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Certain Inequalities for a General Class of Analytic and Bi-univalent Functions

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ABSTRACT. In this work, the subclass $S_{\Sigma}^{h,p}(\lambda,\delta,\gamma)$ of the function class S of analytic and bi-univalent functions is defined and studied in the open unit disc. Estimates for initial coefficients of Taylor-Maclaurin series of bi-univalent functions belonging this class are obtained. Also, some relevant classes are recognized and connections to previous results are made.

1. Introduction

Assume that $\mathbb{E}=\{z : z\in\mathbb{C} \text{ and } |z|<1\}$ is the open unit disc. Let $\mathcal{A}$ indicate the class of functions f which are analytic in the open unit disc $\mathbb{E}$, of the form

$$f(z)=z+\sum_{n=2}^{\infty}a_n z^n. \tag{1.1}$$

Let S be the subclass of $\mathcal{A}$ consisting of functions which are univalent in $\mathbb{E}$. Due to the fact that the univalent functions are one to one, these functions are invertible. While the inverse of univalent functions are invertible they do not need to be defined on the entire unit disc $\mathbb{E}$. According to the Koebe one-quarter theorem, a disc of radius $\frac{1}{4}$ is in the image of $\mathbb{E}$ under every function $f\in\mathcal{S}$. Thus, every function $f\in\mathcal{S}$ has an inverse function and this inverse function can be defined on a disc of radius $\frac{1}{4}$. The inverse function of f can be given by

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$$g(w) = f^{-1}(w) = w - a_2 w^2 + \left(2a_2^2 - a_3\right) w^3 - \left(5a_2^3 - 5a_2 a_3 + a_4\right) w^4 + \cdots . \tag{1.2}$$

If f and f^{-1} are univalent in $\mathbb{E}$, then the function $f \in \mathcal{A}$ is said to be bi-univalent in $\mathbb{E}$. The class of bi-univalent functions defined in $\mathbb{E}$ is symbolized by Σ. We refer for the basic definitions of the analytic and bi-univalent function class and their properties and interesting of functions in the class Σ, to the study of Srivastava et al. [15] and the references in it. Lewin [12] was the first mathematician working on this subject, obtaining the bound 1.51 for the modulus of the second coefficient $|a_2|$. Later, Netanyahu [13] showed that $\max |a_2| = \frac{4}{3}$. Then, Brannan and Clunie [7] conjectured that $|a_2| \leqq \sqrt{2}$ for $f \in \Sigma$. Subsequently, Brannan and Taha [6] introduced certain subclasses of class Σ analogous subclasses $\mathcal{S}^{\star}(\beta)$ of starlike functions and $\mathcal{K}(\beta)$ convex functions of order $\beta\,(0 \leqq \beta < 1)$ in $\mathbb{E}$ in turn (see [13]). The classes $\mathcal{S}^{\star}_{\Sigma}(\beta)$ and $\mathcal{K}_{\Sigma}(\beta)$ of bi-starlike functions of order β in $\mathbb{E}$ and bi-convex functions of order β in $\mathbb{E}$, corresponding to the function classes $\mathcal{S}^{\star}(\beta)$ and $\mathcal{K}(\beta)$, were also introduced congruently. For each of the function classes $\mathcal{S}^{\star}_{\Sigma}(\beta)$ and $\mathcal{K}_{\Sigma}(\beta)$, these mathematicians obtained some estimates for the initial coefficients but these estimates were not sharp. Recently, motivated substantially by the following work on this area by Srivastava *et al.* [15], many authors searched the coefficient bounds for diversified subclasses of bi-univalent functions (see [1–5, 9, 10]). Dealing with the bounds on the general coefficient $|a_n|$ for $n \geqq 4$, there is not enough knowledge. In the literature, only a few works has been made to identify the general coefficient bounds for $|a_n|$ for the analytic bi-univalent functions. Today, the problem of identifying the coefficient for each of the coefficients $|a_n|\,(n \in \mathbb{N} \setminus \{1, 2\}\,;\ \mathbb{N} = \{1, 2, 3, \cdots\})$ is an unsolved problem.

Motivated by above mentioned studies, we define a subclass of function class $\mathcal{S}$ in such a way.

