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σ-Connes Amenability and Pseudo-(Connes) Amenability of Beurling Algebras

Zahra Hasanzadeh[1] and Amin Mahmoodi[2*]

Abstract. In this paper, pseudo-amenability and pseudo-Connes amenability of weighted semigroup algebra $\ell^1(S,\omega)$ are studied. It is proved that pseudo-Connes amenability and pseudo-amenability of weighted group algebra $\ell^1(G,\omega)$ are the same. Examples are given to show that the class of σ-Connes amenable dual Banach algebras is larger than that of Connes amenable dual Banach algebras.

1. Introduction

Let $\mathcal{A}$ be a Banach algebra, and let E be a Banach $\mathcal{A}$-bimodule. A bounded linear map $D : \mathcal{A} \to E$ is a derivation if $D(ab) = a \cdot D(b) + D(a) \cdot b$ for $a, b \in \mathcal{A}$, and it is inner derivation if there exists an element $x \in E$ such that $D(a) = a \cdot x - x \cdot a$ for all $a \in \mathcal{A}$. A Banach algebra $\mathcal{A}$ is amenable if for every Banach $\mathcal{A}$-bimodule E, every derivation $D : \mathcal{A} \to E^*$ is inner [9]. For a Banach algebra $\mathcal{A}$, it is known that there exists a continuous linear $\mathcal{A}$-bimodule homomorphism π from the projective tensor product $\mathcal{A}\hat{\otimes}\mathcal{A}$ into $\mathcal{A}$ such that $\pi(a \otimes b) = ab$ for $a, b \in \mathcal{A}$. It is known that amenability of a a Banach algebra $\mathcal{A}$ is equivalent to existence of a bounded approximate diagonal for $\mathcal{A}$, that is, a net $(m_\alpha) \subseteq \mathcal{A}\hat{\otimes}\mathcal{A}$ such that $a \,.\, m_\alpha - m_\alpha \,.\, a \longrightarrow 0$ and $\pi(m_\alpha)a \longrightarrow a$ for each $a \in \mathcal{A}$ [8]. In [7], Ghahramani and Zhang introduced the concept of pseudo-amenability for Banach algebras which modifies Johnson's original definition of amenability by relaxing the structure of the diagonals. A Banach algebra $\mathcal{A}$ is pseudo-amenable if it admits an (not necessarily bounded) approximate diagonal. Motivated by this notion, the second

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* Corresponding author.

author in [12] introduced the concept of pseudo-Connes amenability for dual Banach algebras. The original definition of Connes amenability systematically introduced by Runde [16], however it had been studied previously under different names.

A generalized version of amenability was introduced by Mirzavaziri and Moslehian in [13] (see also [15]). Let $\mathcal{A}$ be a Banach algebra. We write $Hom(\mathcal{A})$ for the set of all continuous homomorphisms from $\mathcal{A}$ to itself. Let E be a Banach $\mathcal{A}$-bimodule, and let $\sigma \in Hom(\mathcal{A})$. From [14], we recall that a bounded linear map $D : \mathcal{A} \to E$ is a σ-derivation if $D(ab) = \sigma(a)\cdot D(b) + D(a)\cdot\sigma(b), (a, b \in \mathcal{A})$, and it is σ-inner derivation if there is an element $x \in E$ such that $D(a) = \sigma(a)\cdot x - x\cdot\sigma(a)$ for all $a \in \mathcal{A}$. A Banach algebra $\mathcal{A}$ is σ-amenable if for every Banach $\mathcal{A}$-bimodule E, every σ-derivation $D : \mathcal{A} \to E^*$ is σ-inner.

The organization of the paper is as follows. In section 2, we study pseudo-amenability and pseudo-Connes amenability of Beurling algebras $\ell^1(S,\omega)$, where S is a discrete semigroup and ω is a weight on S, that is, a function $\omega : S \longrightarrow (0,\infty)$ for which $\omega(st) \leq \omega(s)\omega(t)$ $(s, t \in S)$. We prove that $\ell^1(G,\omega)$ is pseudo-Connes amenable if and only if it is pseudo-amenable, where G is a discrete group.

In section 3, we continue the investigation of σ-Connes amenability begun in [13]. In particular, we introduce a class of Banach algebras to show difference between σ-Connes amenability and Connes amenability.

2. Pseudo-amenability and Pseudo-Connes Amenability of Beurling Algebras

Throughout, we use the term unital for a semigroup (or an algebra) X with (if any) an identity element e_X. We write $\mathcal{A}^\sharp$ for the forced unitization of an algebra $\mathcal{A}$. For Banach spaces E and F, we also write $\mathcal{L}(E, F)$ for the Banach space of all bounded linear maps from E into F.

Let $\mathcal{A}$ be a Banach algebra, and let E be a Banach $\mathcal{A}$-bimodule. A derivation $D : \mathcal{A} \longrightarrow E^*$ is w^*-approximately inner if there exists a net $(\phi_i) \subseteq E^*$ such that $D(a) = w^* - \lim_i(a\ .\ \phi_i - \phi_i\ .\ a)$ for $a \in \mathcal{A}$, the limit is in the w^*-topology on E^*. A Banach algebra $\mathcal{A}$ is w^*-approximately amenable if for each Banach $\mathcal{A}$-bimodule E, every derivation $D : \mathcal{A} \longrightarrow E^*$ is w^*-approximately inner. In [6], it was shown that w^*-approximate amenability and approximate amenability are the same for Banach algebras.

The following is immediate by the classical argument in [5, Theorem 2.1].

