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Approximate Duals of g-frames and Fusion Frames in Hilbert C^*-modules

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Abstract. In this paper, we study approximate duals of g-frames and fusion frames in Hilbert C^*-modules. We get some relations between approximate duals of g-frames and biorthogonal Bessel sequences, and using these relations, some results for approximate duals of modular Riesz bases and fusion frames are obtained. Moreover, we generalize the concept of $Q-$approximate duality of g-frames and fusion frames to Hilbert C^*-modules, where Q is an adjointable operator, and obtain some properties of this kind of approximate duals.

1. Introduction

Frames for Hilbert spaces were first introduced by Duffin and Schaeffer [5] in 1952 to study some problems in nonharmonic Fourier series, reintroduced in 1986 by Daubechies, Grossmann and Meyer [4]. Various generalizations of frames have been introduced. Fusion frames [2] and g-frames [16] are two important generalizations of frames.

In [6], Frank and Larson presented a general approach to the frame theory in Hilbert C^*–modules. Also, fusion frames and g-frames have been introduced in Hilbert C^*–modules (see [9]).

Approximate duality of frames in Hilbert spaces was recently investigated in [3] and it has been introduced for g-frames in Hilbert spaces in [11]. Also, approximate duals of frames and g-frames have been generalized to Hilbert C^*-modules in [13] (see also [14]). Moreover, $Q-$approximate duals for g-frames and fusion frames have been introduced in [15].

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In this paper, we obtain some new results about approximate duals and $Q-$approximate duals of g-frames and fusion frames in Hilbert C^*-modules. First in the following section, we recall the definitions of frames, g-frames and fusion frames in Hilbert C^*-modules.

2. Frames, Fusion Frames and g-frames in Hilbert C^*-modules

Suppose that $\mathfrak{A}$ is a unital C^*–algebra and E is a left $\mathfrak{A}$–module such that the linear structures of $\mathfrak{A}$ and E are compatible. E is a pre-Hilbert $\mathfrak{A}$–module if it is equipped with an $\mathfrak{A}$–valued inner product $\langle .,. \rangle : E \times E \to \mathfrak{A}$, such that

(i) $\langle \alpha x + \beta y, z \rangle = \alpha \langle x, z \rangle + \beta \langle y, z \rangle$, for each $\alpha, \beta \in \mathbb{C}$ and $x, y, z \in E$;
(ii) $\langle ax, y \rangle = a \langle x, y \rangle$, for each $a \in \mathfrak{A}$ and $x, y \in E$;
(iii) $\langle x, y \rangle = \langle y, x \rangle^*$, for each $x, y \in E$;
(iv) $\langle x, x \rangle \geq 0$, for each $x \in E$ and if $\langle x, x \rangle = 0$, then $x = 0$.

For each $x \in E$, we define $\|x\| = \| \langle x, x \rangle \|^{\frac{1}{2}}$ and $|x| = \langle x, x \rangle^{\frac{1}{2}}$. If E is complete with $\|.\|$, it is called a Hilbert $\mathfrak{A}$–module or a Hilbert C^*–module over $\mathfrak{A}$. We call $\mathcal{Z}(\mathfrak{A}) = \{a \in \mathfrak{A} : ab = ba, \forall b \in \mathfrak{A}\}$, the center of $\mathfrak{A}$. Note that if $a \in \mathcal{Z}(\mathfrak{A})$, then $a^* \in \mathcal{Z}(\mathfrak{A})$, and if a is an invertible element of $\mathcal{Z}(\mathfrak{A})$, then $a^{-1} \in \mathcal{Z}(\mathfrak{A})$, also if a is a positive element of $\mathcal{Z}(\mathfrak{A})$, since $a^{\frac{1}{2}}$ is in the closure of the set of polynomials in a, we have $a^{\frac{1}{2}} \in \mathcal{Z}(\mathfrak{A})$. Let E and F be Hilbert $\mathfrak{A}$–modules. An operator $T : E \to F$ is called adjointable if there exists an operator $T^* : F \to E$ such that $\langle T(x), y \rangle = \langle x, T^*(y) \rangle$, for each $x \in E$ and $y \in F$. Every adjointable operator T is bounded and $\mathfrak{A}$–linear (that is, $T(ax) = aT(x)$ for each $x \in E$ and $a \in \mathfrak{A}$). We denote the set of all adjointable operators from E into F by $\mathfrak{L}(E, F)$. Note that $\mathfrak{L}(E, E)$ is a C^*–algebra and it is denoted by $\mathfrak{L}(E)$. For a unital C^*–algebra $\mathfrak{A}$, $\ell^2(I, \mathfrak{A})$ which is defined by

$$\ell^2(I, \mathfrak{A}) = \left\{ \{a_i\}_{i \in I} \subseteq \mathfrak{A} : \sum_{i \in I} a_i {a_i}^* \text{ converges in } \|.\| \right\},$$

is a Hilbert $\mathfrak{A}$–module with inner product

$$\langle \{a_i\}_{i \in I}, \{b_i\}_{i \in I} \rangle = \sum_{i \in I} a_i b_i^*.$$