Definition 1.1. Let the functions $h, p : \mathbb{E} \to \mathbb{C}$ be so constrained that

$$\min \{\operatorname{Re}(h(z)), \operatorname{Re}(p(z))\} > 0,$$

and

$$h(0) = p(0) = 1.$$

Also, let the function f defined by (1.1) be analytic in the function class $\mathcal{A}$. The class $S^{h,p}_{\Sigma}(\lambda, \delta, \gamma)$ consists of the function $f \in \Sigma$ satisfying the following conditions:

$$1 + \frac{1}{\gamma}\left[(1-\lambda)\frac{f(z)}{z} + \lambda f'(z) + \delta z f''(z) - 1\right] \in h(\mathbb{E}), \quad (z \in \mathbb{E}), \tag{1.3}$$

(1.4)
$$1+\frac{1}{\gamma}\left[(1-\lambda)\frac{g(w)}{w}+\lambda g'(w)+\delta w g''(w)-1\right]\in p(\mathbb{E}), \quad (w\in\mathbb{E}),$$

where $f\in\Sigma$, $\lambda\geq 1$, $\delta\geq 0$, $0\neq\gamma\in\mathbb{C}$ and $g(w)=f^{-1}(w)$ given in (1.2).

By choosing the special values for $\delta,\gamma,\lambda,$ and the class $S_{\Sigma}^{h,p}(\lambda,\delta,\gamma)$ reduces to several earlier known classes of analytic and bi-univalent functions studied in the literature.

Remark 1.2. (i) For $\delta=0$ and $\gamma=1$, we have the class

$$\mathcal{B}_{\Sigma}^{h,p}(\lambda)=S_{\Sigma}^{h,p}(\lambda,0,1).$$

The class $\mathcal{B}_{\Sigma}^{h,p}(\lambda)$ was introduced and studied in recent years by Xu et all. [18]. This class consists of the functions $f\in\Sigma$ satisfying

$$(1-\lambda)\frac{f(z)}{z}+\lambda f'(z)\in h(\mathbb{E}), \quad (z\in\mathbb{E}),$$

and

$$(1-\lambda)\frac{g(w)}{w}+\lambda g'(w)\in p(\mathbb{E}), \quad (w\in\mathbb{E}).$$

(ii) For $\delta=0,\lambda=1$ and $\gamma=1$, we have the class

$$\mathcal{H}_{\Sigma}^{h,p}=S_{\Sigma}^{h,p}(1,0,1).$$

The class $\mathcal{H}_{\Sigma}^{h,p}$ introduced and studied recently by Xu et all. [17]. This class consists of the functions $f\in\Sigma$ satisfying

$$f'(z)\in h(\mathbb{E}),$$

and

$$g'(w)\in p(\mathbb{E}).$$

(iii) For $\gamma=1$, we have a new class

$$S_{\Sigma}^{h,p}(\lambda,\delta)=S_{\Sigma}^{h,p}(\lambda,\delta,1).$$

The class $S_{\Sigma}^{h,p}(\lambda,\delta)$ consists of the functions $f\in\Sigma$ satisfying

$$(1-\lambda)\frac{f(z)}{z}+\lambda f'(z)+\delta z f''(z)\in h(\mathbb{E}), \quad (z\in\mathbb{E}), \tag{1.5}$$

and

$$(1-\lambda)\frac{g(w)}{w}+\lambda g'(w)+\delta w g''(w)\in p(\mathbb{E}), \quad (w\in\mathbb{E}), \tag{1.6}$$

where $g(w)=f^{-1}(w)$ is given in (1.2).

Choosing the functions $h(z)$ and $p(z)$ in a different way we can obtain impressive subclasses of $\mathcal{A}$. Let us show how we can obtain the earlier classes from this new class $S_{\Sigma}^{h,p}(\lambda,\delta,\gamma)$.

Remark 1.3. Choosing

$$h(z)=p(z)=\frac{1+(1-2\beta)z}{1-z}, \quad (0\leq\beta<1),$$

we have a new class $S_{\Sigma}^{\beta}(\lambda,\delta,\gamma)$. Checking that the functions $h(z)$ and $p(z)$ satisfy the hypotheses of Definition 1.1 is very easy. So, taking $f\in S_{\Sigma}^{\beta}(\lambda,\delta,\gamma)$, we have

$$\operatorname{Re}\left(1+\frac{1}{\gamma}\left[(1-\lambda)\frac{f(z)}{z}+\lambda f'(z)+\delta z f''(z)-1\right]\right)>\beta,$$

and

$$\operatorname{Re}\left(1+\frac{1}{\gamma}\left[(1-\lambda)\frac{g(w)}{w}+\lambda g'(w)+\delta w g''(w)-1\right]\right)>\beta,$$

where $0\leq\beta<1,\ \ z\in\mathbb{E}, w\in\mathbb{E}$, and $g(w)=f^{-1}(w)$ is defined by (1.2). This is a new class $S_{\Sigma}^{\beta}(\lambda,\delta,\gamma)$ which includes many well known classes which are as follows:

