_2^0\end{pmatrix} + M\begin{pmatrix}x\\y\end{pmatrix} + \begin{pmatrix}q_1\\q_2\end{pmatrix}\right]\\
&\quad = \nu\left[\begin{pmatrix}r_1^0\\r_2^0\end{pmatrix} + M\begin{pmatrix}x^0-x^*\\y^0-y^*\end{pmatrix}\right]\\
&\quad\quad - \left[\nu\begin{pmatrix}r_1^0\\r_2^0\end{pmatrix} + M\begin{pmatrix}x^0-x^*\\y^0-y^*\end{pmatrix}\right]\\
&\quad = M\left[\nu\begin{pmatrix}x^0\\y^0\end{pmatrix} + (1-\nu)\begin{pmatrix}x^*\\y^*\end{pmatrix} - \begin{pmatrix}x\\y\end{pmatrix}\right].\end{aligned}$$

Since M has the Cartesian positive semidefinite property we obtain

$$\begin{aligned}0 &\leq \left\langle \left[\nu\begin{pmatrix}x^0\\y^0\end{pmatrix} + (1-\nu)\begin{pmatrix}x^*\\y^*\end{pmatrix} - \begin{pmatrix}x\\y\end{pmatrix}\right], \left[\nu\begin{pmatrix}s^0\\0\end{pmatrix} + (1-\nu)\begin{pmatrix}s^*\\0\end{pmatrix} - \begin{pmatrix}s\\0\end{pmatrix}\right]\right\rangle\\
&= \nu^2\left\langle \begin{pmatrix}x^0\\y^0\end{pmatrix}, \begin{pmatrix}s^0\\0\end{pmatrix}\right\rangle + \nu(1-\nu)\left\langle \begin{pmatrix}x^0\\y^0\end{pmatrix}, \begin{pmatrix}s^*\\0\end{pmatrix}\right\rangle\end{aligned}$$

$$
\begin{aligned}
&- \left\langle \nu \begin{pmatrix} x^0 \\ y^0 \end{pmatrix}, \begin{pmatrix} s \\ 0 \end{pmatrix}) \right\rangle \\
&+ \nu(1-\nu) \left\langle \begin{pmatrix} x^* \\ y^* \end{pmatrix}, \begin{pmatrix} s^0 \\ 0 \end{pmatrix} \right\rangle + (1-\nu)^2 \left\langle \begin{pmatrix} x^* \\ y^* \end{pmatrix}, \begin{pmatrix} s^* \\ 0 \end{pmatrix} \right\rangle \\
&- \left\langle (1-\nu) \begin{pmatrix} x^* \\ y^* \end{pmatrix}, \begin{pmatrix} s \\ 0 \end{pmatrix} \right\rangle - \left\langle \begin{pmatrix} x \\ y \end{pmatrix}, \nu \begin{pmatrix} s^0 \\ 0 \end{pmatrix} \right\rangle \\
&- \left\langle \begin{pmatrix} x \\ y \end{pmatrix}, (1-\nu) \begin{pmatrix} s^* \\ 0 \end{pmatrix} \right\rangle + \left\langle \begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} s \\ 0 \end{pmatrix} \right\rangle.
\end{aligned}
$$

After some simple calculations, due to definition of the canonical inner product, we have

$$
\begin{aligned}
0 \leq \nu^2 \langle x^0, s^0 \rangle + \langle x, s \rangle + \nu\,(1-\nu) \left(\langle x^0, s^* \rangle + \langle s^0, x^* \rangle \right) \\
- \nu \left(\langle x^0, s \rangle + \langle s^0, x \rangle \right) - (1-\nu) \left(\langle s, x^* \rangle + \langle x, s^* \rangle \right).
\end{aligned}
$$