Theorem 2.1. *Let $\mathcal{A}$ be a Banach algebra. Then the following are equivalent:*

(i) *$\mathcal{A}$ is w^*-approximately amenable.*
(ii) *There exists a net $(M_\alpha)_\alpha \subseteq (\mathcal{A}^\sharp \hat{\otimes} \mathcal{A}^\sharp)^{**}$ such that for every $a \in \mathcal{A}^\sharp$, $a \,.\, M_\alpha - M_\alpha \,.\, a \xrightarrow{w^*} 0$ and $\pi^{**}(M_\alpha) \xrightarrow{w^*} e_{\mathcal{A}^\sharp}$ in $(\mathcal{A}^\sharp \hat{\otimes} \mathcal{A}^\sharp)^{**}$ and $(\mathcal{A}^\sharp)^{**}$, respectively.*
(iii) *There exists a net $(M'_\alpha)_\alpha \subseteq (\mathcal{A}^\sharp \hat{\otimes} \mathcal{A}^\sharp)^{**}$ such that for every $a \in \mathcal{A}^\sharp$, $a \,.\, M'_\alpha - M'_\alpha \,.\, a \xrightarrow{w^*} 0$ in $(\mathcal{A}^\sharp \hat{\otimes} \mathcal{A}^\sharp)^{**}$, and $\pi^{**}(M'_\alpha) = e_{\mathcal{A}^\sharp}$ for all α.*

To proceed further we state a part of the nice result [7, Proposition 3.2].

Theorem 2.2. *For a Banach algebra with a bounded approximate identity, pseudo-amenability and (w^*-) approximate amenability are the same.*

Let S be a discrete semigroup, and let ω be a weight on S. Following [3], we consider the Beurling algebra

$$\ell^1(S,\omega) = \left\{ (a_g)_{g\in S} \subseteq \mathbb{C} | ||(a_g)_g|| = \sum_{g\in S} |a_g|\omega(g) < \infty \right\},$$

as the Banach space $\ell^1(S)$ with the product $\delta_g \star_\omega \delta_h := \delta_{gh}\Omega(g,h)$, where

$$\Omega(g,h) := \frac{\omega(gh)}{\omega(g)\omega(h)}, \quad (g,h \in S),$$

and extend $\star_\omega$ to $\ell^1(S)$ by linearity and continuity. We recall that $\ell^1(S)\hat{\otimes}\ell^1(S) = \ell^1(S\times S)$, where $\delta_g \otimes \delta_h$ is identified with $\delta_{(g,h)}$ for $g,h \in S$. Thus we have $\mathcal{L}(\ell^1(S),\ell^\infty(S)) = (\ell^1(S)\hat{\otimes}\ell^1(S))^* = \ell^1(S\times S)^* = \ell^\infty(S\times S)$, for which $T \in \mathcal{L}(\ell^1(S),\ell^\infty(S))$ is identified with $(T_{(g,h)})_{(g,h)\in S\times S} \in \ell^\infty(S\times S)$, where $T_{(g,h)} = \langle \delta_h, T(\delta_g)\rangle$. Further, it is readily seen that $\ell^\infty(S\times S)$ has a natural $\ell^1(S)$-bimodule structure. If S is unital, without loss of generality, we may assume that $\omega(e_S) = 1$.

Theorem 2.3. *Let S be a discrete unital semigroup and let ω be a weight on S. Then the following are equivalent:*

(i) *$\ell^1(S,\omega)$ is pseudo-amenable.*
(ii) *There exists a net $(M_\alpha)_\alpha \subseteq \ell^\infty(S\times S)^*$ such that*

$$\left\langle (f(hk,g)\Omega(h,k) - f(h,kg)\Omega(k,g))_{(g,h)\in S\times S}, M_\alpha \right\rangle \longrightarrow 0,$$

for every $k \in S$ and $f \in \ell^\infty(S\times S)$, and

$$\left\langle (f_{gh}\Omega(g,h))_{(g,h)\in S\times S}, M_\alpha \right\rangle \longrightarrow f_{e_S},$$

for all $f \in \ell^\infty(S)$.

(iii) *There exists a net* $(M_\alpha^{'})_\alpha \subseteq \ell^\infty(S\times S)^*$ *such that*

$$\left\langle (f(hk,g)\Omega(h,k) - f(h,kg)\Omega(k,g))_{(g,h)\in S\times S}, M_\alpha^{'} \right\rangle \longrightarrow 0,$$

for every $k\in S$ *and* $f\in \ell^\infty(S\times S)$, *and*

$$\left\langle (f_{gh}\Omega(g,h))_{(g,h)\in S\times S}, M_\alpha^{'} \right\rangle = f_{e_S},$$

for all $f\in\ell^\infty(S)$ *and* α.

Proof. First, we notice that for every $f=(f_g)_{g\in S}\in\ell^\infty(S)$

$$\pi^*(f) = (\langle \delta_{gh}, f\rangle\,\Omega(g,h))_{(g,h)\in S\times S} \in \ell^\infty(S\times S).$$

Next, for every $T\in\ell^\infty(S\times S)$ and every $k\in S$ we have

$$\langle \delta_g\otimes\delta_h, \delta_k\ .\ T - T.\ \delta_k\rangle = \langle \delta_g, T(\delta_{hk})\rangle\,\Omega(h,k) - \langle \delta_{kg}, T(\delta_h)\rangle\,\Omega(k,g).$$

We also observe that $e_{\mathcal{A}} = \delta_{e_S}$ and therefore $\langle f, e_{\mathcal{A}}\rangle = f_{e_S}$.
(i) $\longrightarrow$ (ii) Suppose that $\mathcal{A}$ is pseudo-amenable. Using Theorem 2.2, we may take the net $(M_\alpha)_\alpha\subseteq\ell^\infty(S\times S)^*$ as in Theorem 2.1 (ii). For every $f\in\ell^\infty(S)$ we have

$$\left\langle (f_{gh}\Omega(g,h))_{(g,h)\in S\times S}, M_\alpha\right\rangle - f_{e_S} = \langle f, \pi^{**}(M_\alpha) - e_{\mathcal{A}}\rangle \longrightarrow 0.$$