(i) Taking $\gamma=1$, the class $S_{\Sigma}^{\beta}(\lambda,\delta,\gamma)$ reduces to the class $N_{\Sigma}(\lambda,\delta,\beta)$ defined by Bulut ([8]). This class consists of the functions $f\in\Sigma$ satisfying

$$\operatorname{Re}\left((1-\lambda)\frac{f(z)}{z}+\lambda f'(z)+\delta z f''(z)\right)>\beta,$$

and

$$\operatorname{Re}\left((1-\lambda)\frac{g(w)}{w}+\lambda g'(w)+\delta w g''(w)\right)>\beta,$$

where $0\leq\beta<1$ and $g(w)=f^{-1}(w)$ is defined by (1.2).

(ii) Taking $\gamma=1$ and $\delta=0$, the class $S_{\Sigma}^{\beta}(\lambda,\delta,\gamma)$ reduces to the class $\mathcal{B}_{\Sigma}(\beta,\lambda),(\lambda\geq1)$ defined by Frasin and Aouf ([11, Definition 3.1]). This class consists of the functions $f\in\Sigma$ satisfying

$$\operatorname{Re}\left((1-\lambda)\frac{f(z)}{z}+\lambda f'(z)\right)>\beta,$$

and

$$\operatorname{Re}\left((1-\lambda)\frac{g(w)}{w}+\lambda g'(w)\right)>\beta,$$

where $0\leq\beta<1$ and $g(w)=f^{-1}(w)$ is defined by (1.2).

(iii) Taking $\gamma = 1$, $\delta = 0$ and $\lambda = 1$, the class $S_{\Sigma}^{\beta}(\lambda, \delta, \gamma)$ reduces to the class $\mathcal{H}_{\Sigma}(\beta)$ defined and investigated by Srivastava et al. [15, Definition 2]. This class consists of the functions $f \in \Sigma$ satisfying

$$\operatorname{Re}\left(f'(z)\right) > \beta$$

and

$$\operatorname{Re}\left(g'(w)\right) > \beta$$

where $0 \leq \beta < 1$ and $g(w) = f^{-1}(w)$ is given in (1.2).

(iv) Taking $\gamma = 1$ and $\lambda = 1$, the class $S_{\Sigma}^{\beta}(\lambda, \delta, \gamma)$ reduces the class $\mathcal{H}_{\Sigma}(\beta, \delta)$ defined and investigated by Srivastava et al. [16]. This class consists of the functions $f \in \Sigma$ satisfying

$$\operatorname{Re}\left(f'(z) + \delta z f''(z)\right) > \beta,$$

and

$$\operatorname{Re}\left(g'(w) + \delta w g''(w)\right) > \beta,$$

where $0 \leq \beta < 1$ and $g(w) = f^{-1}(w)$ is given in (1.2).

Remark 1.4. Choosing

$$\varphi(z) = \left(\frac{1+z}{1-z}\right)^{\alpha}, \quad 0 < \alpha \leq 1,\ \lambda \geq 1,$$

we have a new class $S_{\Sigma}^{\alpha}(\gamma, \lambda, \delta)$. Checking that the functions $h(z)$ and $p(z)$ satisfy the hypotheses of Definition 1.1 is very easy. So, taking $f \in S_{\Sigma}^{\alpha}(\gamma, \lambda, \delta), f \in \Sigma$, we get

$$\left|\arg\left(1 + \frac{1}{\gamma}\left[(1-\lambda)\frac{f(z)}{z} + \lambda f'(z) + \delta z f''(z) - 1\right]\right)\right| < \frac{\alpha\pi}{2}, \tag{1.7}$$

and

$$\left|\arg\left(1 + \frac{1}{\gamma}\left[(1-\lambda)\frac{g(w)}{w} + \lambda g'(w) + \delta w g''(w) - 1\right]\right)\right| < \frac{\alpha\pi}{2}, \tag{1.8}$$

where $0 < \alpha \leq 1$, $z \in \mathbb{E}, w \in \mathbb{E}$, and $g(w) = f^{-1}(w)$ is given in (1.2). This new class $S_{\Sigma}^{\alpha}(\lambda, \delta, \gamma)$ includes many well known classes, which are as follows:

