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Multi-Frame Vectors for Unitary Systems in Hilbert C^*-modules

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Abstract. In this paper, we focus on the structured multi-frame vectors in Hilbert C^*- modules. More precisely, it will be shown that the set of all complete multi-frame vectors for a unitary system can be parameterized by the set of all surjective operators, in the local commutant. Similar results hold for the set of all complete wandering vectors and complete multi-Riesz vectors, when the surjective operator is replaced by unitary and invertible operators, respectively. Moreover, we show that new multi-frames (resp. multi-Riesz bases) can be obtained as linear combinations of known ones using coefficients which are operators in a certain class.

1. Introduction

Frames in Hilbert spaces were originally introduced by Duffin and Schaeffer [19] to deal with some problems in nonharmonic Fourier analysis. Apparently, the importance of this concept was not well realized by the mathematical community, and it took at least 30 years before the next treatment appeared in print. In 1985, Daubechies et al. [17] brought attention to it, which showed that Duffin and Schaeffer's definition is an abstraction of the concept introduced by Gabor [21] in 1946.

Frames have been used as a powerful alternative to Hilbert bases, and they allow a deep theory (for an overview see [9, 11, 24]). They are also very important for applications, e.g. in physics [1, 13], signal processing [5, 7, 8], numerical treatment of operator equations [14, 32] and acoustics [6, 30].

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There have been numerous generalizations of the concept of frames, see e.g. [1, 10, 12]. One of the most important generalizations of frame theory is the concept of modular frames which is introduced by Frank and Larson [20].

Recently, there are many mathematicians who apply operator theory tools to do research on frame theory in Hilbert spaces, in particular, they apply operator techniques to consider the construction of frames, which is a basic problem in theory and applications.

As mentioned, frame theory is a powerful tool in data processing such as image compression, denoising, etc. Before, only singly generated frames were used. Alpert [2] and Hervé [25], applied multi-frames in their works for the first time. Multi-frames naturally generalize the singly generated frames and gave some advantages in comparison to singly generated ones. For example, such features as short support, orthogonality, symmetry, vanishing moments are known to be important in signal processing. A singly generated frame cannot posses all these properties at the same time [16]. On the other hand, a multi-frame system can have all them simultaneously. This suggests that multi-frames can provide perfect reconstruction (orthogonality), good performance at the boundaries (symmetry) and high order of approximation (vanishing moments), so they could perform better than singly generated ones. There are several literatures on the theory and applications of multi-frames (see [18, 22, 23, 29]).

This paper is organized as follows. In Section 2, we state some definitions and preliminaries. In Section 3, we characterize the set of all complete wandering vectors, complete multi-frame vectors and complete multi-Riesz vectors in terms of unitary, surjective and invertible operators, respectively, in the local commutant. Finally, Section 4 is devoted to the linear combinations of multi-frame vectors for unitary systems.

2. Notation and Preliminaries

In this section, we recall some definitions and basic properties of Hilbert C^*-modules and their frames. Throughout this paper, A is a unital C^*-algebra and E, F are finitely or countably generated Hilbert A-modules. Moreover, by N we mean a countable index set.

A (left) Hilbert C^*-module over the C^*-algebra A is a left A-module E equipped with an A-valued inner product $\langle \cdot, \cdot \rangle : E \times E \to A$ satisfying the following conditions:

(i) $\langle x, x \rangle \geq 0$ for every $x \in E$ and $\langle x, x \rangle = 0$ if and only if $x = 0$,
(ii) $\langle x, y \rangle = \langle y, x \rangle^*$ for every $x, y \in E$,
(iii) $\langle \cdot, \cdot \rangle$ is A-linear in the first argument,
(iv) E is complete with respect to the norm $\|x\|^2 = \| \langle x, x \rangle \|_A$.

Given Hilbert C^*-modules E and F, we denote by $\mathcal{L}(E,F)$ the set of all adjointable operators from E to F (i.e. of all maps $T : E \to F$ such that there exists $T^* : F \to E$ with the property $\langle Tx, y\rangle = \langle x, T^*y\rangle$ for all $x \in E, y \in F$). Specially, if $E = F$, we write $\mathcal{L}(E)$.

It is well-known that each adjointable operator is necessarily bounded and A-linear in the sense $T(ax) = aT(x)$, for all $a \in A, x \in E$.

An operator $U \in \mathcal{L}(E,F)$ is said to be unitary if

$$U^*U = 1_E, \qquad UU^* = 1_F.$$

A Hilbert A-module E is called finitely generated (resp. countably generated) if there exists a finite subset $\{x_1, \ldots, x_n\}$ (resp. countable set $\{x_n\}_{n\in N}$) of E such that E equals the closed A-linear hull of this set.

Let A be a C^*-algebra. Consider

$$\ell^2(A) := \left\{ \{a_n\}_{n\in N} \subseteq A : \quad \sum_{n\in N} a_n a_n^* \text{ converges in norm in } A \right\}.$$

It is easy to see that $\ell^2(A)$ with pointwise operations and the inner product

$$\left\langle \{a_n\}_{n\in N}, \{b_n\}_{n\in N} \right\rangle = \sum_{n\in N} a_n b_n^*,$$

becomes a Hilbert C^*-module which is called the standard Hilbert C^*-module over A. For more detail about Hilbert C^*-modules, we refer the interested readers to the books [28, 31].

Now, we recall the concept of frames in Hilbert C^*-modules which is defined in [20]. Let E be a countably generated Hilbert module over a unital C^*-algebra A. A sequence $\{x_n\}_{n\in N} \subset E$ is said to be a frame if there exist two constants $C, D > 0$ such that

$$C\langle x, x\rangle \le \sum_{n\in N} \langle x, x_n\rangle \langle x_n, x\rangle \le D\langle x, x\rangle, \tag{2.1}$$

for every $x \in E$. The numbers C, D are called frame bounds.

