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Bounded Approximate Character Amenability of Banach Algebras

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Abstract. The bounded approximate version of φ-amenability and character amenability are introduced and studied. These new notions are characterized in several different ways, and some hereditary properties of them are established. The general theory for these concepts is also developed. Moreover, some examples are given to show that these notions are different from the others. Finally, bounded approximate character amenability of some Banach algebras related to locally compact groups are investigated.

1. Introduction

Let A be a Banach algebra and let $\varphi \in \sigma(A)$, the space of non-zero characters of A. The concepts of φ-amenable Banach algebras and character amenable Banach algebras were introduced by Kaniuth, Lau and Pym in [11] and by Sangani Monfared in [12]. Character contractibility is also considered in [9]. The first named author, Shi and Wu introduced the concept of approximate character amenability and characterized this notion in several ways [1]. We recall these definitions below.

For a Banach A-bimodule X, a derivation is a bounded linear map $D : A \to X$ such that

$$D(ab) = a \cdot D(b) + D(a) \cdot b, \quad (a, b \in A).$$

Each $x \in X$ defines a derivation given by $D(a) = a \cdot x - x \cdot a$ for $a \in A$, which is called an inner derivation and is denoted by ad_x. Let $\mathcal{M}_\varphi^A$ $({}_\varphi\mathcal{M}^A)$ denote the class of Banach A-bimodules such that $x \cdot a = \varphi(a)x$

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($a \cdot x = \varphi(a)x$) for all $a \in A$ and $x \in X$. Obviously, $X \in {}_{\varphi}\mathcal{M}^A$ if and only if $X^* \in \mathcal{M}_{\varphi}^A$, where X^* denotes the dual space of X.

Definition 1.1. A Banach algebra A is called φ-amenable, for some $\varphi \in \sigma(A) \cup \{0\}$, if for each $X \in {}_{\varphi}\mathcal{M}^A$, every derivation $D : A \to X^*$ is inner. A is called character amenable if it is φ-amenable for each $\varphi \in \sigma(A) \cup \{0\}$.

In [4], Ghahramani and Loy generalized the theory of classical amenable Banach algebras, introduced by Johnson in 1972 [10], to approximate amenability. According to [4], a derivation $D : A \to X$ is (sequentially) approximately inner if there is a (sequence) net $(x_\alpha) \subset X$ such that $D(a) = \lim_\alpha \mathrm{ad}_{x_\alpha}(a)$ for all $a \in A$. A Banach algebra A is called approximately amenable if every derivation $D : A \to X^*$ is approximately inner for each Banach A-bimodule X, and is called boundedly approximately amenable if for every Banach A-bimodule X, and every derivation $D : A \to X^*$, there is a net $(\xi_i) \subset X^*$ such that the net (ad_{ξ_i}) is norm bounded in $\mathcal{B}(A, X^*)$ and such that $D(a) = \lim_i \mathrm{ad}_{\xi_i}(a)$, $a \in A$, where $\mathcal{B}(A, X^*)$ is the space of bounded operators from A into X^*.

Definition 1.2 ([1])**.** Let A be a Banach algebra.

(i) A is called (sequentially) approximately φ-amenable, for some $\varphi \in \sigma(A) \cup \{0\}$, if for each $X \in {}_{\varphi}\mathcal{M}^A$, every derivation $D : A \to X^*$ is (sequentially) approximately inner.

(ii) A is called (sequentially) approximately character amenable if it is approximately φ-amenable for each $\varphi \in \sigma(A) \cup \{0\}$.

It is shown through lines of the proof of [4, Lemma 2.2] that a Banach algebra is approximately 0-amenable if and only if it has a right approximate identity. Every Banach algebra that has an unbounded right approximate identity and does not have a bounded right approximate identity is approximately 0-amenable but not necessarily 0-amenable. For example, ℓ^1, $A(\mathbb{F}_2)$ and every non-trivial Segal algebra on a locally compact infinite group enjoy this property, where ℓ^1 is the Banach algebra of absolutely summable sequences with pointwise product and $A(\mathbb{F}_2)$ is the Fourier algebra of the free group $\mathbb{F}_2$ on two generators.

Motivated by the earlier generalizations of amenability, we define the bounded version of approximate φ-amenability and character amenability. We characterize each of these notions in several different ways and develop the general theory of them. We give some examples to show that the class of boundedly approximately character amenable Banach algebras is larger than the class of character amenable Banach algebras.