(i) Taking $\gamma = 1$, the class $S_{\Sigma}^{\alpha}(\gamma, \lambda, \delta)$ reduces to the new class $S_{\Sigma}(\lambda, \delta, \alpha)$ for functions $f \in \Sigma$ satisfying:

$$\left|\arg\left((1-\lambda)\frac{f(z)}{z} + \lambda f'(z) + \delta z f''(z)\right)\right| < \frac{\alpha\pi}{2}, \tag{1.9}$$

and

$$\left|\arg\left((1-\lambda)\frac{g(w)}{w} + \lambda g'(w) + \delta w g''(w)\right)\right| < \frac{\alpha\pi}{2}, \tag{1.10}$$

where $0<\alpha\leq 1, z\in\mathbb{E}, w\in\mathbb{E}$, and $g(w)=f^{-1}(w)$ is defined by (1.2).

(ii) Taking $\gamma=1$ and $\delta=0$, the class $S_{\Sigma}^{\alpha}(\gamma,\lambda,\delta)$ reduces to the class $\mathcal{B}_{\Sigma}(\alpha,\lambda)$ defined by Frasin and Aouf ([11, Definition 2.1]). This class consists of the fuctions $f\in\Sigma$ satisfying

$$\left|\arg\left((1-\lambda)\frac{f(z)}{z}+\lambda f'(z)\right)\right|<\frac{\alpha\pi}{2},$$

and

$$\left|\arg\left((1-\lambda)\frac{g(w)}{w}+\lambda g'(w)\right)\right|<\frac{\alpha\pi}{2},$$

where $0<\alpha\leq 1$ and $g(w)=f^{-1}(w)$ is defined by (1.2).

(iii) Taking $\gamma=1,\delta=0$ and $\lambda=1$, the class $S_{\Sigma}^{\alpha}(\gamma,\lambda,\delta)$ reduces to the class $\mathcal{H}_{\Sigma}^{\alpha}$ defined and investigated by Srivastava et al. [15, Definition 1]. This class consists of the functions $f\in\Sigma$ satisfying

$$\left|\arg f'(z)\right|<\frac{\alpha\pi}{2},$$

and

$$\left|\arg g'(w)\right|<\frac{\alpha\pi}{2},$$

where $0<\alpha\leq 1$ and $g(w)=f^{-1}(w)$ is defined by (1.2).

2. A set of Coefficient Estimates for the Function Class $S_{\Sigma}^{h,p}(\lambda,\delta,\gamma)$

We will state the Lemma 2.1 to obtain our result. Later, we will express and prove our general results. These results involve the bi-univalent function class $S_{\Sigma}^{h,p}(\lambda,\delta,\gamma)$ given in Definition 1.1.

Lemma 2.1 ([14]). *If $p\in\mathcal{P}$ then $|p_i|\leq 2$ for each i, where $\mathcal{P}$ is the family functions p, analytic in $\mathbb{E}$, for which*

$$\operatorname{Re}\{p(z)\}>0,\quad (z\in\mathbb{E}),$$

where

$$p(z)=1+p_1z+p_2z^2+\cdots,\quad (z\in\mathbb{E}).$$

The following theorem is one of the results of this section.

Theorem 2.2. *If the function $f\in\Sigma$ given by (1.1) belongs to the class $S_{\Sigma}^{h,p}(\lambda,\delta,\gamma)$ then*

$$|a_2|\leq\min\left\{\sqrt{\frac{|h'(0)|^2+|p'(0)|^2}{2(1+\lambda+2\delta)^2}\,|\gamma|^2},\sqrt{\frac{|h''(0)|+|p''(0)|}{4(1+2\lambda+6\delta)}\,|\gamma|}\right\},\tag{2.1}$$

and

(2.2)
$$|a_3| \leq \min\left\{\left[\tfrac{|h'(0)|^2+|p'(0)|^2}{2(1+\lambda+2\delta)^2}\,|\gamma| + \tfrac{|h''(0)|+|p''(0)|}{4(1+2\lambda+6\delta)}\right]|\gamma|\,, \frac{|h''(0)|}{2\,(1+2\lambda+6\delta)}\,|\gamma|\right\},$$

where $0 \neq \gamma \in \mathbb{C}, \lambda \geq 1, \delta \geq 0, z, w \in \mathbb{E}$.