Take $f\in\ell^\infty(S\times S)$, $k\in S$ and consider $T\in\ell^\infty(S\times S)$ defined by $\langle\delta_h, T(\delta_g)\rangle = f(g,h)$. Then we see that

$$\begin{aligned}
&\left\langle (f(hk,g)\Omega(h,k) - f(h,kg)\Omega(k,g))_{(g,h)\in S\times S}, M_\alpha\right\rangle\\
&= \left\langle ((\langle\delta_g\otimes\delta_h, \delta_k\ .\ T - T\ .\ \delta_k\rangle)_{(g,h)\in S\times S}, M_\alpha\right\rangle\\
&= \langle \delta_k\ .\ T - T\ .\ \delta_k, M_\alpha\rangle\\
&= \langle T, \delta_k\ .\ M_\alpha - M_\alpha\ .\ \delta_k\rangle \longrightarrow 0.
\end{aligned}$$

Hence, all in all, we have the clause (ii).

Similarly, we may prove the implications (ii) $\longrightarrow$ (i) and (i) $\leftrightarrow$ (iii). □

Let $\mathcal{A}$ be a Banach algebra. A Banach $\mathcal{A}$-bimodule E is dual if there is a closed submodule E_* of E^* such that $E=(E_*)^*$. We call E_* the predual of E. A Banach algebra $\mathcal{A}$ is dual if it is dual as a Banach $\mathcal{A}$-bimodule. We write $\mathcal{A}=(\mathcal{A}_*)^*$ if we wish to stress that $\mathcal{A}$ is a dual Banach algebra with predual $\mathcal{A}_*$. Let $\mathcal{A}$ be a dual Banach algebra. A dual Banach $\mathcal{A}$-bimodule E is normal if the module actions of $\mathcal{A}$ on E are w^*-continuous. A dual Banach algebra $\mathcal{A}$ is Connes-amenable if every w^*-continuous derivation from $\mathcal{A}$ into a normal dual Banach $\mathcal{A}$-bimodule is inner. The reader is referred to [3, 4, 10–12, 16–18] for more information and basic properties of Connes amenable dual Banach algebras. Let $\mathcal{A}=(\mathcal{A}_*)^*$ be a dual Banach algebra and let E be a

Banach $\mathcal{A}$-bimodule. We write $\sigma wc(E)$ for the set of all elements $x \in E$ such that the maps

$$\mathcal{A} \longrightarrow E \quad , \qquad a \longmapsto \begin{cases} a \cdot x \\ x \cdot a \end{cases} ,$$

are w^*-weak continuous. The space $\sigma wc(E)$ ia a closed submodule of E. It is shown in [17, Corollary 4.6], that $\pi^*(\mathcal{A}_*) \subseteq \sigma wc(\mathcal{A}\hat{\otimes}\mathcal{A})^*$. Taking adjoint, we can extend π to an $\mathcal{A}$-bimodule homomorphism $\pi_{\sigma wc}$ from $\sigma wc((\mathcal{A}\hat{\otimes}\mathcal{A})^*)^*$ to $\mathcal{A}$. A σwc-virtual diagonal for a dual Banach algebra $\mathcal{A}$ is an element $U \in \sigma wc((\mathcal{A}\hat{\otimes}\mathcal{A})^*)^*$ such that $a \,.\, U = U \,.\, a$ and $a\pi_{\sigma wc}(U) = a$ for $a \in \mathcal{A}$. From [17], we know that Connes amenability of a dual Banach algebra $\mathcal{A}$ is equivalent to the existence of a σwc-virtual diagonal for $\mathcal{A}$. A dual Banach algebra $\mathcal{A}$ is w^*-approximately Connes-amenable if for every normal, dual Banach $\mathcal{A}$-bimodule E every w^*-continuous derivation $D : \mathcal{A} \longrightarrow E$ is w^*-approximately inner [12].

We state the following which is [12, Proposition 2.5].

Proposition 2.4. *Let $\mathcal{A}$ be a dual Banach algebra. Then the following are equivalent:*

(i) *$\mathcal{A}$ is w^*-approximately Connes-amenable.*

(ii) *There exists a net $(M_\alpha)_\alpha \subseteq \sigma wc((\mathcal{A}^\sharp\hat{\otimes}\mathcal{A}^\sharp)^*)^*$ such that for every $a \in \mathcal{A}^\sharp$, $a \,.\, M_\alpha - M_\alpha \,.\, a \xrightarrow{w^*} 0$ and $\pi_{\sigma wc}M_\alpha \xrightarrow{w^*} e_{\mathcal{A}^\sharp}$, in $\sigma wc((\mathcal{A}^\sharp\hat{\otimes}\mathcal{A}^\sharp)^*)^*$ and $\mathcal{A}^\sharp$, respectively.*

(iii) *There exists a net $(M'_\alpha)_\alpha \subseteq \sigma wc((\mathcal{A}^\sharp\hat{\otimes}\mathcal{A}^\sharp)^*)^*$ such that for every $a \in \mathcal{A}^\sharp$, $a \,.\, M'_\alpha - M'_\alpha \,.\, a \xrightarrow{w^*} 0$, in $\sigma wc((\mathcal{A}^\sharp\hat{\otimes}\mathcal{A}^\sharp)^*)^*$ and $\pi_{\sigma wc}M'_\alpha = e_{\mathcal{A}^\sharp}$ for all α.*

Suppose that $\mathcal{A}$ is a dual Banach algebra. A net (m_α) in $\mathcal{A}\hat{\otimes}\mathcal{A}$ is an approximate σwc-diagonal for $\mathcal{A}$ if for every $a \in \mathcal{A}$, $a \,.\, m_\alpha - m_\alpha \,.\, a \xrightarrow{w^*} 0$ in $\sigma wc((\mathcal{A}\hat{\otimes}\mathcal{A})^*)^*$, and $a\pi_{\sigma wc}(m_\alpha) \xrightarrow{w^*} a$ in $\mathcal{A}$.