This follows the desired result. $\square$

Lemma 4.10. *Let $(x,\, y,\, s)$ be feasible for the perturbed problem P_ν and $\delta(v)$ is defined as in (3.3) and $\left(x^0,\, y^0,\, s^0\right)$ as defined in (4.16). Then we have*

$$\|x\|_1 \leq (3 + \delta\,(v))\, r_2 \rho_p, \tag{4.25}$$

$$\|s\|_1 \leq (3 + \delta\,(v))\, r_2 \rho_d. \tag{4.26}$$

Proof. Since $x,\, s,\, x^*$ and s^* belong to $\mathrm{int}\mathcal{K}$, it implies that $\langle s, x^* \rangle + \langle x, s^* \rangle$ is positive. Therefore, Lemma 4.9 implies

$$\langle x^0, s \rangle + \langle s^0, x \rangle \leq \nu \langle x^0, s^0 \rangle + \frac{\langle x, s \rangle}{\nu} + (1-\nu) \left(\langle x^0, s^* \rangle + \langle s^0, x^* \rangle \right).$$

On the other hand, according to (4.15) and (4.16), we have

$$\langle x^0, s^* \rangle + \langle s^0, x^* \rangle \leq \rho_d \,\langle x^0, e \rangle + \rho_p \,\langle s^0, e \rangle = 2\, r_2\, \rho_p\, \rho_d.$$

Also $\langle x^0, s^0 \rangle = r_2\, \rho_p\, \rho_d$. Hence, we get

$$
\begin{aligned}
\langle x^0, s \rangle + \langle s^0, x \rangle &\leq \frac{\langle x, s \rangle}{\nu} + 2\, r_2\, \rho_p\, \rho_d - \nu\, r_2\, \rho_p\, \rho_d \\
&\leq \frac{\langle x, s \rangle}{\nu} + 2\, r_2\, \rho_p\, \rho_d \\
&= \frac{\mu \langle v, v \rangle}{\nu} + 2\, r_2\, \rho_p\, \rho_d \\
&= \mu^0 \lambda^2_{\max}(v) \langle e, e \rangle + 2\, r_2\, \rho_p\, \rho_d \\
&\leq (2 + q^2(\delta)) r_2\, \rho_p\, \rho_d \\
&= (3 + \delta) r_2\, \rho_p\, \rho_d.
\end{aligned}
$$

Since x^0, s^0, $x, s \in \mathrm{int}\mathcal{K}$, we obtain

$$\langle x^0, s\rangle \leq (3+\delta)r_2\, \rho_p\, \rho_d,$$
$$\langle s^0, x\rangle \leq (3+\delta)r_2\, \rho_p\, \rho_d.$$

On the other hand, since $x^0 = \rho_p e$ and $s^0 = \rho_d e$, we have

$$\|x\|_1 \leq (3+\delta)\, r_2\, \rho_p, \qquad \|s\|_1 \leq (3+\delta)\, r_2\, \rho_d,$$

which proves the lemma. □

By substituting the result of Lemma 4.10 into (4.24), we derive an upper bound for $\left\|p\,(w)^{\frac{1}{2}}\, \tilde{r}\right\|_F$ as follows.

$$\left\|p\,(w)^{\frac{1}{2}}\, \tilde{r}\right\|_F \leq (1+3\sqrt{r_2})\, r_2\, \rho_p\rho_d \frac{\left(2+q^2\,(\delta)\right)}{\sqrt{\mu}q(\delta)}. \tag{4.27}$$

Now, we are ready to obtain an upper bound for $\left\|d_x^f\right\|_F^2 + \left\|d_s^f\right\|_F^2$. Substituting (4.27) into (4.14) we conclude

$$\begin{aligned}\left\|d_x^f\right\|_F^2 + \left\|d_s^f\right\|_F^2 \leq \frac{1}{q^2\,(\delta)}\Bigg[&(\delta+\theta\sqrt{r_2})^2 \\ &+ 2\left(\theta r_2\,(1+3\sqrt{r_2})\,(3+\delta)\right)\Big((\delta+\theta\sqrt{r_2}) \\ &+ (\theta r_2 (1+3\sqrt{r_2})(3+\delta))\Big)\Bigg].\end{aligned} \tag{4.28}$$

