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On the Linear Combinations of Slanted Half-Plane Harmonic Mappings

Ahmad Zireh[1] and Mohammad Mehdi Shabani[2*]

ABSTRACT. In this paper, the sufficient conditions for the linear combinations of slanted half-plane harmonic mappings to be univalent and convex in the direction of $(-\gamma)$ are studied. Our result improves some recent works. Furthermore, a illustrative example and imagine domains of the linear combinations satisfying the desired conditions are enumerated.

1. Introduction

A continuous function $f=u+iv$ is a complex valued harmonic function in a complex domain $\Omega\subset\mathbb{C}$ if both u and v are real harmonic in Ω.

In any simply connected domain $\Omega\subset\mathbb{C}$, we can write $f=h+\overline{g}$, where h and g are analytic in Ω. We call h the analytic part and g the co-analytic part of f. A necessary and sufficient condition for f to be locally univalent and sense-preserving in Ω is that $|h^{'}(z)|>|g^{'}(z)|$ in Ω (see [2]).

Denote by $\mathcal{S}_{\mathcal{H}}$ the class of functions $f=h+\overline{g}$ that are harmonic univalent and sense-preserving in $\mathbb{D}=\{z\in\mathbb{C}:|z|<1\}$ for which $f(0)=f_z(0)-1=0$. Then for $f=h+\overline{g}\in\mathcal{S}_{\mathcal{H}}$, we may express the analytic functions h and g as

$$h(z)=z+\sum_{k=2}^{\infty}a_kz^k,\qquad g(z)=\sum_{k=1}^{\infty}b_kz^k,\ |b_1|<1, \tag{1.1}$$

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* Corresponding author.

that is,

$$f(z) = \sum_{k=1}^{\infty} \left(a_k z^k + \overline{b_k z^k} \right), \quad (a_1 = 1, |b_1| < 1, \ z \in \mathbb{D}). \tag{1.2}$$

A domain Ω is said to be convex in a direction γ, where $0 \leq \gamma < \pi$, if for all $a \in \mathbb{C}$, the set $\Omega \cap \{a + te^{i\gamma} : t \in \mathbb{R}\}$ is either connected or empty. In particular, a domain is said to be convex in the horizontal direction (CHD) if its intersection with each horizontal lines is connected (or empty). A univalent harmonic mapping is called a CHD mapping if its range is a CHD domain.

An effective way of constructing univalent harmonic mappings with given dilatations, known as the shear construction, was introduced by Clunie and Sheil-Small [2].

Theorem 1.1. *A harmonic function $f = h + \overline{g}$ locally univalent in $\mathbb{D}$ is a univalent mapping of $\mathbb{D}$ onto a domain convex in the direction γ $(0 \leq \gamma < \pi)$ if and only if $h - e^{2i\gamma} g$ is a conformal univalent mapping of $\mathbb{D}$ onto a domain convex in the direction γ.*

Next, there is a useful remark by Pommerenke [5] concerning analytic mappings convex in one direction. Using a particular case of this, we have the following result.

Theorem 1.2. *Let f(z) be an analytic function in $\mathbb{D}$ with $f(0) = 0$ and $f'(0) \neq 0$, and let*

$$\varphi(z) = \frac{z}{(1 + ze^{i\theta})(1 + ze^{-i\theta})},$$

where $\theta \in \mathbb{R}$. If

$$\mathfrak{Re}\left\{ \frac{zf'(z)}{\varphi(z)} \right\} > 0, \quad \forall z \in \mathbb{D},$$

then f is a CHD mapping.

Furthermore, we investigate the linear combination of two suitable harmonic maps. Note that if $f_1 = h_1 + \overline{g_1}$ and $f_2 = h_2 + \overline{g_2}$ are two harmonic univalent mappings in $\mathbb{D}$, the linear combination $\lambda f_1 + (1-\lambda) f_2$ need not be univalent (for details, see [4]).

Recently, Wang et al.[7] derived several sufficient conditions on harmonic univalent functions f_1 and f_2 so that their linear combination is univalent and convex in the direction of the real axis. In particular, they established:

Theorem 1.3. *Let $f_j = h_j + \overline{g_j} \in \mathcal{S}_{\mathcal{H}}$ with $h_j(z) + g_j(z) = z/(1 - z)$ for $j = 1, 2$. Then $\lambda f_1 + (1 - \lambda) f_2$, $0 \leq \lambda \leq 1$, is univalent and convex in the direction of the real axis.*

The purpose of this paper is to prove Theorem 1.3, without the conditions $h_j(z) + g_j(z) = z/(1 - z)$, $j = 1, 2$, for a subclass of harmonic mappings.