Definition 2.5 ([12])**.** A dual Banach algebra $\mathcal{A}$ is pseudo-Connes amenable if it admits an (not necessarily bounded) approximate σwc-diagonal.

The following is a part of [12, Theorem 5.3].

Theorem 2.6. *Let $\mathcal{A}$ be a unital dual Banach algebra. Then $\mathcal{A}$ is pseudo-Connes amenable if and only if it is w^*-approximately Connes-amenable.*

For a Banach algebra $\mathcal{A}$, we denote by $\imath_{\mathcal{A}}$, the canonical map from $\mathcal{A}$ to $\mathcal{A}^{**}$ defined by $\langle \mu, \imath_{\mathcal{A}}(x)\rangle = \langle x, \mu\rangle$ for $\mu \in \mathcal{A}^*$, $x \in E$. Next, it is standard that $(\mathcal{A}\hat{\otimes}\mathcal{A})^* \cong \mathcal{L}(\mathcal{A}, \mathcal{A}^*)$. Hence we obtain a bimodule structure on

$\mathcal{L}(\mathcal{A},\mathcal{A}^*)$ given by $(a\ .\ T)(b)=T(ba)$ and $(T\ .\ a)(b)=T(b)\ .\ a$ for $a,b\in\mathcal{A}$, and $T\in\mathcal{L}(\mathcal{A},\mathcal{A}^*)$. The reader may see [1] for more details.

Proposition 2.7. *Let $\mathcal{A}$ be a dual Banach algebra. Then the following are equivalent:*

(i) *$\mathcal{A}$ is w^*-approximately Connes-amenable.*

(ii) *There exists a net $(M_\alpha)_\alpha\subseteq(\mathcal{A}^\sharp\hat{\otimes}\mathcal{A}^\sharp)^{**}$ such that for every $a\in\mathcal{A}^\sharp$ and $T\in\sigma wc(\mathcal{L}(\mathcal{A},\mathcal{A}^*))$, $\langle T,a\ .\ M_\alpha-M_\alpha\ .\ a\rangle\longrightarrow 0$ and $\imath^*_{(\mathcal{A}^\sharp)_*}\pi^{**}M_\alpha\xrightarrow{w^*}e_{\mathcal{A}^\sharp}$, in $\mathcal{A}^\sharp$.*

(iii) *There exists a net $(M'_\alpha)_\alpha\subseteq(\mathcal{A}^\sharp\hat{\otimes}\mathcal{A}^\sharp)^{**}$ such that for every $a\in\mathcal{A}^\sharp$ and $T\in\sigma wc(\mathcal{L}(\mathcal{A},\mathcal{A}^*))$, $\left\langle T,a\ .\ M'_\alpha-M'_\alpha\ .\ a\right\rangle\longrightarrow 0$ and $\imath^*_{(\mathcal{A}^\sharp)_*}\pi^{**}M'_\alpha=e_{\mathcal{A}^\sharp}$ for all α.*

Proof. As $\sigma wc((\mathcal{A}^\sharp\hat{\otimes}\mathcal{A}^\sharp)^*)^*$ is a quotient of $(\mathcal{A}^\sharp\hat{\otimes}\mathcal{A}^\sharp)^{**}$, this is just a re-statement of Proposition 2.4. □

A semigroup S is weakly cancellative if for each $s\in S$, the maps L_s and R_s defined by $L_s(t)=st$ and $R_s(t)=ts$ are finite-to-one. If S is weakly cancellative, then it is known that $\ell^1(S,\omega)$ with the convolution product is a dual Banach algebra [3, Proposition 5.1].

The following is [3, Proposition 5.5].

Proposition 2.8. *Let S be a weakly cancellative semigroup, let ω be a weight on S, and let $\mathcal{A}:=\ell^1(S,\omega)$. Let $T\in\mathcal{L}(\mathcal{A},\mathcal{A}^*)$ be such that $T(\mathcal{A})\subseteq\imath_{c_0(S)}(c_0(S))$ and $T^*(\imath_{\mathcal{A}}(\mathcal{A}))\subseteq\imath_{c_0(S)}(c_0(S))$. Then $T\in\mathcal{W}(\mathcal{A},\mathcal{A}^*)$, and $T\in WAP(\mathcal{W}(\mathcal{A},\mathcal{A}^*))$ if and only if for each sequence (k_n) of distinct elements of S, and each sequence (g_m,h_m) of distinct elements of $S\times S$ such that the repeated limits*

$$\lim_n\lim_m\langle\delta_{k_ng_m},\quad T(\delta_{h_m})\rangle,\quad \lim_n\lim_m\Omega(k_n,g_m),$$
$$\lim_n\lim_m\langle\delta_{g_m},T(\delta_{h_mk_n})\rangle,\quad \lim_n\lim_m\Omega(h_m,k_n).$$

all exist, then we have at least one repeated limit in each row which is zero.