The optimal constants (i.e. maximal for C and minimal for D) are called optimal frame bounds. If the sum in (2.1) converges in norm, the frame is called standard frame. If $C = D = 1$, then $\{x_n\}_{n\in N}$ is said to be a standard normalized tight frame. The sequence $\{x_n\}_{n\in N}$ is called a Bessel sequence with bound D if the upper inequality in (2.1) holds for every $x \in E$.

A Riesz basis in a Hilbert C^*-module E is a frame $\{x_n\}_{n\in N}$ such that for each $n \in N$, $x_n \neq 0$ and if an A-linear combination $\sum_{n\in M\subseteq N} a_n x_n$ is equal to zero, it concludes that every summand $a_n x_n$ is equal to zero.

Suppose that $\{x_n\}_{n\in N}$ is a standard Bessel sequence of a Hilbert A-module E. The operator $T : E \to \ell^2(A)$ defined by

$$Tx = \{\langle x, x_n\rangle\}_{n\in N},$$

is called the analysis operator. The adjoint operator $T^* : \ell^2(A) \to E$ is given by

$$T^*\left(\{a_n\}_{n\in N}\right) = \sum_{n\in N} a_n x_n,$$

and it is called the synthesis operator. If $\{x_n\}_{n\in N}$ is a standard frame for E with bounds C and D, then the frame operator $S : E \to E$ defined by:

$$Sx = T^*Tx = \sum_{n\in N} \langle x, x_n\rangle x_n,$$

is a well-defined, positive, invertible and adjointable operator. Further, it satisfies $C \le S \le D$ and $D^{-1} \le S^{-1} \le C^{-1}$. Also, the the reconstruction formula holds as follows:

$$x = \sum_{n\in N} \left\langle x, S^{-1}x_n\right\rangle x_n = \sum_{n\in N} \langle x, x_n\rangle S^{-1}x_n, \quad (x \in E)\,. \tag{2.2}$$

The sequence $\{\tilde{x}\}_{n\in N} = \left\{S^{-1}x_n\right\}_{n\in N}$, which is a standard frame with bounds D^{-1} and C^{-1}, is called the canonical dual frame of $\{x_n\}_{n\in N}$. Sometimes, the reconstruction formula for the standard frames is valid with other (standard) frames $\{y_n\}_{n\in N}$ instead of $\left\{S^{-1}x_n\right\}_{n\in N}$. They are said to be alternative dual frames of $\{x_n\}_{n\in N}$.

Following, we discuss about the concept of orthonormal bases in Hilbert modules. Actually, we borrow the definition of an orthonormal system for Hilbert C^*-modules from [4].

Definition 2.1. A collection $\{e_n\}_{n\in N}$ of vectors from E is called orthogonal if $\langle e_n, e_m\rangle = 0$ for all $n \ne m$. The orthogonal system $\{e_n\}_{n\in N}$ is said to be quasi-orthonormal if $\langle e_n, e_n\rangle = p_n$ is a minimal projection in A in the sense that $p_nAp_n = \mathbb{C}p_n$, for all $n \in N$. In a unital C^*-algebra A, $\{e_n\}_{n\in N}$ is said to be orthonormal if $\langle e_n, e_n\rangle = 1_A$, for all $n \in N$.

An orthonormal system $\{e_n\}_{n\in N}$ in E is said to be an orthonormal basis for E if it generates a dense submodule of E.

In [4, Theorem 1], the authors established some results for quasi-orthonormal systems in Hilbert modules, such as Fourier expansion and Parsevals identity. The following proposition gives similar results for orthonormal systems in Hilbert modules. Because of the similar process, we have omitted the proof.

Proposition 2.2. *Let E be a Hilbert C^*-module over a unital C^*-algebra A and let $\{e_n\}_{n\in N}$ be an orthonormal system in E. The following statements are mutually equivalent:*

(i) $\{e_n\}_{n\in N}$ *is an orthonormal basis for E.*
(ii) $x = \sum_{n\in N} \langle x, e_n\rangle e_n$ *for every $x \in E$.*
(iii) $\langle x, x\rangle = \sum_{n\in N} \langle x, e_n\rangle \langle e_n, x\rangle$ *for every $x \in E$.*
(iv) $\langle x, y\rangle = \sum_{n\in N} \langle x, e_n\rangle \langle e_n, y\rangle$ *for all $x, y \in E$.*

It should be mentioned that, contrasting to the Hilbert space situation, an arbitrary Hilbert C^*-module need not possess an orthonormal basis. The following lemma characterizes all orthonormal bases for a Hilbert module E which contains an orthonormal basis.

Lemma 2.3. *Let $\{e_n\}_{n\in N}$ be an orthonormal basis for Hilbert module E. Then the orthonormal bases for E are precisely the sets $\{Ue_n\}_{n\in N}$, where $U : E \to E$ is a unitary operator.*

Proof. Suppose $\{f_n\}_{n\in N}$ is an orthonormal basis for E. Define the operators U_1 and U_2 from E to $\ell^2(A)$ as

$$U_1(x) := \{\langle x, e_n\rangle\}_{n\in N}, \qquad U_2(x) := \{\langle x, f_n\rangle\}_{n\in N}.$$

Since $\{e_n\}_{n\in N}$ and $\{f_n\}_{n\in N}$ are both standard normalized tight frames, the operators U_1 and U_2 are well-defined and adjointable. Let $U := U_2^* U_1$. Then $Ux = \sum_{n\in N} \langle x, e_n\rangle f_n$ and so $Ue_n = f_n$, for each $n \in N$.

Now, we show that U is unitary. For each $x, y \in E$:

$$\begin{aligned} \langle U^*Ux, y\rangle &= \langle Ux, Uy\rangle \\ &= \left\langle \sum_{n\in N} \langle x, e_n\rangle f_n, \sum_{m\in N} \langle y, e_m\rangle f_m \right\rangle \\ &= \sum_{n\in N} \langle x, e_n\rangle \langle e_n, y\rangle \\ &= \langle x, y\rangle . \end{aligned}$$

Therefore U is an isometry. Due to the fact that U is also a surjective operator, by [27], it is obtained that U is unitary.