The unitization of A will be denoted by $A^{\#}$, which is $A \oplus \mathbb{C}$ with the product given by

$$(a, \alpha) \cdot (b, \beta) = (ab + \beta a + \alpha b, \alpha\beta), \quad (a, b \in A,\ \alpha, \beta \in \mathbb{C}).$$

The projective tensor product of two Banach algebras A and B is denoted by $A\hat{\otimes}B$, which is a Banach algebra with the following usual multiplication:

$$(a_1 \otimes b_1) \cdot (a_2 \otimes b_2) = (a_1 a_2) \otimes (b_1 b_2), \quad (a_1, a_2 \in A,\, b_1, b_2 \in B).$$

The product map of A extends to a map $\Delta : A\hat{\otimes}A \to A$ determined by $\Delta(a \otimes b) = ab$ for all $a, b \in A$. For a Banach algebra A, its second dual, A^{**}, is a Banach algebra under either the first or second Arens products, of which we will always take the first one denoted by $\Box$.

2. Bounded Approximate Character Amenability

In this section, we introduce the bounded version of approximate φ-amenability and character amenability. Then we characterize these notions in several different ways.

Definition 2.1. Let A be a Banach algebra and $\varphi \in \sigma(A) \cup \{0\}$. Then A is termed boundedly approximately φ-amenable if for every $X \in {}_{\varphi}\mathcal{M}^A$ and every derivation $D : A \to X^*$, there exist a net (f_α) in X^* and $c > 0$ such that $D(a) = \lim_\alpha \mathrm{ad}_{f_\alpha}(a) = \lim_\alpha a \cdot f_\alpha - \varphi(a) f_\alpha$ and $\|\mathrm{ad}_{f_\alpha}(a)\| \leq c\|a\|$ for all $a \in A$. In this case, we say that D is boundedly approximately φ-inner.

Let A be a Banach algebra. We say that a net (e_α) is a right approximate identity for A, if $\|ae_\alpha - a\| \to 0$ for all $a \in A$. Left or two-sided approximate identity for A is defined similarly. We call (e_α) a bounded right (left or two-sided, respectively) approximate identity for A, if it is also bounded. We call (e_α) a multiplier-bounded right approximate identity for A if there exists a constant $k > 0$ such that $\|ae_\alpha\| \leq k\|a\|$ for all $a \in A$ and all α.

Corollary 3.4 of [6] shows that a boundedly approximately amenable Banach algebra does not have a (right) bounded approximate identity in general. It then follows that a boundedly approximately character amenable Banach algebra does not have a right bounded approximate identity in general (e.g. see Example 4.2). However, it can be readily seen that if A has a multiplier-bounded right approximate identity, then A is boundedly approximately 0-amenable. For if (e_α) is a multiplier-bounded right approximate identity for A with multiplier bound k and $D : A \to X^*$ is a derivation for some $X \in {}_0\mathcal{M}^A$, then

$$D(a) = \lim_\alpha D(ae_\alpha) = \lim_\alpha aD(e_\alpha) = \lim_\alpha \mathrm{ad}_{D(e_\alpha)}(a),$$

and

$$\|\mathrm{ad}_{D(e_\alpha)}(a)\| = \|aD(e_\alpha)\| = \|D(ae_\alpha)\| \leq \|D\|\|ae_\alpha\| \leq k\|D\|\|a\|.$$

The following proposition is the analogue of [5, Proposition 5.3].

Proposition 2.2. *Let A be a Banach algebra and let $\varphi \in \sigma(A) \cup \{0\}$. Then A is boundedly approximately φ-amenable if and only if there is a constant $L_b > 0$ such that for any $X \in {}_\varphi\mathcal{M}^A$ and any derivation $D : A \to X^*$, there is a net $(\eta_\alpha) \subset X^*$ such that*

(i) $\sup_\alpha \|\mathrm{ad}_{\eta_\alpha}\| \leq L_b \|D\|$;
(ii) $D(a) = \lim_\alpha \mathrm{ad}_{\eta_\alpha}(a)$ *for each* $a \in A$.