Theorem 2.9. *Let S be a discrete, weakly cancellative semigroup, let ω be a weight on S, and let $\ell^1(S,\omega)$ be unital. Then the following are equivalent:*

(i) *$\ell^1(S,\omega)$ exists pseudo-Connes amenable.*

(ii) *There exists a net $(M_\alpha)_\alpha\subseteq\ell^\infty(S\times S)^*$ such that*

$$\left\langle(f(hk,g)\Omega(h,k)-f(h,kg)\Omega(k,g))_{(g,h)\in S\times S},M_\alpha\right\rangle\longrightarrow 0,$$

for each $k\in S$ and each $f\in\ell^\infty(S\times S)$, which is, such that the map T defined by $\langle\delta_h,T(\delta_g)\rangle=f(g,h)$, for $g,h\in S$, satisfies *the conclusions of Proposition 2.8, and*

$$\left\langle (f_{gh}\Omega(g,h))_{(g,h)\in S\times S}, M_\alpha \right\rangle \longrightarrow \langle f, e_{\mathcal{A}}\rangle ,$$

for all $f \in c_0(S)$.

(iii) *There exists a net* $(M_\alpha^{'})_\alpha \subseteq \ell^\infty(S\times S)^*$ *such that*

$$\left\langle (f(hk,g)\Omega(h,k) - f(h,kg)\Omega(k,g))_{(g,h)\in S\times S}, M_\alpha^{'} \right\rangle \longrightarrow 0,$$

for each $k \in S$ *and each* $f \in \ell^\infty(S\times S)$, *which is, such that the map* T *defined by* $\langle \delta_h, T(\delta_g)\rangle = f(g,h)$, *for* $g, h \in S$, *satisfies the conclusions of Proposition 2.8, and*

$$\left\langle (f_{gh}\Omega(g,h))_{(g,h)\in S\times S}, M_\alpha^{'} \right\rangle = \langle f, e_{\mathcal{A}}\rangle ,$$

for all $f \in \ c_0(S)$ *and* α.

Proof. In view of Theorem 2.6, The result follows as Theorem 2.3, but by using Proposition 2.7 in place of Theorem 2.1. For the sake of convenience, we include the proof of (i) $\Rightarrow$ (ii). Suppose that $\ell^1(S,\omega)$ is pseudo-Connes amenable. Then it is w^*-approximately Connes-amenable, by Theorem 2.6. Take the net $(M_\alpha)_\alpha \subseteq \ell^\infty(S\times S)^*$ as in Proposition 2.7 (ii). Let $k \in S$, and let $f : S\times S \longrightarrow \mathbb{C}$ be a bounded function such that the map $T \in \mathcal{L}(\mathcal{A},\mathcal{A}^*) = \ell^\infty(S\times S)$ defined by $\langle \delta_h, T(\delta_g)\rangle = f(g,h)$, $(g, h \in S)$, satisfies the conclusions of Proposition 2.8. Then by [3, Corollary 3.5], $T \in \sigma wc(\mathcal{L}(\mathcal{A},\mathcal{A}^*))$. Therefore

$$\begin{aligned}
&\left\langle (f(hk,g)\Omega(h,k) - f(h,kg)\Omega(k,g))_{(g,h)\in S\times S}, M_\alpha \right\rangle \\
&\quad = \langle \delta_g, T(\delta_{hk})\rangle \Omega(h,k) - \langle \delta_{kg}, T(\delta_h)\rangle \Omega(k,g) \\
&\qquad \left\langle (\langle \delta_g\otimes\delta_h, \delta_k \ . \ T - T \ . \ \delta_k\rangle)_{(g,h)\in S\times S}, M_\alpha \right\rangle \\
&\quad = \langle \delta_k \ . \ T - T \ . \ \delta_k, M\rangle \\
&\quad = \langle T, \delta_k \ . \ M_\alpha - M_\alpha \ . \ \delta_k\rangle \longrightarrow 0,
\end{aligned}$$

as required. Next, for each family $(f_g)_{g\in S} \in c_0(S)$, we have

$$\left\langle \delta_g\otimes\delta_h, \pi^*\imath_{c_0(S)}(f)\right\rangle = \left\langle \delta_{gh}\Omega(g,h), \imath_{c_0(S)}(f)\right\rangle ,$$

so that

$$\pi^*\imath_{c_0(S)}(f) = (\left\langle \delta_{gh}, \imath_{c_0(S)}(f)\right\rangle \Omega(g,h))_{(g,h)\in S\times S} \in \ell^\infty(S\times S).$$

Hence

$$\begin{aligned}
\left\langle (f_{gh}\Omega(g,h))_{(g,h)\in S\times S}, M_\alpha \right\rangle &= \left\langle \pi^*\imath_{c_0(S)}(f), M_\alpha \right\rangle \\
&= \left\langle f, \imath_{c_0(S)}^*\pi^{**}M_\alpha \right\rangle \longrightarrow \langle f, e_{\mathcal{A}}\rangle .
\end{aligned}$$

$\square$

Let G be a discrete group and let $h \in G$. We define $J_h : \ell^\infty(G) \longrightarrow \ell^\infty(G)$ by

$$J_h(f) := (f_{hg}\Omega(h,g)\omega(h)\Omega(g^{-1},h^{-1})\omega(h^{-1}))_{g\in G} \quad (f = (f_g)_g \in \ell^\infty(G)).$$

It is clear that $||J_h(f)|| \leq ||f||\omega(h)\omega(h^{-1})$, so J_h is bounded.

In [3], Daws proved that Connes-amenability and amenability are the same notions for a Beurling algebra $\ell^1(G,\omega)$, where G is a discrete group. We extend it as follows.