2. Main Results

Let

$$H_\gamma = \left\{ z \in \mathbb{C} : \mathfrak{Re}(e^{i\gamma} z) > -\frac{1}{2} \right\},$$

where $0 \leq \gamma < 2\pi$. We denote by $\mathcal{S}_{H_\gamma}$ the subclass of harmonic functions f which map $\mathbb{D}$ onto H_γ. To prove our result, we require the following lemma.

Lemma 2.1. *If $f = h + \overline{g} \in \mathcal{S}_{H_\gamma}$, then*

$$h(z) + e^{-2i\gamma} g(z) = \frac{z}{1 - z e^{i\gamma}}, \quad z \in \mathbb{D}. \tag{2.1}$$

Proof. If $f = h + \overline{g} \in \mathcal{S}_{H_\gamma}$, then

$$\mathfrak{Re} \left\{ e^{i\gamma} \left(h(z) + \overline{g(z)} \right) \right\} > -1/2,$$

which means that

$$\mathfrak{Re} \left\{ e^{i\gamma} h(z) + e^{-i\gamma} g(z) \right\} > -1/2.$$

In other words,

$$\mathfrak{Re} \left\{ e^{i\gamma} \left(h(z) + e^{-2i\gamma} g(z) \right) \right\} > -1/2.$$

Since

$$h(z) + e^{-2i\gamma} g(z) = h(z) - e^{-2i(\pi/2 - \gamma)} g(z),$$

therefore, it follows from Theorem 1.1 that the function $h(z) + e^{-2i\gamma} g(z)$ is convex in the direction $\left(\frac{\pi}{2} - \gamma\right)$ and so is univalent. It is also clear that $z \to h(z) + e^{-2i\gamma} g(z)$ maps $\mathbb{D}$ onto H_γ which implies the result. $\square$

Theorem 2.2. *Let $f_j = h_j + \overline{g_j} \in \mathcal{S}_{H_\gamma}$, $(j = 1, 2)$. Then $f_3 = \lambda f_1 + (1 - \lambda) f_2$, where $0 \leq \lambda \leq 1$, is univalent and convex in the direction $(-\gamma)$.*

Proof. By noting that $g_1' = \omega_1 h_1'$, $g_2' = \omega_2 h_2'$, we have

$$\begin{aligned} \omega_3 &= \frac{\lambda g_1' + (1 - \lambda) g_2'}{\lambda h_1' + (1 - \lambda) h_2'} \\ &= \frac{\lambda \omega_1 h_1' + (1 - \lambda) \omega_2 h_2'}{\lambda h_1' + (1 - \lambda) h_2'}. \end{aligned} \tag{2.2}$$

Now, we divide into two cases to discuss:

(i) If $\omega_1 = \omega_2$, then

$$\omega_3 = \frac{\lambda\omega_1 h_1' + (1-\lambda)\omega_1 h_2'}{\lambda h_1' + (1-\lambda)h_2'} = \omega_1.$$

Therefore in this case $|\omega_3| = |\omega_1| < 1$, which implies that f_3 is locally univalent.

(ii) If $\omega_1 \neq \omega_2$, then by using (2.1), we have

$$h_j(z) + e^{-2i\gamma} g_j(z) = \frac{z}{1 - ze^{i\gamma}}.$$

Therefore

$$h_j' = \frac{1}{(1+\omega_j e^{-2i\gamma})(1-ze^{i\gamma})^2}, \quad j = 1, 2. \tag{2.3}$$

By replacing (2.3) in (2.2), it follows that

$$\begin{aligned}|\omega_3| &= \left|\frac{\lambda\omega_1 h_1' + (1-\lambda)\omega_2 h_2'}{\lambda h_1' + (1-\lambda)h_2'}\right| \\ &= \frac{\left|\lambda\omega_1 + (1-\lambda)\omega_2 + \omega_1\omega_2 e^{-2i\gamma}\right|}{\left|1 + (1-\lambda)\omega_1 e^{-2i\gamma} + \lambda\omega_2 e^{-2i\gamma}\right|}.\end{aligned}$$