Proof. Suppose that A is boundedly approximately φ-amenable. If there is no such L_b, then for every $n \in \mathbb{N}$ there exist $X_n \in {}_\varphi\mathcal{M}^A$ and a derivation $D_n : A \to X_n^*$ with $\|D_n\| = 1$ such that $D_n(a) = \lim_\alpha \mathrm{ad}_{\eta_\alpha}(a)$ and $\sup_\alpha \|\mathrm{ad}_{\eta_\alpha}\| > n$. Consider $X = \ell^1 - \oplus_{n\in\mathbb{N}} X_n \in {}_\varphi\mathcal{M}^A$ with dual $X^* = \ell^\infty - \oplus_{n\in\mathbb{N}} X_n^*$, and the derivation $D : A \to X^*$ by $D(a) = (D_n(a))_{n\in\mathbb{N}}$. Since A is boundedly approximately φ-amenable, there are a net $(\eta_\alpha) \subseteq X^*$ and a constant $c > 0$ such that $D(a) = \lim_\alpha \mathrm{ad}_{\eta_\alpha}(a)$ and $\sup_\alpha \|\mathrm{ad}_{\eta_\alpha}(a)\| \leq c\|a\|$.
Let $P_n : X^* \to X_n^*$ be the canonical projection. Then $D_n = P_n \circ D$ and so $D_n(a) = \lim_\alpha \mathrm{ad}_{P_n\eta_\alpha}(a)$ and $\sup_\alpha \|\mathrm{ad}_{P_n\eta_\alpha}\| \leq \sup_\alpha \|\mathrm{ad}_{\eta_\alpha}\| \leq c$, which is a contradiction.

The converse is trivial. □

The uniform boundedness principle shows that every sequentially approximately φ-amenable Banach algebra is boundedly approximately φ-amenable. In the following proposition we show that the converse of this result holds in separable Banach algebras.

Proposition 2.3. *Suppose that A is a boundedly approximately φ-amenable Banach algebra and $\varphi \in \sigma(A)$. If A is separable as a Banach space, then it is boundedly sequentially approximately φ-amenable.*

Proof. Let $X \in {}_\varphi\mathcal{M}^A$ and let $D : A \to X^*$ be a derivation. Let $\{a_n : n \in \mathbb{N}\}$ be a countable dense subset of A. Since A is boundedly approximately φ-amenable, there is a constant $c > 0$ such that for each $n \in \mathbb{N}$ there is $\psi_n \in X^*$ such that $\|D(a_k) - \mathrm{ad}_{\psi_n}(a_k)\| \leq \frac{1}{n}$, $(k \leq n)$, and

$$\|a \cdot \psi_n - \varphi(a)\psi_n\| \leq c\|a\|, \quad (a \in A).$$

This shows that the sequence $(\psi_n) \subset X^*$ satisfies

$$D(a_k) = \lim_{n\to\infty} (a_k \cdot \psi_n - \varphi(a_k)\psi_n), \quad (k \in \mathbb{N}),$$

and the sequence (ad_{ψ_n}) is bounded in $\mathcal{B}(A, X^*)$. So by density of (a_k) in A,

$$D(a) = \lim_{n\to\infty} (a \cdot \psi_n - \varphi(a)\psi_n), \quad (a \in A).$$

Therefore, D is boundedly sequentially approximately φ-inner. □

Example 2.4. ([5, Example 5.7]) Let S be an uncountable set. Then $c_0(S)$ is amenable, hence boundedly approximately character amenable, but it cannot be sequentially approximately character amenable, for otherwise it would have a sequential approximate identity, which is impossible. So, without separability hypothesis Proposition 2.3 is not true in general.

The next theorem characterizes bounded approximate φ-amenability in terms of approximate φ-mean. The corresponding result characterizing approximate φ-amenability of a Banach algebra is obtained in [1, Proposition 2.2].

Theorem 2.5. *For a Banach algebra A and $\varphi \in \sigma(A)$, the following are equivalent:*

(i) *A is boundedly approximately φ-amenable;*
(ii) *Give $(\ker\varphi)^{**}$ a dual A-bimodule structure by taking the right action to be $m \cdot a = \varphi(a)m$ for $a \in A$, $m \in (\ker\varphi)^{**}$ and taking the left action to be the natural one. Then any derivation $D : A \to (\ker\varphi)^{**}$ is boundedly approximately φ-inner;*
(iii) *There exist a net $(m_\alpha) \subset A^{**}$ and $c > 0$ such that $m_\alpha(\varphi) = 1$, $a \cdot m_\alpha - \varphi(a)m_\alpha \to 0$ and $\|a \cdot m_\alpha - \varphi(a)m_\alpha\| \leq c\|a\|$ for all $a \in A$;*
(iv) *There exist a net $(m_\alpha) \subset A^{**}$ and $c > 0$ such that $m_\alpha(\varphi) \to 1$, $a \cdot m_\alpha - \varphi(a)m_\alpha \to 0$ and $\|a \cdot m_\alpha - \varphi(a)m_\alpha\| \leq c\|a\|$ for all $a \in A$.*

Proof. (i) $\Rightarrow$ (ii) is clear.