Theorem 2.10. *Let G be a discrete group, and let ω be a weight on G. Then the following are equivalent:*

(i) *$\ell^1(G,\omega)$ is pseudo-Connes amenable.*

(ii) *$\ell^1(G,\omega)$ is pseudo-amenable.*

(iii) *There exists a net $(N_\alpha)_\alpha \subseteq \ell^\infty(G)^*$ such that for every $k \in G$, $J_k^*(N_\alpha) - N_\alpha \xrightarrow{w^*} 0$ in $\ell^\infty(G)^*$, and $\langle (\Omega(g,g^{-1}))_{g\in G}, N_\alpha \rangle \longrightarrow 1$.*

(iv) *There exists a net $(N'_\alpha)_\alpha \subseteq \ell^\infty(G)^*$ such that for every $k \in G$, $J_k^*(N'_\alpha) - N'_\alpha \xrightarrow{w^*} 0$ in $\ell^\infty(G)^*$, and $\left\langle (\Omega(g,g^{-1}))_{g\in G}, N'_\alpha \right\rangle = 1$ for all α.*

Proof. The implications (ii) $\Rightarrow$ (i) and (iv) $\Rightarrow$ (iii) are clear.

(i) $\Rightarrow$ (iv): Let the net $(M'_\alpha)_\alpha \subseteq \ell^\infty(G\times G)^*$ be given as in Theorem 2.9 (iii). Define $\phi : \ell^\infty(G) \longrightarrow \ell^\infty(G\times G)$ by

$$\langle \delta_{(g,h)}, \phi(f) \rangle := \begin{cases} f_g, & g = h^{-1} \\ 0, & g \neq h^{-1} \end{cases}.$$

Let $N'_\alpha := \phi^*(M'_\alpha)$. Then we have

$$\phi((\Omega(g,g^{-1}))_{g\in G}) = (\delta_{gh,e_G}\Omega(g,h))_{(g,h)\in G\times G}.$$

Hence

$$\begin{aligned}\left\langle (\Omega(g,g^{-1}))_{g\in G}, N'_\alpha \right\rangle &= \left\langle (\delta_{gh,e_G}\Omega(g,h))_{(g,h)\in G\times G}, M'_\alpha \right\rangle \\ &= \langle (\delta_{gh,e_G})_{(g,h)\in G\times G}, \delta_{e_G} \rangle = \delta_{e_G,e_G} = 1,\end{aligned}$$

by the second condition on $(M'_\alpha)_\alpha$ from Theorem 2.9 (iii).

Fix $k \in G$ and $f \in \ell^\infty(G)$. Define $F : G \times G \longrightarrow \mathbb{C}$ by

$$F(g,h) := \delta_{gh,k}\ f_g\ \omega(k)\ \omega(hk^{-1})\ \omega(h)^{-1} \quad (g,h \in G).$$

It is clear that F is bounded and $||F||_\infty \leq ||f||_\infty \omega(k)\omega(k^{-1})$. Let T be the operator associated with F. The same argument as in the proof of [3, Theorem 5.11] shows that F satisfies the conditions of Proposition 2.8. Notice that

$$\langle \delta_{(g,h)}, \phi(J_k(f)) \rangle = \delta_{gh,e}\ f_{kg}\ \omega(kg)\ \omega(g)^{-1}\ \omega(g^{-1}k^{-1})\ \omega((g)^{-1})^{-1}.$$

Thus, for $f \in \ell^\infty(G)$ we have

$$\begin{aligned}\left\langle f, J_k^*(N_\alpha^{'}) - N_\alpha^{'}\right\rangle &= \left\langle \phi(f) - \phi(J_k(f)), M_\alpha^{'}\right\rangle \\ &= \left\langle (F(hk,g)\Omega(h,k) - F(h,kg)\Omega(k,g))_{(g,h)}, M_\alpha^{'}\right\rangle \longrightarrow 0,\end{aligned}$$

as required.

(iii) $\Rightarrow$ (ii): Let $(N_\alpha)_\alpha \subseteq \ell^\infty(G)^*$ be given as in (iii). Define $\psi : \ell^\infty(G \times G) \longrightarrow \ell^\infty(G)$ by $\langle \delta_g, \psi(F)\rangle := F(g, g^{-1})$, for each $F \in \ell^\infty(G \times G)$ and $g \in G$. Put $M_\alpha := \psi^*(N_\alpha)$ for every α. Then it is enough to show that the net $(M_\alpha)_\alpha$ has desired properties in Theorem 2.3 (ii). First, for every $f \in$ ball$\ell^\infty(G)$, we see that

$$\left\langle (f_{gh}\Omega(g,h))_{(g,h)}, M_\alpha\right\rangle = \left\langle (f_{e_G}\Omega(g,g^{-1}))_g, N_\alpha\right\rangle \longrightarrow f_{e_G}.$$

Next, for an arbitrary bounded function $f : G \times G \longrightarrow \mathbb{C}$ and an element $k \in G$, it is clear that

$$\begin{aligned}\psi((f(hk,g)\Omega(h,k) - f(h,kg)\Omega(k,g))_{(g,h)}) = (&f(g^{-1}k,g)\Omega(g^{-1},k) \\ &- f(g^{-1},kg)\Omega(k,g))_g.\end{aligned}$$

Define $F : G \times G \longrightarrow \mathbb{C}$, by $F(g,h) := f(hk,g)\Omega(h,k)$, for each $g, h \in G$. Hence, it is readily seen that F is bounded and $||F||_\infty \leq ||f||_\infty$. Therefore

$$\begin{aligned}&\left\langle (f(hk,g)\Omega(h,k) - f(h,kg)\Omega(k,g))_{(g,h)}, M_\alpha\right\rangle \\ &\quad = \left\langle (f(g^{-1}k,g)\Omega(g^{-1},k) - f(g^{-1},kg)\Omega(k,g))_g, N_\alpha\right\rangle \\ &\quad = \langle \psi(F) - J_k(\psi(F)), N_\alpha\rangle \\ &\quad = \langle \psi(F), N_\alpha - J_k^*(N_\alpha)\rangle \longrightarrow 0.\end{aligned}$$

□

3. σ-Connes Amenability

Let $\mathcal{A}$ be a dual Banach algebra, and let $\sigma \in \mathrm{Hom}(\mathcal{A})$ be a w^*-continuous homomorphism. From [13], we recall that $\mathcal{A}$ is σ-Connes amenable if for every normal dual Banach $\mathcal{A}$-bimodule E, every w^*-continuous σ-derivation $D : \mathcal{A} \longrightarrow E$ is σ-inner.