Conversely, if U is a unitary operator on E, then

$$\begin{aligned} \langle Ue_n, Ue_m\rangle &= \langle U^*Ue_n, e_m\rangle \\ &= \langle e_n, e_m\rangle \\ &= \delta_{n,m}, \end{aligned}$$

it means that $\{Ue_n\}_{n\in N}$ is an orthonormal system in E, and it is a basis by surjectivity of U. $\square$

Definition 2.4 ([15])**.** A unitary system $\mathcal{U}$ on E is a set of unitary operators acting on E which contains the identity operator.

Definition 2.5 ([15])**.** Let $\mathcal{S} \subseteq \mathcal{L}(E)$. We denote its commutant $\{A \in \mathcal{L}(E) : AS = SA, S \in \mathcal{S}\}$ by $\mathcal{S}'$. Let $\Psi^r = \{\psi_1, \psi_2, \dots, \psi_r\}$ be a tuple of elements in E. The local commutant $C_{\Psi^r}(\mathcal{S})$ is defined by

$$C_{\Psi^r}(\mathcal{S}) = \{A \in \mathcal{L}(E) : AS\psi_n = SA\psi_n, n = 1, \dots, r; S \in \mathcal{S}\}.$$

It is clear that this is a linear subspace of $\mathcal{L}(E)$.

Definition 2.6. A tuple $\Gamma^r = \{\gamma_1, \gamma_2, \dots, \gamma_r\}$ in E is called a complete multi-frame vector (resp. complete multi-normalized tight frame vector, complete multi-Riesz basis vector, complete multi-Bessel sequence vector) of multiplicity r for a unitary system $\mathcal{U}$ if

$$\mathcal{U}\Gamma^r = \{U\gamma_1, U\gamma_2, \dots, U\gamma_r; U \in \mathcal{U}\}$$

is a frame (resp. normalized tight frame, Riesz basis, Bessel sequence) for E. If $\mathcal{U}\Gamma^r$ is an orthonormal basis for E, then Γ^r is called a complete wandering r-tuple for $\mathcal{U}$. The set of all complete wandering r-tuples for $\mathcal{U}$ is denoted by $\mathcal{W}^r(\mathcal{U})$.

3. Parametrization of Multi-Frame Vectors by Operators in a Local Commutant

In this section, the results of [22] are extended to Hilbert modules. Namely, we characterize (complete) multi-frame vectors for a unitary system $\mathcal{U}$ in terms of certain class of operators in a local commutant of $\mathcal{U}$.

Proposition 3.1. *Let $\mathcal{U}$ be a unitary system on a Hilbert module E. Suppose that E has an orthonormal basis and $\Psi^r = \{\psi_1, \psi_2, \dots, \psi_r\}$ is a complete wandering r-tuple for $\mathcal{U}$. For a tuple $\Gamma^r = \{\gamma_1, \gamma_2, \dots, \gamma_r\}$ in E, the following statements are satisfying:*

(i) *Γ^r is a complete wandering r-tuple for $\mathcal{U}$ if and only if there exists a unitary operator $T \in C_{\Psi^r}(\mathcal{U})$ such that $\gamma_i = T\psi_i$ for $i = 1, \dots, r$.*
(ii) *Γ^r is a complete multi-Riesz basis vector with a unique dual frame for $\mathcal{U}$ if and only if there exists an invertible operator $T \in C_{\Psi^r}(\mathcal{U})$ such that $\gamma_i = T\psi_i$ for $i = 1, \dots, r$.*
(iii) *Γ^r is a complete multi-frame vector for $\mathcal{U}$ if and only if there exists a surjective operator $T \in C_{\Psi^r}(\mathcal{U})$ such that $\gamma_i = T\psi_i$ for $i = 1, \dots, r$.*
(iv) *Γ^r is a complete multi-Bessel sequence vector for $\mathcal{U}$ if and only if there exists an operator $T \in C_{\Psi^r}(\mathcal{U})$ such that $\gamma_i = T\psi_i$ for $i = 1, \dots, r$.*

(v) Γ^r *is a complete multi-normalized tight frame vector for* $\mathcal{U}$ *if and only if there exists a co-isometry* $T \in C_{\Psi^r}(\mathcal{U})$ *such that* $\gamma_i = T\psi_i$ *for* $i = 1, \ldots, r$.

Proof. (i) Firstly, suppose there is a unitary operator $T \in C_{\Psi^r}(\mathcal{U})$ such that for each $i = 1, \ldots, r$, $\gamma_i = T\psi_i$. Then, $U\gamma_i = UT\psi_i = TU\psi_i$, for $U \in \mathcal{U}$. Since $\{U\psi_1, \ldots, U\psi_r, U \in \mathcal{U}\}$ is an orthonormal basis for E and T is a unitary operator, it follows, by Lemma 2.3, that $\{U\gamma_1, \ldots, U\gamma_r, U \in \mathcal{U}\}$ is also an orthonormal basis for E and so Γ^r is a complete wandering r-tuple for $\mathcal{U}$.

Conversely, let Γ^r be a complete wandering r-tuple for $\mathcal{U}$. So $\{U\gamma_1, \ldots, U\gamma_r, U \in \mathcal{U}\}$ and $\{U\psi_1, \ldots, U\psi_r, U \in \mathcal{U}\}$ are two orthonormal bases for E. Hence, by Lemma 2.3, there exists a unitary operator T on E such that $U\gamma_i = TU\psi_i$, for each $i = 1, \ldots, r$ and $U \in \mathcal{U}$. Particularly, for $U = I$, we have $\gamma_i = T\psi_i$. Finally, for $U \in \mathcal{U}$, we have $UT\psi_i = U\gamma_i = TU\psi_i$, $i = 1, \ldots, r$, so $T \in C_{\Psi^r}(\mathcal{U})$.