Proof. Assume that $f \in S_{\Sigma}^{h,p}(\lambda,\delta,\gamma)$. Using the argument inequalities in (1.3) and (1.4) we have

$$1+\frac{1}{\gamma}\left[(1-\lambda)\frac{f(z)}{z}+\lambda f'(z)+\delta z f''z)-1\right]=h\,(z)\,, \tag{2.3}$$

and

$$1+\frac{1}{\gamma}\left[(1-\lambda)\frac{g(w)}{w}+\lambda g'(w)+\delta w g''(w)-1\right]=p\,(w)\,, \tag{2.4}$$

where the conditions of Definiton 1.1 are satisfied for the functions $h\,(z)$ and $p\,(w)$. So, we can write the Taylor-Maclaurin series expansions of the functions $h\,(z)$ and $p\,(w)$ as

$$h\,(z)=1+h_1 z+h_2 z^2+\cdots,$$

and

$$p\,(w)=1+p_1 w+p_2 w^2+\cdots.$$

By using the form of the functions f and g, which are giben by (1.1) and (1.2), we have

$$\begin{aligned} &1+\frac{1}{\gamma}\left[(1-\lambda)\frac{f(z)}{z}+\lambda f'(z)+\delta z f''(z)-1\right] \\ &\qquad =1+\frac{1}{\gamma}\left[\sum_{n=2}^{\infty}\left[1+(n-1)\lambda+n(n-1)\delta\right]a_n z^{n-1}\right], \end{aligned} \tag{2.5}$$

and

$$\begin{aligned} &1+\frac{1}{\gamma}\left[(1-\lambda)\frac{g(w)}{w}+\lambda g'(w)+\delta w g''(w)-1\right] \\ &\qquad =1+\frac{1}{\gamma}\left[\sum_{n=2}^{\infty}\left[1+(n-1)\lambda+n(n-1)\delta\right]a_n w^{n-1}\right], \end{aligned} \tag{2.6}$$

respectively. Thus, upon comparing the corresponding coefficients in (2.4) and (2.5) with those of $h\,(z)$ and $p\,(w)$ respectively, we get

$$\frac{1}{\gamma}\left(1+\lambda+2\delta\right)a_2=h_1, \tag{2.7}$$

$$\frac{1}{\gamma}\left(1+2\lambda+6\delta\right)a_3=h_2, \tag{2.8}$$

$$-\frac{1}{\gamma}\left(1+\lambda+2\delta\right)a_2 = p_1, \tag{2.9}$$

$$\frac{1}{\gamma}\left(1+2\lambda+6\delta\right)\left(2a_2^2-a_3\right) = p_2. \tag{2.10}$$

From (2.7) and (2.9) we obtain

$$h_1 = -p_1, \tag{2.11}$$

and

$$\frac{2}{\gamma^2}\left(1+\lambda+2\delta\right)^2 a_2^2 = h_1^2 + p_1^2. \tag{2.12}$$

Now, by adding (2.8) to (2.10), we can write

$$\frac{2}{\gamma}\left(1+2\lambda+6\delta\right)a_2^2 = h_2 + p_2. \tag{2.13}$$

Thus, we obtain from relations (2.12) and (2.13) that

$$\left|a_2\right|^2 \leq \frac{\left|h_1^2\right| + \left|p_1^2\right|}{2\left(1+\lambda+2\delta\right)^2}\left|\gamma\right|^2, \tag{2.14}$$

and

$$\left|a_2\right|^2 \leq \frac{\left|h_2\right| + \left|p_2\right|}{2\left(1+2\lambda+6\delta\right)}\left|\gamma\right|, \tag{2.15}$$

respectively. These inequalities give us the desired estimate on $|a_2|$ as given in (2.1). Subsequently, to find the bound on $|a_3|$, by subtracting (2.10) from (2.8), we obtain

$$\frac{2}{\gamma}\left(1+2\lambda+6\delta\right)\left(a_3-a_2^2\right) = h_2 - p_2. \tag{2.16}$$