(ii) $\Rightarrow$ (iii): Choose $b \in A$ with $\varphi(b) = 1$. Then $Da = ab - \varphi(a)b$, $a \in A$, defines a derivation from A into $(\ker\varphi)^{**}$. By (ii), D is approximately inner. It follows that there exist a net $\{n_\alpha\} \subset (\ker\varphi)^{**}$ and $k > 0$ such that $D(a) = \lim_\alpha a \cdot n_\alpha - \varphi(a)n_\alpha$ and $\|a \cdot n_\alpha - \varphi(a)n_\alpha\| \leq k\|a\|$ for all $a \in A$. Set $m_\alpha = b - n_\alpha$. Then $m_\alpha(\varphi) = 1$ and for all $a \in A$, $a \cdot m_\alpha - \varphi(a)m_\alpha \to 0$ and

$$\begin{aligned}\|a \cdot m_\alpha - \varphi(a)m_\alpha\| &= \|ab - a \cdot n_\alpha - \varphi(a)b + \varphi(a)n_\alpha\| \\ &\leq \|ab - \varphi(a)b\| + \|a \cdot n_\alpha - \varphi(a)n_\alpha\| \\ &= \|D(a)\| + \|a \cdot n_\alpha - \varphi(a)n_\alpha\| \\ &\leq (\|D\| + k)\|a\|.\end{aligned}$$

(iii) $\Rightarrow$ (iv) is obvious.

(iv) $\Rightarrow$ (i): Let $(m_\alpha) \subset A^{**}$ satisfy (iv). Without loss of generality one can assume that $|m_\alpha(\varphi) - 1| < 1$ for each α. Now let $X \in {}_\varphi\mathcal{M}^A$ and $D : A \to X^*$ be a derivation. Define $D' := D^*|_X : X \to A^*$ and

$g_\alpha := (D')^*(m_\alpha) \in X^*$. Then for all $a, b \in A$ and $x \in X$,

$$\begin{aligned} D'(x \cdot a)(b) &= D(b)(x \cdot a) \\ &= D(ab)(x) - D(a)(b \cdot x) \\ &= (D'(x) \cdot a)(b) - D(a)(x)\varphi(b), \end{aligned}$$

and hence $D'(x \cdot a) = D'(x) \cdot a - D(a)(x)\varphi$. This implies that

$$\begin{aligned} (a \cdot g_\alpha)(x) &= g_\alpha(x \cdot a) \\ &= (D')^*(m_\alpha)(x \cdot a) \\ &= m_\alpha(D'(x \cdot a)) \\ &= m_\alpha(D'(x) \cdot a) - D(a)(x)m_\alpha(\varphi) \\ &= (a \cdot m_\alpha)(D'(x)) - D(a)(x)m_\alpha(\varphi). \end{aligned}$$

It follows that

$$\begin{aligned} &||(a \cdot g_\alpha)(x) - \varphi(a)g_\alpha(x) + D(a)(x)|| \\ &\quad \leq ||\varphi(a)m_\alpha(D'(x)) - (a \cdot m_\alpha)(D'(x))|| + ||D(a)(x) - D(a)(x)m_\alpha(\varphi)|| \\ &\quad \leq c\|a\|\|D'\|\|x\| + \|D\|\|a\|\|x\|. \end{aligned}$$

Hence, $D(a) = \lim_\alpha \mathrm{ad}_{-g_\alpha}(a)$ for each $a \in A$, and $\|\mathrm{ad}_{-g_\alpha}\| \leq c\|D'\| + 2\|D\|$ for each α. Therefore, A is boundedly approximately φ-amenable. □

If A is a Banach algebra, the corresponding diagonal operator $\Delta : A\widehat{\otimes}A \to A$ is an A-bimodule homomorphism by the following module actions:

$$a \cdot (b \otimes c) = ab \otimes c, \ (b \otimes c) \cdot a = b \otimes ca \qquad (a, b, c \in A).$$

Theorem 2.6. *Let A be a Banach algebra and let $\varphi \in \sigma(A)$. Then the following assertions are equivalent:*

(i) *A is boundedly approximately φ-amenable.*

(ii) *There exist a net $(M_\alpha) \subset (A\widehat{\otimes}A)^{**}$ and $c > 0$ such that $(\Delta^{**}M_\alpha)(\varphi) = 1$, $a \cdot M_\alpha - \varphi(a)M_\alpha \to 0$ and $\|a \cdot M_\alpha - \varphi(a)M_\alpha\| \leq c\|a\|$ for all $a \in A$.*