(ii) Assume that there is an invertible operator $T \in C_{\Psi^r}(\mathcal{U})$ such that $\gamma_i = T\psi_i, i = 1, \ldots, r$. We show that $\{U\gamma_1, \ldots, U\gamma_r, U \in \mathcal{U}\}$ is a Riesz basis for E which has a unique dual frame. To do this, by [26, Theorem4.9], it is enough to prove $\{U\gamma_1, \ldots, U\gamma_r, U \in \mathcal{U}\}$ is a frame for E and the associated analysis operator ${\mathcal{T}^*}_{\{U\gamma_i\}}$ is surjective. Since Ψ^r is a complete wandering r-tuple for $\mathcal{U}$, so $\{U\psi_1, \ldots, U\psi_r, U \in \mathcal{U}\}$ is an orthonormal basis for E. Hence, by Proposition 2.2, it is also a normalized tight frame for E. Now, since for each $i = 1, \ldots, r$, $UT\psi_i = TU\psi_i$, so it is concluded, by [20, Theorem 5.3], that $\{U\gamma_1, \ldots, U\gamma_r, U \in \mathcal{U}\} = \{UT\psi_1, \ldots, UT\psi_r, U \in \mathcal{U}\}$ is a frame for E. Moreover, regarding $\{U\psi_1, \ldots, U\psi_r, U \in \mathcal{U}\}$ is a generating set for E, clearly $\{U\gamma_i, i = 1, \ldots, r; U \in \mathcal{U}\} = \{UT\psi_i = TU\psi_i, i = 1, \ldots, r; U \in \mathcal{U}\}$ is also a generating set for E.

To complete the proof it remains to prove that the synthesis operator $\mathcal{T}_{\{U\gamma_i\}}$ is injective. Assume that $\sum_{i=1}^{r} \sum_{U \in \mathcal{U}} a_U^i U\gamma_i = 0$. So

$$\begin{aligned} 0 &= \sum_{i=1}^{r} \sum_{U \in \mathcal{U}} a_U^i UT\psi_i \\ &= \sum_{i=1}^{r} \sum_{U \in \mathcal{U}} a_U^i TU\psi_i \\ &= T\left(\sum_{i=1}^{r} \sum_{U \in \mathcal{U}} a_U^i U\psi_i\right). \end{aligned}$$

Since T is injective, we have

$$\sum_{i=1}^{r}\sum_{U\in\mathcal{U}} a_U^i U\psi_i = 0.$$

Hence, for some index j and $V \in \mathcal{U}$,

$$\begin{aligned} 0 &= \sum_{i=1}^{r}\sum_{U\in\mathcal{U}} a_U^i \langle U\psi_i, V\psi_j\rangle \\ &= a_V^j \langle V\psi_j, V\psi_j\rangle \\ &= a_V^j. \end{aligned}$$

Since j and V are arbitrary, it follows that the synthesis operator $\mathcal{T}_{\{U\gamma_i\}}$ is injective and so $\mathcal{T}^*_{\{U\gamma_i\}}$ is surjective. Therefore, we conclude $\Gamma^r = T\Psi^r$ is a complete multi-Riesz basis vector with a unique dual frame.

Conversely, let Γ^r be a complete multi-Riesz basis vector for $\mathcal{U}$ with a unique dual frame. Define the operator T on E as:

$$Tx := \sum_{i=1}^{r}\sum_{U\in\mathcal{U}} \langle x, U\psi_i\rangle U\gamma_i. \tag{3.1}$$

Then T is well-defined and adjointable with

$$T^*x = \sum_{i=1}^{r}\sum_{U\in\mathcal{U}} \langle x, U\gamma_i\rangle\, U\psi_i.$$

Since $\{U\gamma_1, \ldots, U\gamma_r, U \in \mathcal{U}\}$ is a Riesz basis (and so a frame) for E, so there exist $0 < C \le D < \infty$ such that for each $x \in E$,

$$C\,\langle x, x\rangle \le \sum_{i=1}^{r}\sum_{U\in\mathcal{U}} \langle x, U\gamma_i\rangle\,\langle U\gamma_i, x\rangle \le D\,\langle x, x\rangle\,.$$

Now, by the fact that

$$\langle T^*x, T^*x\rangle = \sum_{i=1}^{r}\sum_{U\in\mathcal{U}} \langle x, U\gamma_i\rangle\,\langle U\gamma_i, x\rangle\,,$$

it follows from [3, Proposition 2.1] that T is surjective. For injectivity, let

$$Tx := \sum_{i=1}^{r}\sum_{U\in\mathcal{U}} \langle x, U\psi_i\rangle\, U\gamma_i = 0,$$

for some $x \in E$. Since $\{U\gamma_1, \ldots, U\gamma_r, U \in \mathcal{U}\}$ is a Riesz basis with a unique dual frame, it follows by [26, Theorem 4.9] that

$\langle x, U\psi_i\rangle = 0$, for each $i \in I, U \in \mathcal{U}$. Hence

$$x = \sum_{i=1}^{r} \sum_{U\in\mathcal{U}} \langle x, U\psi_i\rangle U\psi_i = 0.$$

Therefor, T is also injective. To complete the proof, it remains to prove that $\gamma_i = T\psi_i, i = 1, \ldots, r$ and $T \in C_{\Psi^r}(\mathcal{U})$. For every $V, U \in \mathcal{U}$ and $i = 1, \ldots, r$,

$$\begin{aligned}\langle V\gamma_i, TU\psi_i\rangle &= \left\langle V\gamma_i, \sum_{j=1}^{r} \sum_{W\in\mathcal{U}} \langle U\psi_i, W\psi_j\rangle W\gamma_j \right\rangle \\ &= \sum_{j=1}^{r} \sum_{W\in\mathcal{U}} \langle V\gamma_i, W\gamma_j\rangle \langle W\psi_j, U\psi_i\rangle \\ &= \langle V\gamma_i, U\gamma_i\rangle .\end{aligned}$$

Hence, $TU\psi_i = U\gamma_i$, for $U \in \mathcal{U}$ and $i = 1, \ldots, r$. Particularly, for $U = I$, we have $T\psi_i = \gamma_i$ and $TU\psi_i = U\gamma_i = UT\psi_i$. The proof is complete.