A Hilbert $\mathfrak{A}$–module E is finitely generated if there exists a finite set $\{x_1, \dots, x_n\} \subseteq E$ such that every element $x \in E$ can be expressed as an $\mathfrak{A}$–linear combination $x = \sum_{i=1}^n a_i x_i, a_i \in \mathfrak{A}$. A Hilbert $\mathfrak{A}$–module E is countably generated if there exists a countable set $\{x_i\}_{i \in I} \subseteq E$ such that E equals the norm-closure of the $\mathfrak{A}$–linear hull of $\{x_i\}_{i\in I}$. For more details about Hilbert C^*–modules, see [12].

Let E be a Hilbert $\mathfrak{A}$–module. A family $\mathcal{F}=\{f_i\}_{i\in I}\subseteq E$ is a frame for E, if there exist real constants $0<A_{\mathcal{F}}\leq B_{\mathcal{F}}<\infty$, such that for each $x\in E$,

$$A_{\mathcal{F}}\langle x,x\rangle \leq \sum_{i\in I}\langle x,f_i\rangle\langle f_i,x\rangle \leq B_{\mathcal{F}}\langle x,x\rangle\,, \tag{2.1}$$

i.e., there exist real constants $0<A_{\mathcal{F}}\leq B_{\mathcal{F}}<\infty$, such that the series $\sum_{i\in I}\langle x,f_i\rangle\langle f_i,x\rangle$ converges in the ultraweak operator topology to some element in the universal enveloping Von Neumann algebra of $\mathfrak{A}$ such that the inequality holds, for each $x\in E$. The numbers $A_{\mathcal{F}}$ and $B_{\mathcal{F}}$ are called the lower and upper bounds of the frame, respectively. In this case we call it an $(A_{\mathcal{F}},B_{\mathcal{F}})$ frame. If only the second inequality is required, we call it a Bessel sequence. If the sum in (2.1) converges in norm, the frame is called standard.

For a standard Bessel sequence $\mathcal{F}=\{f_i\}_{i\in I}$ with an upper bound $B_{\mathcal{F}}$, the operator $T_{\mathcal{F}}:\ell^2(I,\mathfrak{A})\to E$ defined by

$$T_{\mathcal{F}}(\{a_i\}_{i\in I})=\sum_{i\in I}a_if_i$$

is called the synthesis operator of $\mathcal{F}$. It is adjointable with $T^*_{\mathcal{F}}(x)=\{\langle x,f_i\rangle\}_{i\in I}$ and $\|T_{\mathcal{F}}\|\leq\sqrt{B_{\mathcal{F}}}$. $T^*_{\mathcal{F}}$ is the analysis operator of $\mathcal{F}$. Now we define the operator $S_{\mathcal{F}}:E\to E$ by

$$\begin{aligned}S_{\mathcal{F}}(x)&=T_{\mathcal{F}}T^*_{\mathcal{F}}(x)\\&=\sum_{i\in I}\langle x,f_i\rangle f_i.\end{aligned}$$

If $\mathcal{F}$ is a standard $(A_{\mathcal{F}},B_{\mathcal{F}})$ frame, then $A_{\mathcal{F}}.Id_E\leq S_{\mathcal{F}}\leq B_{\mathcal{F}}.Id_E$. The operator $S_{\mathcal{F}}$ is called the frame operator of $\mathcal{F}$. Let $\mathcal{F}=\{f_i\}_{i\in I}$ and $\mathcal{G}=\{g_i\}_{i\in I}$ be standard Bessel sequences in E. Then we say that $\mathcal{G}$ is an alternate dual or a dual of $\mathcal{F}$, if $x=\sum_{i\in I}\langle x,f_i\rangle g_i$ or equivalently $x=\sum_{i\in I}\langle x,g_i\rangle f_i$, for each $x\in E$ (see [7, Proposition 3.8]).