(iii) *There exist a net $(N_\alpha) \subset (A\widehat{\otimes}A)^{**}$ and $c > 0$ such that $(\Delta^{**}N_\alpha)(\varphi) \to 1$, $a \cdot N_\alpha - \varphi(a)N_\alpha \to 0$ and $\|a \cdot N_\alpha - \varphi(a)N_\alpha\| \leq c\|a\|$ for all $a \in A$.*

Proof. (i) ⇒ (ii): Let A be boundedly approximately φ-amenable. Consider $A\hat{\otimes}A$ as an element of $\mathcal{M}_\varphi^A$ and choose $m \in (A\hat{\otimes}A)^{**}$ such that $m(\varphi \otimes \varphi) = 1$. Then the image of the derivation $\mathrm{ad}_m : A \to (A\hat{\otimes}A)^{**}$ is included in the closed subspace $\{\mathbb{C}(\varphi \otimes \varphi)\}^\perp \cong ((A\hat{\otimes}A)^*/\{\mathbb{C}(\varphi \otimes \varphi)\})^*$

of $(A\hat{\otimes}A)^{**}$. Since $\varphi\cdot a=\varphi(a)\varphi$ for $a\in A$, we see that $\{\mathbb{C}(\varphi\otimes\varphi)\}^{\perp}$ is a closed submodule of $(A\hat{\otimes}A)^{**}$. So there are a net $(m_\alpha)\subset\{\mathbb{C}(\varphi\otimes\varphi)\}^{\perp}$ and $c>0$ such that $\mathrm{ad}_m(a)=\lim_\alpha \mathrm{ad}_{m_\alpha}(a)$ and $\|\mathrm{ad}_{m_\alpha}(a)\|\le c\|a\|$ for each $a\in A$. Set $M_\alpha=m-m_\alpha$. Then

$$\begin{aligned} a\cdot M_\alpha-\varphi(a)M_\alpha &= (a\cdot m-\varphi(a)m)-(a\cdot m_\alpha-\varphi(a)m_\alpha)\\ &= \mathrm{ad}_m(a)-\mathrm{ad}_{m_\alpha}(a)\to 0,\end{aligned}$$

$$\begin{aligned} \Delta^{**}(M_\alpha)(\varphi) &= \langle\varphi\otimes\varphi, M_\alpha\rangle=\langle\varphi\otimes\varphi, m-m_\alpha\rangle\\ &= 1-\langle\varphi\otimes\varphi, m_\alpha\rangle=1,\end{aligned}$$

and $\|a\cdot M_\alpha-\varphi(a)M_\alpha\|\le(2\|m\|+c)\|a\|$ for all $a\in A$.

(ii) $\Rightarrow$ (iii) is obvious.

(iii) $\Rightarrow$ (i): Let $X\in {}_\varphi\mathcal{M}^A$ and let $D:A\to X^*$ be a derivation. For each $x\in X$ define $\mu_x\in(A\hat{\otimes}A)^*$ by $\mu_x(a\otimes b)=\varphi(b)\langle x, D(a)\rangle$ (note that $\|\mu_x\|\le\|D\|\cdot\|x\|$), and for each α define $f_\alpha\in X^*$ by $f_\alpha(x)=N_\alpha(\mu_x)$. It can be easily checked that

$$\mu_{x\cdot a-\varphi(a)x}(n)=(\mu_x\cdot a-\varphi(a)\mu_x)(n)-\varphi(\Delta(n))\langle x, D(a)\rangle\quad (n\in A\hat{\otimes}A).$$

Now for each α take a net $(n_{i,\alpha})\subset A\hat{\otimes}A$ converging to N_α in the w^*-topology. Then

$$\begin{aligned}(a\cdot f_\alpha-\varphi(a)f_\alpha)(x) &= f_\alpha(x\cdot a-\varphi(a)x)\\ &= N_\alpha(\mu_{x\cdot a-\varphi(a)x})\\ &= \lim_i \mu_{x\cdot a-\varphi(a)x}(n_{i,\alpha})\\ &= \lim_i(\mu_x\cdot a-\varphi(a)\mu_x)(n_{i,\alpha})-\lim_i\varphi(\Delta(n_{i,\alpha}))\langle x, D(a)\rangle\\ &= N_\alpha(\mu_x\cdot a-\varphi(a)\mu_x)-\langle\varphi,\Delta^{**}(N_\alpha)\rangle\langle x, D(a)\rangle\\ &= (a\cdot N_\alpha-\varphi(a)N_\alpha)(\mu_x)-\langle\varphi,\Delta^{**}(N_\alpha)\rangle\langle x, D(a)\rangle.\end{aligned}$$