(iii) Suppose that there exists a surjective operator $T \in C_{\Psi^r}(\mathcal{U})$ on E such that $\gamma_i = T\psi_i, i = 1, \ldots, r$. Thus $U\gamma_i = UT\psi_i = TU\psi_i$, for $i = 1, \ldots, r$. Since $\{U\psi_1, \ldots, U\psi_r; U \in \mathcal{U}\}$ is an orthonormal basis (and so a normalized tight frame) for E, and T is surjective, it follows that $\{U\gamma_1, \ldots, U\gamma_r; U \in \mathcal{U}\}$ is a frame for E, by [3, Theorem 3.5], and hence Γ^r is a complete multi-frame vector for $\mathcal{U}$.

Conversely, let Γ^r be a complete multi-frame vector for $\mathcal{U}$ with lower bound C. Define the operator T on E as (3.1). Thus, for each $x \in E$, we have:

$$\begin{aligned}\langle T^*x, T^*x\rangle &= \left\langle \sum_{i=1}^{r} \sum_{U\in\mathcal{U}} \langle x, U\gamma_i\rangle U\psi_i, \sum_{j=1}^{r} \sum_{V\in\mathcal{U}} \langle x, V\gamma_j\rangle V\psi_j \right\rangle \\ &= \sum_{i=1}^{r} \sum_{U\in\mathcal{U}} \langle x, U\gamma_i\rangle \langle U\gamma_i, x\rangle \\ &\geq C \langle x, x\rangle .\end{aligned}$$

Hence, by [3, Proposition 2.1], T is a surjective operator. The rest of proof is similar to the proof of part (ii).

(iv) It is obtained by a similar argument as part (iii).

(v) Suppose there is a co-isometry $T \in C_{\Psi^r}(\mathcal{U})$ such that $\Gamma^r = T\Psi^r$. Then, for every $x \in E$:

$$\langle x, x\rangle = \langle T^*x, T^*x\rangle$$

$$
\begin{aligned}
&= \sum_{i=1}^{r} \sum_{U \in \mathcal{U}} \langle T^* x, U\psi_i \rangle \langle U\psi_i, T^* x \rangle \\
&= \sum_{i=1}^{r} \sum_{U \in \mathcal{U}} \langle x, TU\psi_i \rangle \langle TU\psi_i, x \rangle \\
&= \sum_{i=1}^{r} \sum_{U \in \mathcal{U}} \langle x, UT\psi_i \rangle \langle UT\psi_i, x \rangle \\
&= \sum_{i=1}^{r} \sum_{U \in \mathcal{U}} \langle x, U\gamma_i \rangle \langle U\gamma_i, x \rangle ,
\end{aligned}
$$

which implies that $\{U\gamma_i\}$ is a normalized tight frame for E and hence Γ^r is a complete multi-normalized tight frame vector for $\mathcal{U}$.

Conversely, define the operator T on E as (3.1). Then, for every $x \in E$:

$$
\begin{aligned}
\langle T^* x, T^* x \rangle &= \left\langle \sum_{i=1}^{r} \sum_{U \in \mathcal{U}} \langle x, U\gamma_i \rangle U\psi_i, \sum_{j=1}^{r} \sum_{U \in \mathcal{U}} \langle x, U\gamma_j \rangle U\psi_j \right\rangle \\
&= \sum_{i=1}^{r} \sum_{U \in \mathcal{U}} \langle x, U\gamma_i \rangle \langle U\gamma_i, x \rangle \\
&= \langle x, x \rangle .
\end{aligned}
$$

Therefore, T is a co-isometry on E. The proof of $T \in C_{\Psi^r}(\mathcal{U})$ and $\gamma_i = T\psi_i$, $i = 1, \dots, r$, is similar to the previous parts.

□

Remark 3.2. It is well-known that in Hilbert spaces every Riesz basis has a unique dual which is also a Riesz basis. But in Hilbert C^*-modules, due to the zero divisors, not all Riesz bases have unique duals and not every dual is a Riesz basis. For example, let $A = M_{2\times 2}(\mathbb{C})$ denote the C^*-algebra of all 2×2 complex matrices. Let $E = A$ and for any $B, C \in E$ define

$$\langle B, C \rangle = BC^*.$$

Then E is a Hilbert A-module. Let F_{ij} be the 2×2 matrix with 1 in the ij-th entry and 0 elsewhere, where $1 \leq i, j \leq 2$. Then $\{F_{11}, F_{22}\}$ is a Riesz basis of E and it is a dual of itself. One can check that $\{F_{11} + F_{21}, F_{22}\}$ is also a dual Riesz basis of $\{F_{11}, F_{22}\}$.

In part (2) of [26, Proposition 5.1], it was claimed that for a unitary system $\mathcal{U}$ on a Hilbert module E, every complete Riesz basis is the image of a complete wandering vector under an invertible and adjointable operator. It seems that this is not true in general. Indeed, assume that

$\mathcal{U}$ is a unitary system on E and γ is a complete Riesz basis vector for $\mathcal{U}$. So, part (2) of [26, Proposition 5.1] implies that there is an invertible and adjointable operator T on E such that $\{U\gamma : U \in \mathcal{U}\} = \{TU\psi : U \in \mathcal{U}\}$, where ψ is a complete wandering vector in E. Now, if $\sum_{U\in\mathcal{U}} a_U U\gamma = 0$, for some $\{a_U\}_U \in \ell^2_{\mathcal{U}}(A)$, then

$$0 = \sum_{U\in\mathcal{U}} a_U U\gamma = T\left(\sum_{U\in\mathcal{U}} a_U U\psi\right).$$

Since T is injective, we have $\sum_{U\in\mathcal{U}} a_U U\psi = 0$. Therefore, for every $V \in \mathcal{U}$,

$$0 = \left\langle \sum_{U\in\mathcal{U}} a_U U\psi, V\psi \right\rangle = a_V.$$

Now, [26, Theorem 4.9] implies that $\{U\gamma\}_{U\in\mathcal{U}}$ has a unique dual frame, which does not hold in general.