Let $\mathcal{A}$ be a Banach algebra. Consider the map $\tau : \mathcal{A}^\sharp \to \mathcal{A}^\sharp$ defined by $\tau(a + \lambda e_{\mathcal{A}^\sharp}) = \lambda$, $(a \in \mathcal{A}, \lambda \in \mathbb{C})$. Then $\tau \in Hom(\mathcal{A}^\sharp)$. If $\mathcal{A}$ is dual, then it is routinely checked that τ is w^*-continuous.

Theorem 3.1. *Let $\mathcal{A}$ be a Banach algebra with a right [left] approximate identity, and let E be a Banach $\mathcal{A}^\sharp$-bimodule. Then, every τ-derivation $D : \mathcal{A}^\sharp \longrightarrow E$ is zero.*

Proof. Let $(e_i)_i$ be a right approximate identity for $\mathcal{A}$. A simple calculation shows that $D(ae_i) = 0$ for each $a \in \mathcal{A}$ and i, and consequently $D(a) = 0$. Hence, $D(a + \lambda e_{\mathcal{A}^\sharp}) = D(a) + \lambda D(e_{\mathcal{A}^\sharp}) = 0$, i.e., $D = 0$. □

Corollary 3.2. *Let $\mathcal{A}$ be a Banach algebra with a right [left] approximate identity. Then $\mathcal{A}^\sharp$ is τ-amenable. Moreover, if $\mathcal{A}$ is dual then $\mathcal{A}^\sharp$ is τ-Connes amenable.*

Now, we give a class of σ-Connes amenable Banach algebras which are not Connes amenable.

Example 3.3. Let $\mathcal{A}$ be a non-Connes amenable Banach algebra with a right [left] approximate identity. It is known that $\mathcal{A}^\sharp$ is not Connes amenable, however it is τ-Connes amenable by Corollary 3.2.

Lemma 3.4. *Let $\mathcal{A}$ be a unital Banach algebra, E be a Banach $\mathcal{A}$-bimodule, $\sigma \in Hom(\mathcal{A})$, and let $D : \mathcal{A} \longrightarrow E^*$ be a σ-derivation. Then there exist a σ-derivation $D_1 : \mathcal{A} \longrightarrow e_\mathcal{A} \cdot E^* \cdot e_\mathcal{A}$, and $\eta \in E^*$ such that $D = D_1 + ad_\eta$, where ad_η stands for the inner derivation $a \longmapsto a\cdot\eta - \eta\cdot a$.*

Proof. The argument of [5, Lemma 2.3] suffices. We just notice that D_j's and ad_η are σ-derivations. □

Let $\mathcal{A}$ be a Banach algebra, and let $\sigma \in Hom(\mathcal{A})$. We define $\sigma^\sharp \in Hom(\mathcal{A}^\sharp)$ by $\sigma^\sharp(a+\lambda e_{\mathcal{A}^\sharp}) = \sigma(a)+\lambda e_{\mathcal{A}^\sharp}$. Note that if σ is w^*-continuous, then so is $\sigma^\sharp$.

Theorem 3.5. *Let $\mathcal{A}$ be a Banach algebra, and let $\sigma \in Hom(\mathcal{A})$. Then $\mathcal{A}$ is σ-amenable if and only if $\mathcal{A}^\sharp$ is $\sigma^\sharp$-amenable.*

Proof. $\Rightarrow$ Let $D : \mathcal{A}^\sharp \longrightarrow E^*$ be a $\sigma^\sharp$-derivation. Choose D_1 and η as in Lemma 3.4, so that $D = D_1 + ad_\eta$. We have $D_1(e_{\mathcal{A}^\sharp}) = \sigma^\sharp(e_{\mathcal{A}^\sharp})$. $D_1(e_{\mathcal{A}^\sharp}) + D_1(e_{\mathcal{A}^\sharp})\cdot\sigma^\sharp(e_{\mathcal{A}^\sharp}) = e_{\mathcal{A}^\sharp}\cdot D_1(e_{\mathcal{A}^\sharp}) + D_1(e_{\mathcal{A}^\sharp})\cdot e_{\mathcal{A}^\sharp} = 2D_1(e_{\mathcal{A}^\sharp})$, therefore $D_1(e_{\mathcal{A}^\sharp}) = 0$. Since $D_1 \mid_\mathcal{A}$ is σ-derivation, there exists $\xi \in E^*$ such that $D_1 = ad_\xi$. Hence, $D = ad_{\eta+\xi}$, as required.