It is easy to see that if $\mathcal{F}$ is an $(A_{\mathcal{F}},B_{\mathcal{F}})$ standard frame, then $\widetilde{\mathcal{F}}=\{S^{-1}_{\mathcal{F}}f_i\}_{i\in I}$ is an $(\frac{1}{B_{\mathcal{F}}},\frac{1}{A_{\mathcal{F}}})$ standard frame with

$$\begin{aligned}x&=\sum_{i\in I}\left\langle x,S^{-1}_{\mathcal{F}}f_i\right\rangle f_i\\&=\sum_{i\in I}\langle x,f_i\rangle S^{-1}_{\mathcal{F}}f_i,\end{aligned}$$

for each $x\in E$. Hence $\widetilde{\mathcal{F}}=\{S^{-1}_{\mathcal{F}}f_i\}_{i\in I}$ is a dual of $\mathcal{F}$ called the canonical dual of $\mathcal{F}$.

Note that a closed submodule M of E is orthogonally complemented if $E = M \oplus M^{\perp}$. In this case $\pi_M \in \mathfrak{L}(E, M)$, where $\pi_M : E \to M$ is the orthogonal projection onto M.

Suppose that $\{\omega_i : i \in I\} \subseteq \mathfrak{A}$ is a family of weights, i.e., each ω_i is a positive, invertible element of the center of $\mathfrak{A}$, and $\{W_i : i \in I\}$ is a family of orthogonally complemented submodules of E. Then $\mathcal{W} = \{(W_i, \omega_i)\}_{i\in I}$ is a fusion frame if there exist real constants $0 < A_{\mathcal{W}} \leq B_{\mathcal{W}} < \infty$ such that

$$\begin{aligned} A_{\mathcal{W}} \langle x, x\rangle &\leq \sum_{i\in I} \omega_i^2 \langle \pi_{W_i}(x), \pi_{W_i}(x)\rangle \\ &\leq B_{\mathcal{W}} \langle x, x\rangle , \end{aligned}$$

for each $x \in E$. In this case we call it an $(A_{\mathcal{W}}, B_{\mathcal{W}})$ fusion frame. If we only require to have the upper bound, then $\mathcal{W} = \{(W_i, \omega_i)\}_{i\in I}$ is called a Bessel fusion sequence with upper bound $B_{\mathcal{W}}$.

Note that if $\{E_i : i \in I\}$ is a sequence of Hilbert $\mathfrak{A}$–modules, then $\oplus_{i\in I} E_i$ which is the set

$$\left\{ \{x_i\}_{i\in I} : x_i \in E_i, \left\{ \langle x_i, x_i\rangle^{\frac{1}{2}} \right\}_{i\in I} \in \ell^2(I, \mathfrak{A}) \right\},$$

is a Hilbert $\mathfrak{A}$–module with pointwise operations and $\mathfrak{A}$-valued inner product

$$\langle \{x_i\}_{i\in I}, \{y_i\}_{i\in I}\rangle = \sum_{i\in I} \langle x_i, y_i\rangle .$$

If $E_i = E$, for each $i \in I$, then $\oplus_{i\in I} E_i$ is denoted by $\ell^2(I, E)$.

A sequence $\Lambda = \{\Lambda_i \in \mathfrak{L}(E, E_i) : i \in I\}$ is called a g-frame for E with respect to $\{E_i : i \in I\}$ if there exist positive constants A_Λ, B_Λ with

$$A_\Lambda \langle x, x\rangle \leq \sum_{i\in I} \langle \Lambda_i x, \Lambda_i x\rangle \leq B_\Lambda \langle x, x\rangle ,$$

for each $x \in E$. In this case, we call it an (A_Λ, B_Λ) g-frame. If only the second inequality is required, then Λ is called a g-Bessel sequence. Note that standard fusion frames and g-frames are defined similar to the standard frames. For a standard g-Bessel sequence Λ, the operator $T_\Lambda : \oplus_{i\in I} E_i \to E$ which is defined by

$$T_\Lambda (\{x_i\}_{i\in I}) = \sum_{i\in I} \Lambda_i^*(x_i),$$

is called the synthesis operator of Λ. T_Λ is adjointable with $T_\Lambda^*(x) = \{\Lambda_i x\}_{i\in I}$. Now we define the operator $S_\Lambda : E \to E$ by

$$\begin{aligned} S_\Lambda x &= T_\Lambda T_\Lambda^*(x) \\ &= \sum_{i\in I} \Lambda_i^* \Lambda_i(x). \end{aligned}$$

If Λ is a standard (A_Λ, B_Λ) g-frame, then $A_\Lambda.Id_E \leq S_\Lambda \leq B_\Lambda.Id_E$. The operator S_Λ is called the g-frame operator of Λ.