Next, we show that $|\omega_3| < 1$. Let

$$\begin{aligned}\omega_j &= r_j e^{i\theta_j} \\ &= r_j(\cos\theta_j + i\sin\theta_j), \quad (0 \le r_j < 1, j = 1, 2).\end{aligned}$$

Suppose that

$$\begin{aligned}\varphi(\lambda) &= \left|1 + (1-\lambda)\omega_1 e^{-2i\gamma} + \lambda\omega_2 e^{-2i\gamma}\right|^2 - \left|\lambda\omega_1 + (1-\lambda)\omega_2 + \omega_1\omega_2 e^{-2i\gamma}\right|^2 \\ &= \left|1 + (1-\lambda)r_1 e^{i(\theta_1 - 2\gamma)} + \lambda r_2 e^{i(\theta_2 - 2\gamma)}\right|^2 \\ &\quad - \left|\lambda r_1 e^{i\theta_1} + (1-\lambda)r_2 e^{i\theta_2} + r_1 r_2 e^{i(\theta_1+\theta_2-2\gamma)}\right|^2 \\ &= [1 + (1-\lambda)r_1\cos(\theta_1 - 2\gamma) + \lambda r_2\cos(\theta_2 - 2\gamma)]^2 \\ &\quad + [(1-\lambda)r_1\sin(\theta_1 - 2\gamma) + \lambda r_2\sin(\theta_2 - 2\gamma)]^2 \\ &\quad - [\lambda r_1\cos\theta_1 + (1-\lambda)r_2\cos\theta_2 + r_1 r_2\cos(\theta_1+\theta_2-2\gamma)]^2 \\ &\quad - [\lambda r_1\sin\theta_1 + (1-\lambda)r_2\sin\theta_2 + r_1 r_2\sin(\theta_1+\theta_2-2\gamma)]^2 \\ &= 1 + (1-\lambda)^2 r_1^2\cos^2(\theta_1 - 2\gamma) + \lambda^2 r_2^2\cos^2(\theta_2 - 2\gamma) \\ &\quad + 2(1-\lambda)r_1\cos(\theta_1 - 2\gamma) + 2\lambda r_2\cos(\theta_2 - 2\gamma) \\ &\quad + 2\lambda(1-\lambda)r_1 r_2\cos(\theta_1 - 2\gamma)\cos(\theta_2 - 2\gamma)(1-\lambda)^2 r_1^2\sin^2(\theta_1 - 2\gamma) \\ &\quad + \lambda^2 r_2^2\sin^2(\theta_2 - 2\gamma) + 2\lambda(1-\lambda)r_1 r_2\sin(\theta_1 - 2\gamma)\sin(\theta_2 - 2\gamma) \\ &\quad - \big[\lambda^2 r_1^2\cos^2\theta_1 + (1-\lambda)^2 r_2^2\cos^2\theta_2 + r_1^2 r_2^2\cos^2(\theta_1+\theta_2-2\gamma)\end{aligned}$$

$$
\begin{aligned}
&+ 2\lambda(1-\lambda)r_1r_2\cos\theta_1\cos\theta_2 + 2\lambda r_1^2 r_2\cos\theta_1\cos(\theta_1+\theta_2-2\gamma)\\
&+ 2(1-\lambda)r_1r_2^2\cos\theta_2\cos(\theta_1+\theta_2-2\gamma)\lambda^2 r_1^2\sin^2\theta_1\\
&+ (1-\lambda)^2 r_2^2\sin^2\theta_2 + r_1^2r_2^2\sin^2(\theta_1+\theta_2-2\gamma)\\
&+ 2\lambda(1-\lambda)r_1r_2\sin\theta_1\sin\theta_2 + 2\lambda r_1^2r_2\sin\theta_1\sin(\theta_1+\theta_2-2\gamma)\\
&+2(1-\lambda)r_1r_2^2\sin\theta_2\sin(\theta_1+\theta_2-2\gamma)\big]\\
=\ & 1 + r_1^2 - r_2^2 - r_1^2r_2^2 + 2r_1\cos(\theta_1-2\gamma) - 2r_1r_2^2\cos(\theta_1-2\gamma)\\
&+ 2\lambda\left(r_2\cos(\theta_2-2\gamma) - r_1\cos(\theta_1-2\gamma) - r_1^2r_2\cos(\theta_2-2\gamma)\right.\\
&\left.+r_1r_2^2\cos(\theta_1-2\gamma) + r_2^2 - r_1^2\right).
\end{aligned}
$$