If we compute the value of a_2^2 from (2.12) and it into (2.16), then we get

$$\left|a_3\right| \leq \left[\frac{\left|h_1^2\right| + \left|p_1^2\right|}{2\left(1+\lambda+2\delta\right)^2}\left|\gamma\right| + \frac{\left|h_2\right| + \left|p_2\right|}{2\left(1+2\lambda+6\delta\right)}\right]\left|\gamma\right|.$$

In other respects, upon substituting the value of a_2^2 from (2.13) into (2.16), we obtain

$$\left|a_3\right| \leq \frac{\left|h_2\right|}{\left(1+2\lambda+6\delta\right)}\left|\gamma\right|.$$

Thus, the proof of Theorem 2.2 is completed. $\square$

3. Corollaries and Consequences

By setting $\gamma = 1$ in Theorem 2.2, we obtain the following corollary.

Corollary 3.1. *If the function $f \in \Sigma$ given in (1.1) is in the class $S_\Sigma^{h,p}(\lambda, \delta, 1), \gamma = 1, \lambda \geq 1, \delta \geq 0$ and $z, w \in \mathbb{E}$, then*

$$|a_2| \leq \min\left\{\sqrt{\frac{|h'(0)|^2+|p'(0)|^2}{2(1+\lambda+2\delta)^2}}, \sqrt{\frac{|h''(0)|+|p''(0)|}{4(1+2\lambda+6\delta)}}\right\}, \tag{3.1}$$

and

$$|a_3| \leq \min\left\{\frac{|h'(0)|^2+|p'(0)|^2}{2(1+\lambda+2\delta)^2} + \frac{|h''(0)|+|p''(0)|}{4(1+2\lambda+6\delta)}, \frac{|h''(0)|}{2\,(1+2\lambda+6\delta)}\right\}. \tag{3.2}$$

Corollary 3.2. *If the function $f \in \Sigma$ given in (1.1) is in the class $\mathcal{B}_\Sigma^{h,p}(\lambda), (\lambda \geq 1, z, w \in \mathbb{E})$, then*

$$|a_2| \leq \min\left\{\sqrt{\frac{|h'(0)|^2+|p'(0)|^2}{2(1+\lambda)^2}}, \sqrt{\frac{|h''(0)|+|p''(0)|}{4(1+2\lambda)}}\right\},$$

and

$$|a_3| \leq \min\left\{\frac{|h'(0)|^2+|p'(0)|^2}{2(1+\lambda)^2} + \frac{|h''(0)|+|p''(0)|}{4(1+2\lambda)}, \frac{|h''(0)|}{2\,(1+2\lambda)}\right\}.$$

Remark 3.3. Corollary 3.2 is an advancement of the [18, Corollary 10].

Corollary 3.4 ([18])**.** *If the function $f \in \Sigma$ given in (1.1) is in the class $\mathcal{B}_\Sigma^{h,p}(\lambda), \lambda \geq 1 \quad (z, w \in \mathbb{E})$, then*

$$|a_2| \leq \sqrt{\frac{|h''(0)|+|p''(0)|}{4(1+2\lambda)}},$$

and

$$|a_3| \leq \frac{|h''(0)|}{2\,(1+2\lambda)}.$$

Choosing $\gamma = 1, \lambda = 1$ and $\delta = 0$ in Theorem 2.2, we obtain the following corollary.

Corollary 3.5. *If the function $f \in \Sigma$ given in (1.1) is in the class $\mathcal{H}_\Sigma^{h,p}, (z, w \in \mathbb{E})$, then*

$$|a_2| \leq \min\left\{\sqrt{\frac{|h'(0)|^2+|p'(0)|^2}{8}}, \sqrt{\frac{|h''(0)|+|p''(0)|}{12}}\right\},$$

and

$$|a_3| \leq \min\left\{\frac{|h'(0)|^2+|p'(0)|^2}{8} + \frac{|h''(0)|+|p''(0)|}{12}, \frac{|h''(0)|}{6}\right\}.$$

Remark 3.6. Corollary 3.5 is an advancement of the [17, Corollary 13].

Corollary 3.7 ([17]). *If the function $f \in \Sigma$ given in* (1.1) *is in the class* $\mathcal{H}_{\Sigma}^{h,p}, (z, w \in \mathbb{E})$, *then*

$$|a_2| \leq \tfrac{|h''(0)|+|p''(0)|}{12},$$

and

$$|a_3| \leq \frac{|h''(0)|}{6}.$$

Assuming

$$h(z) = p(z) = \frac{1 + (1-2\beta) z}{1-z}, \quad (0 \leq \beta < 1),$$

in Theorem 2.2, we can readily deduce the following corollary.