Example 3.3. Let $\{e_n\}_{n=-\infty}^{+\infty}$ be an orthonormal basis for a Hilbert module E and $S \in \mathcal{L}(E)$ be the bilateral shift operator of multiplicity 2; $Se_n = e_{n+2}$, for any $n \in N$. Let $\mathcal{U} = \{S^n : n \in \mathbb{Z}\}$. Then $\mathcal{U}$ is a unitary system on E. Clearly, $\Psi^2 = \{e_0, e_1\}$ is a complete wandering 2-tuple vector for $\mathcal{U}$. Now, Proposition 3.1 implies that

$$\mathcal{N} = \left\{V\Psi^2 = \{Ve_0, Ve_1\}; V \in \mathcal{C}_{\Psi^2}(\mathcal{U}) \text{ and } V \text{ is a co-isometry on } E\right\},$$

$$\mathcal{F} = \left\{V\Psi^2 = \{Ve_0, Ve_1\}; V \in \mathcal{C}_{\Psi^2}(\mathcal{U}) \text{ and } V \text{ is a surjective operator on } E\right\},$$

are the sets of complete multi-normalized tight frame vector and complete multi-frame vector for $\mathcal{U}$, respectively.

As it is seen, in part (iii) of Proposition 3.1, the set of complete multi-frame vectors was characterized by surjective operators in local commutant. The following proposition gives another characterization of complete multi-frame vectors for unitary systems.

Proposition 3.4. *Let Ψ^r be a complete multi-normalized tight frame vector for a unitary system $\mathcal{U}$ and let Γ^r be a tuple of elements of E. Then, Γ^r is a complete multi-frame vector for $\mathcal{U}$ if and only if there exists an invertible operator $T \in C_{\Psi^r}(\mathcal{U})$ such that $\Gamma^r = T\Psi^r$, i.e, $\gamma_i = T\psi_i, i = 1, \ldots, r$.*

Proof. Assume that Γ^r is a complete multi-frame vector for $\mathcal{U}$. Then, $\{U\gamma_1, \ldots, U\gamma_r, U \in \mathcal{U}\}$ is a frame for E. So, by [20, Theorem 5.3], there is an invertible operator T on E such that $U\gamma_i = TU\psi_i, i = 1, \ldots, r$. Hence, for $U = I$, $\gamma_i = T\psi_i, i = 1, \ldots, r$, and then $TU\psi_i = U\gamma_i = UT\psi_i, i = 1, \ldots, r$, so $T \in C_{\Psi^r}(\mathcal{U})$.

Conversely, if there exists an invertible operator $T \in C_{\Psi^r}(\mathcal{U})$ such that $\gamma_i = T\psi_i, i = 1, \ldots, r$. Then $U\gamma_i = UT\psi_i = TU\psi_i, i = 1, \ldots, r$. So, by [20, Theorem 5.3], $\{U\gamma_1, \ldots, U\gamma_r, U \in \mathcal{U}\}$ is a frame for E and then Γ^r is a complete multi-frame vector for $\mathcal{U}$. □

The following proposition shows that if $\mathcal{S}$ is a unitary system which is not a group, and if $\mathcal{S}$ has a complete multi-normalized tight frame vector, then $\mathcal{S}$ could be a group. This generalizes [26, Proposition 5.2].

Proposition 3.5. *Let $\mathcal{S}$ be a unital semigroup of unitary operators on a Hilbert module E. Suppose that $\mathcal{S}$ has a complete multi-normalized tight frame vector. Then $\mathcal{S}$ is a group.*

Proof. Let $U_0 \in \mathcal{S}$ and $\Gamma^r = \{\gamma_1, \ldots, \gamma_r\}$ be a complete multi-normalized tight frame. Then for every $x \in E$,

$$\begin{aligned}\sum_{i=1}^{r}\sum_{V\in\mathcal{S}} \langle x, V\gamma_i\rangle \langle V\gamma_i, x\rangle &= \langle x, x\rangle \\ &= \langle U_0^* x, U_0^* x\rangle \\ &= \sum_{i=1}^{r}\sum_{V\in\mathcal{S}} \langle U_0^* x, V\gamma_i\rangle \langle V\gamma_i, U_0^* x\rangle \\ &= \sum_{i=1}^{r}\sum_{V\in\mathcal{S}} \langle x, U_0 V\gamma_i\rangle \langle U_0 V\gamma_i, x\rangle .\end{aligned}$$

Since $U_0\mathcal{S} \subseteq \mathcal{S}$, it follows that

$$\sum_{i=1}^{r}\sum_{V\in\mathcal{S}-U_0\mathcal{S}} \langle x, V\gamma_i\rangle \langle V\gamma_i, x\rangle = 0. \tag{3.2}$$

Assume by contradiction that $U_0^{-1} \notin \mathcal{S}$. Then, $Id \notin U_0\mathcal{S}$ and so by (3.2), it is concluded that $\langle x, \gamma_i\rangle \langle \gamma_i, x\rangle = 0$. Particularly, for $x = \gamma_i$, we have $\gamma_i = 0$, a contradiction. □

4. Linear Combinations of Multi-Frame Vectors for Unitary Systems in Hilbert C^*-modules

The main idea in operator-theoretic interpolation of frames is new frames that can be obtained as linear combinations of known ones using coefficients which are operators in a certain class. Both the ideas and the essential computations extend naturally to more general unitary systems and wandering vectors. In the sequel, by given results in section [3], we investigate some conditions under which some linear combinations of complete wandering (multi-Riesz basis) vectors also belong to the same class of vectors.

The following two results are the most elementary case of operator-theoretic interpolation of multi-vectors.