Therefore,

$$\begin{aligned}\|(a\cdot f_\alpha-\varphi(a)f_\alpha)(x)+\langle x, D(a)\rangle\| &\le \|a\cdot N_\alpha-\varphi(a)N_\alpha\|\|D\|\|x\|\\ &\quad+|1-\langle\varphi,\Delta^{**}(N_\alpha)\rangle|\|D(a)\|\|x\|.\end{aligned}$$

Hence $D(a)=\lim_\alpha \mathrm{ad}_{-f_\alpha}$. Also, by assuming $|1-\langle\varphi,\Delta^{**}(N_\alpha)\rangle|<1$ for all α, we obtain

$$\|(a\cdot f_\alpha-\varphi(a)f_\alpha)(x)\|\le(c+1)\|a\|\|D\|\|x\|.$$

Therefore, A is boundedly approximately φ-amenable. $\square$

Definition 2.7. A net $(N_\alpha)\subset(A\widehat{\otimes}A)^{**}$ satisfying condition (iii) of Theorem 2.6 is called a multiplier-bounded approximate φ-diagonal.

3. Hereditary Properties of Bounded Approximate Character Amenability

In this section, we are concerned with hereditary properties of bounded approximate character amenability and give another characterization of bounded approximate φ-amenability when $\varphi \in \sigma(A)$.

The following lemma helps us to explore the relation between bounded approximate φ-amenability of A and $A^{\#}$.

Lemma 3.1. *Let A be a unital Banach algebra with identity e, $\varphi \in \sigma(A)$, $X \in {}_{\varphi}\mathcal{M}^A$ and $D : A \to X^*$ be a derivation. Then*

(i) *there are $f \in X^* \cdot (1-e)$ and a derivation $D_0 : A \to X^* \cdot e$ such that $D = \mathrm{ad}_f + D_0$;*

(ii) *if moreover A is boundedly approximately φ-amenable, then there are $f \in X^* \cdot (1-e)$ and a net $(f_\alpha) \subset X^* \cdot e$ such that $\| \mathrm{ad}_{f_\alpha} \|$ is bounded and $D(a) = \mathrm{ad}_f(a) + \lim_\alpha \mathrm{ad}_{f_\alpha}(a)$ for all $a \in A$.*

Proof. (i) Obviously we have $X^* = X^* \cdot e \oplus X^* \cdot (1-e)$, $X^* \cdot e \cong (e \cdot X)^*$ and $X^* \cdot (1-e) \cong ((1-e) \cdot X)^*$. Let P and Q be the canonical projections from X^* onto $X^* \cdot e$ and $X^* \cdot (1-e)$, respectively. Then $P \circ D : A \to X^* \cdot e$ and $Q \circ D : A \to X^* \cdot (1-e)$ are derivations and $Q \circ D = \mathrm{ad}_{Q \circ D(e)}$ is inner. Now, choosing f and D_0 with $f = Q \circ D(e)$ and $D_0 = P \circ D$ will finish the proof.

(ii) Since A is boundedly approximately φ-amenable, there is a net $(f_\alpha) \subset X^* \cdot e$ such that $\| \mathrm{ad}_{f_\alpha} \|$ is bounded and $D_0(a) = \lim_\alpha \mathrm{ad}_{f_\alpha}(a)$ for all $a \in A$. So $D(a) = \mathrm{ad}_f(a) + \lim_\alpha \mathrm{ad}_{f_\alpha}(a)$ for all $a \in A$. □

We denote by φ_e the unique extension of $\varphi \in \sigma(A) \cup \{0\}$ to $A^{\#}$. Then each $\varphi \in \sigma(A) \cup \{0\}$ can be extended uniquely to an element φ_e of $\sigma(A^{\#})$.

Proposition 3.2. *Let A be a Banach algebra and let $\varphi \in \sigma(A)$. Then A is boundedly approximately φ-amenable if and only if $A^{\#}$ is boundedly approximately φ_e-amenable.*

Proof. Let $X \in {}_{\varphi_e}\mathcal{M}^{A^{\#}}$ and let $D : A^{\#} \to X^*$ be a derivation. By Lemma 3.1, there are $f \in X^* \cdot (1-e)$ and $D_0 : A^{\#} \to X^* \cdot e$ with $D = \mathrm{ad}_f + D_0$. Since $D_0(e) = 0$ and $D_0|_A$ is boundedly approximately inner, D is boundedly approximately inner too. Conversely, let $X \in {}_{\varphi}\mathcal{M}^A$ and let $D : A \to X^*$ be a derivation. Then by putting $e \cdot x = x \cdot e = x$, X can be considered as an element of ${}_{\varphi_e}\mathcal{M}^{A^{\#}}$. Also, by letting $D(e) = 0$, we extend D to a derivation $\tilde{D}$ on $A^{\#}$. Since $\tilde{D}$ is boundedly approximately inner, so is $D = \tilde{D}|_A$. □

Now we consider bounded approximate character amenability of closed ideals of boundedly approximately character amenable Banach algebras.