Corollary 3.8. *If the function $f \in \Sigma$ given in* (1.1) *is in the new class* $S_{\Sigma}^{\beta}(\lambda, \delta, \gamma)$, *then*

$$|a_2| \leq \min\left\{ \tfrac{2(1-\beta)}{1+\lambda+2\delta} |\gamma|, \sqrt{\tfrac{2(1-\beta)}{1+2\lambda+6\delta} |\gamma|} \right\},$$

and

$$|a_3| \leq \tfrac{2(1-\beta)}{1+2\lambda+6\delta}. \tag{3.3}$$

By setting $\gamma = 1$ in Corollary 3.8, we have the following result.

Corollary 3.9. *If the function $f \in \Sigma$ given by* (1.1) *is in the class* $N_{\Sigma}(\lambda, \delta, \beta)$, *then*

$$|a_2| \leq \min\left\{ \tfrac{2(1-\beta)}{1+\lambda+2\delta}, \sqrt{\tfrac{2(1-\beta)}{1+2\lambda+6\delta}} \right\},$$

and

$$|a_3| \leq \tfrac{2(1-\beta)}{1+2\lambda+6\delta}. \tag{3.4}$$

Remark 3.10. The above estimates for $|a_2|$ and $|a_3|$ show that Theorem 2.2 is an improvement of the estimates obtained by Bulut [8, Theorem 5].

By setting $\delta = 0$ in Corollary 3.9, we have the following corollary.

Corollary 3.11. *If the function $f \in \Sigma$ given by* (1.1) *is in the class* $B_{\Sigma}(\beta, \lambda)$, *then*

$$|a_2| \leq \sqrt{\tfrac{2(1-\beta)}{1+2\lambda}},$$

and

$$|a_3| \leq \tfrac{2(1-\beta)}{1+2\lambda}. \tag{3.5}$$

Remark 3.12. The above estimates for $|a_2|$ and $|a_3|$ show that Theorem 2.2 is an improvement of the estimates obtained by Frasin and Aouf ([11, Theorem 3.2]).

Corollary 3.13. *If $\lambda = 1$, then we get an advancement of the estimates obtained by Srivastava et al. ([15, Theorem 2]).*

Letting

$$\begin{aligned} h(z) &= p(z) \\ &= \left(\frac{1+z}{1-z}\right)^{\alpha} \\ &= 1 + 2\alpha z + 2\alpha^2 z^2 + \cdots, \quad (0 < \alpha \leq 1), \end{aligned}$$

in Theorem 2.2, we can readily deduce the following corollary.

Corollary 3.14. *If the function $f \in \Sigma$ given by* (1.1) *is in the new class $S_{\Sigma}^{\alpha}(\lambda, \delta, \gamma), (\lambda \geq 1, 0 \neq \gamma \in \mathbb{C}, \lambda \geq 1, 0 < \alpha \leq 1, z, w \in \mathbb{E})$, then*

$$|a_2| \leq \min\left\{\tfrac{2\alpha}{1+\lambda+2\delta}\,|\gamma|\,, \sqrt{\tfrac{\alpha}{1+2\lambda+6\delta}\,|\gamma|}\right\},$$

and

$$|a_3| \leq \frac{2\alpha}{1+2\lambda+6\delta}. \tag{3.6}$$

By setting $\gamma = 1$ and $\delta = 0$ in Corollary 3.14, we have the following result.

Corollary 3.15. *If the function $f \in \Sigma$ given by* (1.1) *is in the class $\mathcal{B}_{\Sigma}(\alpha, \lambda)$, then*

$$|a_2| \leq \min\left\{\tfrac{2\alpha}{1+\lambda}, \sqrt{\tfrac{\alpha}{1+2\lambda}}\right\},$$

and

$$|a_3| \leq \frac{2\alpha}{1+2\lambda}. \tag{3.7}$$

Remark 3.16. The above estimates for $|a_2|$ and $|a_3|$ show that Theorem 2.2 is an improvement of the estimates obtained by Frasin and Aouf ([11, Theorem 2.2]).

Corollary 3.17. *If $\lambda = 1$, then we get an improvement of the estimates obtained by Srivastava et al. ([15, Theorem 1]).*

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