Proposition 4.1. *Suppose that $\Psi^r = \{\psi_1, \ldots, \psi_r\}$ and $\Gamma^r = \{\gamma_1, \ldots, \gamma_r\}$ are complete wandering r-tuples for a unitary system $\mathcal{U}$. Then $\Gamma^r + \alpha\Psi^r$ is a complete multi-Riesz basis for $\mathcal{U}$, where $\alpha \in \mathbb{C}$ with $|\alpha| \neq 1$. More generally, if Ψ^r and Γ^r are complete multi-Riesz bases, then there are positive numbers $b > a > 0$ such that $\Gamma^r + \alpha\Psi^r$ is a complete multi-Riesz basis for all $\alpha \in \mathbb{C}$ that either $|\alpha| < a$ or $|\alpha| > b$.*

Proof. By the first part of Proposition 3.1, there exists a unitary operator $T \in C_{\Psi^r}(\mathcal{U})$ such that $\Gamma^r = T\Psi^r$. So

$$\Gamma^r + \alpha\Psi^r = T\Psi^r + \alpha\Psi^r = (T + \alpha I)\,\Psi^r.$$

Since T is a unitary operator, $T + \alpha I$ is invertible if $|\alpha| \neq 1$ and the result is obtained by part (ii) of Proposition 3.1.

Now, assume that Ψ^r and Γ^r are complete multi-Riesz bases. Then, there is an invertible adjointable operator T on E with $T\Psi^r = \Gamma^r$. So

$$\Gamma^r + \alpha\Psi^r = T\Psi^r + \alpha\Psi^r = (T + \alpha I)\,\Psi^r.$$

Since T is invertible, there are $b > a > 0$ such that

$$\sigma(T) \subseteq \{\alpha \in \mathbb{C}, a < |\alpha| < b\},$$

where $\sigma(T)$ denotes the spectrum of T, and the same argument applies. □

Proposition 4.2. *Suppose that $\Psi^r = \{\psi_1, \ldots, \psi_r\}$ and $\Gamma^r = \{\gamma_1, \ldots, \gamma_r\}$ are complete wandering r-tuples for a unitary system $\mathcal{U}$. Then, $\alpha\Psi^r + (1-\alpha)\,\Gamma^r$, $\alpha \in \mathbb{C}$ with $|\alpha| \neq |\alpha - 1|$, is a complete multi-Riesz basis for $\mathcal{U}$.*

Proof. It is assumed that $\alpha \neq 1$. Since Ψ^r and Γ^r are complete wandering r-tuples, there exists a unitary operator T such that $\Gamma^r = T\Psi^r$. Thus, $\alpha\Psi^r + (1-\alpha)\,\Gamma^r = (\alpha I + (1-\alpha)\,T)\,\Psi^r$. It is enough to show that $S = \alpha I + (1-\alpha)\,T$ is invertible. Since T is unitary, $\sigma(T) \subseteq \{\alpha \in \mathbb{C}, |\alpha| = 1\}$. So for $\alpha \in \mathbb{C}$ with $|\alpha| \neq |\alpha - 1|$, we have $\alpha\,(\alpha-1)^{-1} \notin \sigma(T)$ and so

$$S = (1-\alpha)\left[T - \alpha\,(\alpha-1)^{-1}\,I\right],$$

is invertible and the proof is complete. □

In the next proposition, we are going to find the conditions such that the linear combinations of complete wandering multi-vectors for a unitary system $\mathcal{U}$ is also a complete wandering multi-vector.

Proposition 4.3. *Suppose that $\Psi^r = \{\psi_1, \ldots, \psi_r\}, \Gamma^r = \{\gamma_1, \ldots, \gamma_r\}$ and $\Lambda^r = \{\lambda_1, \ldots, \lambda_r\}$ are complete wandering r-tuples for a unitary system $\mathcal{U}$. Suppose that $A_1, A_2 \in C_{\Psi^r}(\mathcal{U})$ are unitary operators such that $\Gamma^r = A_1\Psi^r, \Lambda^r = A_2\Psi^r$. Moreover, let $B_i \in C_{A_i\Psi^r}(\mathcal{U}), (i = 1, 2)$, be two* *normal operators with* $B_iA_i = A_iB_i, (i = 1, 2)$ *and* $B_1B_2^* = B_1^*B_2 = 0$. *Then* $B_1\Gamma^r + B_2\Lambda^r$ *is a complete wandering* r*-tuple for* $\mathcal{U}$ *if and only if* $B_1B_1^* + B_2B_2^* = I$.

Proof. Since $\Gamma^r = A_1\Psi^r$ and $\Lambda^r = A_2\Psi^r$, so $B_1\Gamma^r + B_2\Lambda^r = B_1A_1\Psi^r + B_2A_2\Psi^r = (B_1A_1 + B_2A_2)\Psi^r$. Since $B_i \in C_{A_i\Psi^r}(\mathcal{U})$, so $B_iA_i \in C_{\Psi^r}(\mathcal{U})$, $(i = 1, 2)$ and hence $B_1A_1 + B_2A_2 \in C_{\Psi^r}(\mathcal{U})$. By the first part of Proposition 3.1, $B_1\Gamma^r + B_2\Lambda^r$ is a complete wandering r-tuple if and only if $B_1A_1 + B_2A_2$ is a unitary operator. Since

$$\begin{aligned}
&(B_1A_1 + B_2A_2)(B_1A_1 + B_2A_2)^* \\
&\qquad = (B_1A_1 + B_2A_2)(A_1^*B_1^* + A_2^*B_2^*) \\
&\qquad = B_1A_1A_1^*B_1^* + B_1A_1A_2^*B_2^* + B_2A_2A_1^*B_1^* + B_2A_2A_2^*B_2^* \\
&\qquad = B_1B_1^* + A_1B_1B_2^*A_2^* + A_2B_2B_1^*A_1^* + B_2B_2^* \\
&\qquad = B_1B_1^* + B_2B_2^*,
\end{aligned}$$

and

$$\begin{aligned}
&(B_1A_1 + B_2A_2)^*(B_1A_1 + B_2A_2) \\
&\qquad = (A_1^*B_1^* + A_2^*B_2^*)(B_1A_1 + B_2A_2) \\
&\qquad = A_1^*B_1^*B_1A_1 + A_1^*B_1^*B_2A_2 + A_2^*B_2^*B_1A_1 + A_2^*B_2^*B_2A_2 \\
&\qquad = B_1^*A_1^*A_1B_1 + B_2^*A_2^*A_2B_2 \\
&\qquad = B_1^*B_1 + B_2^*B_2 \\
&\qquad = B_1B_1^* + B_2B_2^*,
\end{aligned}$$

so $B_1A_1+B_2A_2$ is a unitary operator if and only if $B_1B_1^*+B_2B_2^* = I$. □

For complete normalized multi-tight frame vectors, we have the following sufficient conditions for their linear combinations keep to be complete normalized multi-tight frame vectors. Because of the same process as [22, Theorem 4.1], the proof is omitted.