$\Leftarrow$ Let $D : \mathcal{A} \longrightarrow E^*$ be a σ-derivation. Turning E into an $\mathcal{A}^\sharp$-bimodule with identity module actions $e_{\mathcal{A}^\sharp}\cdot x = x\cdot e_{\mathcal{A}^\sharp} = x$ $(x \in E)$, we extend D to $\tilde{D} : \mathcal{A}^\sharp \longrightarrow E^*$ by setting $\tilde{D}(a+\lambda e) = D(a)$. It is easily seen that $\tilde{D}$ is $\sigma^\sharp$-derivation. Now by the assumption, $\tilde{D}$ is $\sigma^\sharp$-inner and so D is σ-inner. □

Theorem 3.6. *Let $\mathcal{A}$ be a dual Banach algebra, and let $\sigma \in Hom(\mathcal{A})$ be w^*-continuous. Then $\mathcal{A}$ is σ-Connes amenable if and only if $\mathcal{A}^\sharp$ is $\sigma^\sharp$-Connes amenable.*

Proof. It is similar to the proof of Theorem 3.5. We just notice that if $D : \mathcal{A}^\sharp \longrightarrow E^*$ is w^*-continuous, then D_1 and ad_η are also w^*-continuous. Next, if E is a normal dual Banach $\mathcal{A}$-bimodule, so is $e_\mathcal{A}\cdot E\cdot e_\mathcal{A}$. The extension $\tilde{D}$ defined in the proof of Theorem 3.5 is also w^*-continuous. □

Proposition 3.7. *Let $\mathcal{A}$, $\mathcal{B}$ be dual Banach algebras, let $\sigma \in Hom(\mathcal{A})$, $\tau \in Hom(\mathcal{B})$ be w^*-continuous, and let $\theta : \mathcal{A} \longrightarrow \mathcal{B}$ be a w^*-continuous homomorphism with w^*-dense range such that $\tau\theta = \theta\sigma$. If $\mathcal{A}$ is σ-Connes amenable, then $\mathcal{B}$ is τ-Connes amenable.*

Proof. Let E be a normal dual Banach $\mathcal{B}$-bimodule, and let $d : \mathcal{B} \longrightarrow E$ be a w^*-continuous τ-derivation. Turning E into an $\mathcal{A}$-bimodule through $a \cdot x = \theta(a) \cdot x$, and $x \cdot a = x \cdot \theta(a)$, $(a \in \mathcal{A}, x \in E)$, we observe that E is a normal dual Banach $\mathcal{A}$-bimodule. Since $\tau\theta = \theta\sigma$, $D := d\theta : \mathcal{A} \longrightarrow E$ is a w^*-continuous σ-derivation. Hence, there exists $x \in E$ such that $D = ad_x$. It follows that $d(\theta(a)) = \tau(\theta(a)) \cdot x - x \cdot \tau(\theta(a))$, $(a \in \mathcal{A})$. Now, w^*-density of $\theta(\mathcal{A})$ in $\mathcal{B}$ implies that $d(b) = \tau(b) \cdot x - x \cdot \tau(b)$ for each $b \in \mathcal{B}$. □

Proposition 3.8. *Let $\mathcal{A}$ be a Banach algebra, let $\mathcal{B}$ be a dual Banach algebra, and let $\theta : \mathcal{A} \longrightarrow \mathcal{B}$ be a homomorphism with w^*-dense range. Let $\sigma \in Hom(\mathcal{A})$, $\tau \in Hom(\mathcal{B})$ such that τ is w^*-continuous, and $\tau\theta = \theta\sigma$. If $\mathcal{A}$ is σ-amenable, then $\mathcal{B}$ is τ-Connes amenable.*

Proof. This is a more or less verbatim of the proof of Proposition 3.7. □

Proposition 3.9. *Let $\mathcal{A}$ be a dual Banach algebra, and let $\sigma \in Hom(\mathcal{A})$ be w^*-continuous. If $\mathcal{A}$ is σ-Connes amenable, then it is $\tau\sigma$-Connes amenable, for any w^*-continuous homomorphism $\tau \in Hom(\mathcal{A})$.*

Proof. Let E be a normal dual Banach $\mathcal{A}$-bimodule, and let $D : \mathcal{A} \longrightarrow E$ be a w^*-continuous $\tau\sigma$-derivation. It is easy to verify that E equipped with $\mathcal{A}$-module actions

$$a \diamond x = \tau(a) \cdot x \quad \text{and} \quad x \diamond a = x \cdot \tau(a) \qquad (a \in \mathcal{A}, x \in E)$$

is still a normal dual Banach $\mathcal{A}$-bimodule, and then D obviously is a σ-derivation. Therefore, there exists $x \in E$ such that for every $a \in \mathcal{A}$, $Da = \sigma(a) \diamond x - x \diamond \sigma(a) = \tau\sigma(a) \cdot x - x \cdot \tau\sigma(a)$, as required. □

Corollary 3.10. *Let $\mathcal{A}$ be a dual Banach algebra. If $\mathcal{A}$ is Connes amenable, then it is σ-Connes amenable for every w^*-continuous homomorphism $\sigma \in Hom(\mathcal{A})$.*

The following is a converse for Corollary 3.10.

Proposition 3.11. *Let $\mathcal{A}$ be a dual Banach algebra, and let $\sigma \in Hom(\mathcal{A})$ be w^*-continuous with w^*-dense range. If $\mathcal{A}$ is σ-Connes amenable, then it is Connes amenable.*

Proof. Let E be a normal dual Banach $\mathcal{A}$-bimodule, and let $D : \mathcal{A} \longrightarrow E$ be a w^*-continuous derivation. Clearly, $d := D\sigma : \mathcal{A} \longrightarrow E$ is a w^*-continuous σ-derivation. Thus, there exists $x \in E$ such that $da = \sigma(a) \cdot x - x \cdot \sigma(a)$ for $a \in \mathcal{A}$. Then, normality of E and w^*-continuity of d yield that $Da = a \cdot x - x \cdot a$, $a \in \mathcal{A}$. □

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[1] Department of Mathematics, Central Tehran Branch, Islamic Azad University, Tehran, Iran.

E-mail address: zh_hasanzadeh@yahoo.com

[2] Department of Mathematics, Central Tehran Branch, Islamic Azad University, Tehran, Iran.

E-mail address: a_mahmoodi@iauctb.ac.ir