4.2. **Value for θ.** As we mentioned before, if (4.7) is satisfied then $\delta\left(v^f\right) \leq \frac{1}{2}$ certainly holds. Then, according to (4.28), inequality (4.7) holds if

$$\begin{aligned}\frac{1}{q^2\,(\delta)}\Bigg[&(\delta+\theta\sqrt{r_2})^2 + 2\left(\theta r_2\,(1+3\sqrt{r_2})\,(3+\delta)\right)\Big((\delta+\theta\sqrt{r_2}) \\ &+ \left(\theta r_2\,(1+3\sqrt{r_2})\,(3+\delta)\right)\Big)\Bigg] \leq 1-\theta.\end{aligned}$$

Obviously, the left hand side of the above inequality is monotonically increasing with respect to δ. Using this, one may easily verify that the above inequality is satisfied if

$$\delta = \frac{1}{16}, \qquad \theta = \frac{1}{66 r_2}. \tag{4.29}$$

4.3. **Complexity Analysis.** We have found values for the parameters in the algorithm such that if at the start of an iteration the iterates satisfy $\delta(x, s; \mu) \le \tau = \frac{1}{16}$, then after the feasibility step the iterates satisfy $\delta(x^f, s^f; \mu) \le \frac{1}{2}$. However, after k centering steps we will have iterates (x^+, y^+, s^+) which are still feasible and $\delta(x^+, y^+; \mu^+) \le \left(\frac{1}{2}\right)^{2^k}$. So, k should satisfy $\left(\frac{1}{2}\right)^{2^k} \le \tau$, which gives $k \ge \log_2\left(\log_2 \frac{1}{\tau}\right)$. Then, we can easily find at most 2 centering steps suffice to get iterates that satisfy $\delta(x^+, y^+; \mu^+) \le \tau$.

However, each iteration of algorithm consists of at most 3 so-called inner iterations. It has become a custom to measure the complexity of an IPM by the required number of inner iterations. In each iteration both the duality gap and the norms of the residual vectors are reduced by the factor $1 - \theta$. Hence, the total number of the main iterations is bounded above by

$$\frac{1}{\theta} \log \frac{\max\left\{Tr\left(x^0 \circ s^0\right), \left\|\binom{r_1^0}{r_2^0}\right\|\right\}}{\varepsilon}.$$

Due to (4.29) we may take

$$\theta = \frac{1}{66\, r_2}.$$

Hence the total number of the inner iterations is bounded above by

$$188\, r_2 \log \frac{\max\left\{Tr\left(x^0 \circ s^0\right), \left\|\binom{r_1^0}{r_2^0}\right\|\right\}}{\varepsilon}.$$

Thus we may state without further proof the main result of the paper.

Theorem 4.11. *If the original problem MSLCP has the optimal solution* (x^*, y^*, s^*) *such that* $\|x^*\|_\infty \le \rho_p$ *and* $\|s^*\|_\infty \le \rho_d$*, then after at most*

$$188\, r_2 \log \frac{\max\left\{Tr\left(x^0 \circ s^0\right), \left\|\binom{r_1^0}{r_2^0}\right\|\right\}}{\varepsilon},$$

iterations the algorithm finds an ε*-solution of MSLCP.*

5. Concluding Remarks

In this paper, we proposed an infeasible algorithm for MSLCPs and derived the currently best known iteration bound for the algorithm with small-update method. Indeed, based on using the proximity measure, we presented a simple convergence analysis for MSLCPs which are a general class of complementarity problems. Each main iteration of our algorithm consists of a feasibility step and at most 2 centering steps.

The obtained iteration bound for this algorithm, coincides with the best known bound for IIPMs.

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