Proposition 4.4. *Suppose that* $\Psi^r = \{\psi_1, \ldots, \psi_r\}$ *is complete wandering* r*-tuples for a unitary system* $\mathcal{U}$ *and* $\Gamma^r = \{\gamma_1, \ldots, \gamma_r\}$ *and* $\Lambda^r = \{\lambda_1, \ldots, \lambda_r\}$ *are complete multi-normalized tight frame vectors for* $\mathcal{U}$. *Suppose that* $A_1, A_2 \in C_{\Psi^r}(\mathcal{U})$ *are co-isometries such that* $\Gamma^r = A_1\Psi^r, \Lambda^r = A_2\Psi^r$ *and* $A_1A_2^* = 0$. *Moreover, let* $B_i \in C_{A_i\Psi^r}(\mathcal{U}), (i = 1, 2)$. *Then,* $B_1\Gamma^r + B_2\Lambda^r$ *is a complete multi-normalized tight frame for* $\mathcal{U}$ *if and only if* $B_1B_1^* + B_2B_2^* = I$.

The next proposition gives a sufficient condition under which the linear combination of complete multi-Riesz bases is also a complete multi-Riesz basis.

Proposition 4.5. *Suppose that $\Psi^r = \{\psi_1, \ldots, \psi_r\}$ is a complete wandering r-tuple for a unitary system $\mathcal{U}$, $\Gamma^r = \{\gamma_1, \ldots, \gamma_r\}$ and $\Lambda^r = \{\lambda_1, \ldots, \lambda_r\}$ are complete multi-Riesz bases for $\mathcal{U}$ and $A_1, A_2 \in C_{\Psi^r}(\mathcal{U})$ are invertible operators such that $\Gamma^r = A_1\Psi^r, \Lambda^r = A_2\Psi^r$. Moreover, let $B_i \in C_{A_i\Psi^r}(\mathcal{U}), (i = 1, 2)$ be two operators with $B_iA_i = A_iB_i, (i = 1, 2)$ and $B_1B_2 = B_2B_1 = 0$. Then $B_1\Gamma^r + B_2\Lambda^r$ is a complete multi-Riesz basis for $\mathcal{U}$ if $B_1^2 + B_2^2 = I$.*

Proof. Since $\Gamma^r = A_1\Psi^r$ and $\Lambda^r = A_2\Psi^r$, so $B_1\Gamma^r + B_2\Lambda^r = B_1A_1\Psi^r + B_2A_2\Psi^r = (B_1A_1 + B_2A_2)\Psi^r$. Since $B_i \in C_{A_i\Psi^r}(\mathcal{U})$, so $B_iA_i \in C_{\Psi^r}(\mathcal{U})$, $(i = 1, 2)$ and hence $B_1A_1 + B_2A_2 \in C_{\Psi^r}(\mathcal{U})$. Moreover,

$$\begin{aligned}(B_1A_1 + B_2A_2)\left(A_1^{-1}B_1 + A_2^{-1}B_2\right) &= B_1^2 + B_1A_1A_2^{-1}B_2 + B_2A_2A_1^{-1}B_1 + B_2^2\\ &= B_1^2 + A_1B_1B_2A_2^{-1} + A_2B_2B_1A_1^{-1} + B_2^2\\ &= B_1^2 + B_2^2.\end{aligned}$$

and

$$\begin{aligned}&\left(A_1^{-1}B_1 + A_2^{-1}B_2\right)(B_1A_1 + B_2A_2)\\ &\quad= A_1^{-1}B_1B_1A_1 + A_1^{-1}B_1B_2A_2 + A_2^{-1}B_2B_1A_1 + A_2^{-1}B_2B_2A_2\\ &\quad= B_1A_1^{-1}A_1B_1 + B_2A_2^{-1}A_2B_2\\ &\quad= B_1^2 + B_2^2.\end{aligned}$$

Hence, if $B_1^2 + B_2^2 = I$, it follows that $B_1A_1 + B_2A_2$ is invertible and so by part (ii) of Proposition 3.1, $B_1\Gamma^r + B_2\Lambda^r$ is a complete multi-Riesz basis for $\mathcal{U}$. □

Proposition 4.6. *Let Ψ^r be a complete wandering r-tuple for a unitary system $\mathcal{U}$, Γ^r be a complete multi-normalized tight frame vector for $\mathcal{U}$ and $U \in C_{\Psi^r}(\mathcal{U})$ be a co-isometry for which $\Gamma^r = U\Psi^r$. Moreover, suppose that $V \in \mathcal{U}'$ is a co-isometry operator. Then $V\Gamma^r - \Psi^r$ is a complete multi-normalized tight frame vector for $\mathcal{U}$ if and only if $VU + U^*V^* = I$.*

Proof. We have:

$$V\Gamma^r - \Psi^r = VU\Psi^r - \Psi^r = (VU - I)\Psi^r.$$

It is obvious that $VU - I \in C_{\Psi^r}$. By part (v) of Proposition 3.1, it is enough to check that $VU - I$ is a co-isometry operator.

$$\begin{aligned}(VU - I)(VU - I)^* &= (VU - I)(U^*V^* - I)\\ &= VUU^*V^* - VU - U^*V^* + I\\ &= I,\end{aligned}$$

and so the proof is complete. □

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