The proof of the following proposition is a modified version of the proof of [1, Proposition 3.6] and we give it for the sake of completeness.

Proposition 3.3. *Let A be a Banach algebra and J a closed ideal of it with a bounded approximate identity. Let also $\varphi \in \sigma(A)$ and $\varphi|_J \neq 0$. If A is boundedly approximately φ-amenable, then J is boundedly approximately $\varphi|_J$-amenable.*

Proof. Let (e_i) be a bounded approximate identity for J. Without loss of generality suppose that $e_i \to E$ in the w^*-topology of J^{**}. Then E is a right identity for J^{**}, $E \cdot a = a \cdot E = \iota(a)$ for all $a \in J$, where $\iota : J \to J^{**}$ is the canonical embedding. By Theorem 2.6, there exists a net $(M_\alpha) \subset (A\widehat{\otimes}A)^{**}$ such that $(\Delta^{**}M_\alpha)(\varphi) \to 1$, $a \cdot M_\alpha - \varphi(a)M_\alpha \to 0$ and $\|a \cdot M_\alpha - \varphi(a)M_\alpha\| \leq c\|a\|$ for some $c > 0$ and for all $a \in A$. Define $N_\alpha = E \cdot M_\alpha \cdot E \in (J\hat{\otimes}J)^{**}$. Then for each $a \in J$,

$$a \cdot N_\alpha - \varphi(a)N_\alpha = a \cdot M_\alpha \cdot E - \varphi(a)E \cdot M_\alpha \cdot E = E \cdot (a \cdot M_\alpha - \varphi(a)M_\alpha) \cdot E \to 0.$$

Now let $a \in J$ be such that $\varphi(a) \neq 0$. Then by noting that $\langle \varphi|_J, E\rangle = \langle \varphi, E\rangle = 1$, we have

$$\begin{aligned}
\varphi(a)^2(\Delta_J^{**}(N_\alpha)(\varphi|_J) - 1) &= \varphi(a)^2(\Delta_J^{**}(N_\alpha) - E)(\varphi|_J)\\
&= \varphi(a)^2(E \cdot \Delta^{**}(M_\alpha) \cdot E - E)(\varphi)\\
&= (aE \cdot \Delta^{**}(M_\alpha) \cdot Ea - aEa)(\varphi)\\
&= (a\Delta^{**}(M_\alpha)a - aEa)(\varphi)\\
&= (a(\Delta^{**}(M_\alpha) - E)a)(\varphi)\\
&= \varphi(a)^2(\Delta^{**}(M_\alpha) - E)(\varphi)\\
&= \varphi(a)^2(\Delta^{**}(M_\alpha)(\varphi) - 1) \to 0,
\end{aligned}$$

hence $\Delta_J^{**}(N_\alpha)(\varphi|_J) \to 1$, where $\Delta_J : J\hat{\otimes}J \to J$ is the diagonal operator. Also,

$$\begin{aligned}
\|a \cdot N_\alpha - \varphi(a)N_\alpha\| &= \|E \cdot (a \cdot M_\alpha - \varphi(a)M_\alpha) \cdot E\|\\
&\leq c\|E\|^2\|a\|.
\end{aligned}$$

Therefore (N_α) is a multiplier-bounded approximate $\varphi|_J$-diagonal for J, and so J is boundedly approximately $\varphi|J$-amenable. □

It is shown in [1, Theorem 3.4] that each $\varphi \in \sigma(J)$ can be extended to an element $\widetilde{\varphi} \in \sigma(A)$ provided that J has a right or left approximate identity. So we have the following corollary.