It is clear that $\varphi(\lambda)$ is a linear function of λ, therefore it is a continuous and monotone function of λ in the interval $[0,1]$. Moreover, we observe that

$$
\begin{aligned}
\varphi(0) &= (1-r_2^2)(r_1^2 + 2r_1\cos(\theta_1-2\gamma)+1)\\
&= (1-r_2^2)\big[(r_1+\cos(\theta_1-2\gamma))^2 + \sin^2(\theta_1-2\gamma)\big] > 0,
\end{aligned}
$$

and

$$
\varphi(1) = (1-r_1^2)\big[(r_2+\cos(\theta_1-2\gamma))^2 + \sin^2(\theta_1-2\gamma)\big] > 0,
$$

which implies that $\varphi(\lambda) > 0$ for all $[0,1]$. It follows that $|\omega_3| < 1$, and then f_3 is locally univalent in $\mathbb{D}$.

Next, we show that $f_3 = \lambda f_1 + (1-\lambda) f_2 = \big[\lambda h_1 + (1-\lambda)h_2\big] + \big[\lambda\overline{g_1} + (1-\lambda)\overline{g_2}\big] = h_3 + \overline{g_3}$ is convex in the direction $(-\gamma)$. Let $F := h_3 - e^{-2i\gamma}g_3$, then we have

$$
\begin{aligned}
F &= h_3 - e^{-2i\gamma}g_3\\
&= (\lambda h_1 + (1-\lambda)h_2) - e^{-2i\gamma}\left(\lambda g_1 + (1-\lambda)g_2\right)\\
&= \lambda\left(h_1 - e^{-2i\gamma}g_1\right) + (1-\lambda)\left(h_2 - e^{-2i\gamma}g_2\right).
\end{aligned}
$$

Hence

$$
\begin{aligned}
F'(z) &= \lambda\left(h_1' - e^{-2i\gamma}g_1'\right) + (1-\lambda)\left(h_2' - e^{-2i\gamma}g_2'\right)\\
&= \lambda\left(h_1' + e^{-2i\gamma}g_1'\right)\left(\frac{h_1' - e^{-2i\gamma}g_1'}{h_1' + e^{-2i\gamma}g_1'}\right)\\
&\quad + (1-\lambda)\left(h_2' + e^{-2i\gamma}g_2'\right)\left(\frac{h_2' - e^{-2i\gamma}g_2'}{h_2' + e^{-2i\gamma}g_2'}\right)\\
&= \frac{\lambda}{(1-e^{i\gamma}z)^2}\cdot P_1(z) + \frac{(1-\lambda)}{(1-e^{i\gamma}z)^2}\cdot P_2(z),
\end{aligned}
$$

where

$$\begin{aligned} P_j(z) &= \frac{h'_j - e^{-2i\gamma} g'_j}{h'_j + e^{-2i\gamma} g'_j} \\ &= \frac{1 - e^{-2i\gamma} \dfrac{g'_j}{h'_j}}{1 + e^{-2i\gamma} \dfrac{g'_j}{h'_j}} \\ &= \frac{1 - e^{-2i\gamma}\omega_j}{1 + e^{-2i\gamma}\omega_j}, \quad j = 1, 2. \end{aligned}$$

Since $|\omega_j| = \left|\dfrac{g'_j}{h'_j}\right| < 1$ for $j = 1, 2$, $\mathfrak{Re}(P_1(z)) > 0$ and $\mathfrak{Re}(P_2(z)) > 0$ in $\mathbb{D}$.

Now for

$$\varphi(e^{i\gamma} z) = e^{i\gamma} \frac{z}{(1 - ze^{i\gamma})^2},$$

we have

$$\begin{aligned} &\mathfrak{Re}\left\{ e^{i\gamma} \frac{zF'(z)}{\varphi(z)} \right\} \\ &\qquad = \mathfrak{Re}\left\{ (1 - ze^{i\gamma})^2 \left(\frac{\lambda}{(1 - ze^{i\gamma})^2} \cdot P_1(z) + \frac{(1-\lambda)}{(1 - ze^{i\gamma})^2} \cdot P_2(z) \right) \right\} \\ &\qquad = \lambda \mathfrak{Re}\left\{ P_1(z) \right\} + (1-\lambda) \mathfrak{Re}\left\{ P_2(z) \right\} \\ &\qquad = \mathfrak{Re}(P_1(z)) > 0. \end{aligned}$$

Therefore by using Theorem 1.2, $e^{i\gamma} F = e^{i\gamma}\left(h_3 - e^{-2i\gamma} g_3\right)$ is CHD. It means $h_3 - e^{-2i\gamma} g_3$ is convex in the direction $(-\gamma)$. Finally, by applying Theorem 1.1 for $F = h_3 - e^{-2i\gamma} g_3$, we get the desired result. ☐

By induction we can get the following result.

Corollary 2.3. *Let $f_j = h_j + g_j \in \mathcal{S}_{H_\gamma}, (j = 1, 2, \ldots, n)$. Then $\lambda_1 f_1 + \cdots + \lambda_n f_n$ is univalent and convex in the direction $(-\gamma)$, where $0 \leq \lambda_j \leq 1 (j = 1, 2, \ldots, n)$ and $\lambda_1 + \lambda_2 + \cdots + \lambda_n = 1$.*

Finally, we give an example to illuminate our main result.

Example 2.4. Let $f_1 = h_1 + \overline{g_1}$, where

$$h_1(z) + e^{-2i\gamma} g_1(z) = \frac{z}{1 - ze^{i\gamma}},$$

and $\omega_1 = -e^{3i\gamma}z$. Then by using shearing technique of Clunie and Shiel-Small [2], we obtain

$$h_1(z) = \frac{z - \frac{1}{2}z^2e^{i\gamma}}{(1-ze^{i\gamma})^2},$$

and

$$g_1(z) = \frac{-\frac{1}{2}z^2e^{3i\gamma}}{(1-ze^{i\gamma})^2}.$$

Also, we suppose that $f_2 = h_2 + \overline{g_2}$, where

$$h_2(z) + e^{-2i\gamma}g_2(z) = \frac{z}{1-ze^{i\gamma}},$$

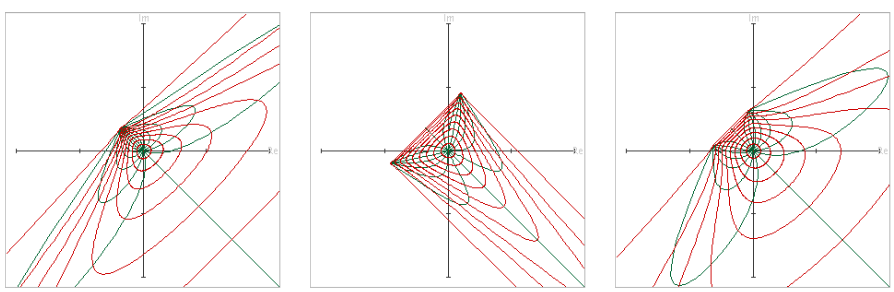
and $\omega_2 = e^{3i\gamma}z$. Then with similar way as above, we get

$$h_2(z) = \frac{1}{4e^{i\gamma}}\left[\log\left(\frac{1+ze^{i\gamma}}{1-ze^{i\gamma}}\right) + \frac{2}{1-ze^{i\gamma}}\right] - \frac{1}{2e^{i\gamma}},$$

and

$$g_2(z) = \frac{e^{i\gamma}}{2}\left(\frac{2ze^{i\gamma}-1}{1-ze^{i\gamma}}\right) - \frac{e^{i\gamma}}{4}\log\left(\frac{1+ze^{i\gamma}}{1-ze^{i\gamma}}\right) + \frac{e^{i\gamma}}{2}.$$

If we take $\gamma = \frac{\pi}{4}$, then f_1 and f_2 belong to $\mathcal{S}_{H_{\pi/4}}$. The images of $\mathbb{D}$ under f_1, f_2, and $f_3 = \lambda f_1 + (1-\lambda)f_2$ with $\lambda = 1/2$ are shown in Fig. 1.



(a) f_1 (b) f_2 (c) $f_3 = \lambda f_1 + (1-\lambda)f_2$

FIGURE 1. Images of $\mathbb{D}$ under f_1, f_2 and f_3 with $\lambda = 1/2$

We see that f_3 is convex in the direction $(-\pi/4)$, it means that Theorem 2.2 is true.

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[1] Department of Mathematics, Shahrood University of Technology, P.O.Box 316-36155, Shahrood, Iran.

E-mail address: `azireh@gmail.com`

[2] Department of Mathematics, Shahrood University of Technology, Shahrood, Iran.

E-mail address: `Mohammadmehdishabani@yahoo.com`