Corollary 3.4. *Let A be a Banach algebra and J a closed ideal of it with a bounded approximate identity. Then bounded approximate character amenability of A implies that of J.*

Proposition 3.5. *Let A and B be Banach algebras, $\theta : A \to B$ a continuous Banach algebra epimorphism, $\psi \in \sigma(B)$ and $\varphi = \psi \circ \theta$. If A is boundedly approximately φ-amenable, then B is boundedly approximately ψ-amenable.*

Proof. Let $(M_\alpha) \subset (A\widehat{\otimes}A)^{**}$ be a multiplier-bounded approximate φ-diagonal for A. So there is a constant $c > 0$ such that $(\Delta_A^{**} M_\alpha)(\varphi) \to 1$, $a \cdot M_\alpha - \varphi(a)M_\alpha \to 0$ and $\|a \cdot M_\alpha - \varphi(a)M_\alpha\| \leq c\|a\|$ for all $a \in A$. Set $N_\alpha = (\theta \otimes \theta)^{**}(M_\alpha)$. Now using the fact that $\theta(a) \cdot (\theta \otimes \theta)^{**}(M_\alpha) = (\theta \otimes \theta)^{**}(a \cdot M_\alpha)$ for each $a \in A$ and $(\theta \otimes \theta)^* \circ \Delta_B^* = \Delta_A^* \circ \theta^*$, it can be easily seen that (N_α) is a multiplier-bounded approximate ψ-diagonal for B. □

Corollary 3.6. *Let A and B be Banach algebras with multiplier-bounded approximate identities and $\theta : A \to B$ a continuous Banach algebra epimorphism. If A is boundedly approximately character amenable, then so is B.*

4. Examples

This short section is devoted to the examples related to locally compact groups and examples which show that the class of boundedly approximately character amenable Banach algebras is different from the class of other known notions of amenability.

Example 4.1. It is shown in [2] that the following commutative Banach algebras have multiplier-bounded approximate identities but no bounded approximate identity. Therefore, they are boundedly approximately 0-amenable, but neither 0-amenable nor boundedly approximately amenable.

(i) $c_0(\omega)$, the space of all sequences $(a_n) \subseteq \mathbb{C}$ such that $|a_n|\omega_n \to 0$, equipped with pointwise multiplication, where $\lim_n \omega_n = \infty$.
(ii) $l^1(\mathbb{N}_{\min}, \omega)$, the weighted convolution algebra of the semilattice $\mathbb{N}_{\min}$, where $\lim_n \omega_n = \infty$.
(iii) The Fourier algebra of $\mathbb{F}_2$, the free group on two generators (see also [7]).

Example 4.2. Consider the Banach algebra $\mathcal{A}$ constructed in [6, Corollary 3.4]. This Banach algebra has an approximate identity and is boundedly approximately amenable. It also possesses a multiplier-bounded left (but not right) approximate identity. Therefore, $\mathcal{A}$ is boundedly approximately character amenable, hence boundedly approximately 0-amenable, and it has no multiplier-bounded right approximate identity.

Example 4.3. Let S be an infinite discrete cancellative semigroup. Then the Banach algebra $l^1(S)^{**}$ is not boundedly approximately character amenable (see [1, Theorem 7.8]).

Example 4.4. Let G be a locally compact group.

(i) Using [1, Theorem 7.1], the group algebra $L^1(G)$ is boundedly approximately character amenable if and only if G is amenable.

(ii) Employing [1, Theorem 7.2], $M(G)$, the measure algebra of G, is boundedly approximately character amenable if and only if G is discrete and amenable.

(iii) By [1, Theorems 7.4 and 7.7], $L^1(G)^{**}$ and $M(G)^{**}$ are boundedly approximately character amenable if and only if G is finite.

(iv) Let $LUC(G)$ denote the Banach space of left uniformly continuous functions on G. Then, by [1, Corollary 7.5], the Banach algebra $LUC(G)^*$ (when it is considered as a closed subalgebra of $L^1(G)^{**}$) is boundedly approximately character amenable if and only if G is finite.

Example 4.5. Let $A_p(G)$ be the Figà-Talamanca-Herz algebra of a locally compact group G, introduced in [8]. It is known that $\sigma(A_p(G)) = G$. Then, by [12, Corollary 2.4], $A_p(G)$ is boundedly approximately φ-amenable for each $\varphi \in \sigma(A_p(G))$. Therefore, $A_p(G)$ is boundedly approximately character amenable if and only if $A_p(G)$ is boundedly approximately 0-amenable.

We remark that if G is an amenable group, by Leptin's theorem, $A_p(G)$ has a bounded approximate identity and so it is boundedly approximately character amenable. On the other hand, $A_2(SL(2,\mathbb{R}) \rtimes \mathbb{R}^n)$, $n \geq 2$, does not have a multiplier-bounded approximate identity, where $SL(2,\mathbb{R}) \rtimes \mathbb{R}^n$ is the semidirect product of $SL(2,\mathbb{R})$ and $\mathbb{R}^n$ (see [3]). Hence $A_2(SL(2,\mathbb{R}) \rtimes \mathbb{R}^n)$ is not boundedly approximately amenable.

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