Recall that if $\Lambda = \{\Lambda_i\}_{i\in I}$ and $\Gamma = \{\Gamma_i\}_{i\in I}$ are standard $g-$Bessel sequences such that $\sum_{i\in I}\Gamma_i^*\Lambda_i x = x$ or equivalently $\sum_{i\in I}\Lambda_i^*\Gamma_i x = x$, for each $x \in E$, then Γ (resp. Λ) is called a g-dual of Λ (resp. Γ).

It is easy to see that if $\mathcal{W} = \{(W_i, \omega_i)\}_{i\in I}$ is a standard Bessel fusion sequence (resp. a standard fusion frame), then $\Lambda_\mathcal{W} = \{\omega_i\pi_{W_i}\}_{i\in I}$, where π_{W_i} is the orthogonal projection on W_i, is a standard g-Bessel sequence (resp. a standard g-frame). We say that $\mathcal{W} = \{(W_i, \omega_i)\}_{i\in I}$ is a dual of $\mathcal{V} = \{(V_i, \upsilon_i)\}_{i\in I}$ if $\Lambda_\mathcal{W} = \{\omega_i\pi_{W_i}\}_{i\in I}$ is a g-dual of $\Lambda_\mathcal{V} = \{\upsilon_i\pi_{V_i}\}_{i\in I}$. Also it is easy to see that if $\mathcal{F} = \{f_i\}_{i\in I}$ is a standard Bessel sequence (resp. a standard frame), then $\Lambda_\mathcal{F} = \{\Lambda_{f_i}\}_{i\in I}$, where $\Lambda_{f_i} : E \to \mathfrak{A}$, $\Lambda_{f_i}(f) = \langle f, f_i\rangle$, is a standard g-Bessel sequence (resp. a standard g-frame) and $\mathcal{G} = \{g_i\}_{i\in I} \subseteq E$ is a dual of $\mathcal{F}$ if $\Lambda_\mathcal{G}$ is a g-dual of $\Lambda_\mathcal{F}$.

For more results about frames, fusion frames and g-frames in Hilbert C^*–modules, see [6, 1, 9, 17].

In the present paper, all C^*–algebras are unital and all Hilbert C^*–modules are finitely or countably generated. All frames, fusion frames, g-frames and Bessel sequences are standard. Throughout this paper $\Lambda = \{\Lambda_i \in \mathfrak{L}(E, E_i) : i \in I\}$ and $\Gamma = \{\Gamma_i \in \mathfrak{L}(E, E_i) : i \in I\}$ are standard g-Bessel sequences with upper bounds B_Λ and B_Γ, respectively. Also $\mathcal{W} = \{(W_i, \omega_i)\}_{i\in I}$ and $\mathcal{V} = \{(V_i, \upsilon_i)\}_{i\in I}$ are standard Bessel fusion sequences for E with upper bounds $B_\mathcal{W}$ and $B_\mathcal{V}$, respectively.

3. Approximate Dulas of g-frames and Biorthogonal Bessel Sequences

First we recall the definition of approximate duality for g-frames in Hilbert C^*-modules from [13].

Definition 3.1. (i) Let Λ and Γ be two standard g-Bessel sequences and $S_{\Gamma\Lambda} = T_\Gamma {T_\Lambda}^*$. Then Λ and Γ are approximately dual g-frames if $\|Id_E - S_{\Gamma\Lambda}\| < 1$ or equivalently $\|Id_E - S_{\Lambda\Gamma}\| < 1$. In this case, we say that Γ (resp. Λ) is an approximate dual g-frame or an approximate g-dual of Λ (resp. Γ).

(ii) Let $\mathcal{W}$ and $\mathcal{V}$ be standard Bessel fusion sequences. Then we say that $\mathcal{W}$ is an approximate dual of $\mathcal{V}$ if $\Lambda_\mathcal{W}$ is an approximate g-dual of $\Lambda_\mathcal{V}$.

(iii) Two standard Bessel sequences $\mathcal{F}$ and $\mathcal{G}$ are approximately dual frames if $\Lambda_\mathcal{F}$ and $\Lambda_\mathcal{G}$ are approximately dual g-frames, i.e., $\|Id_E - S_{\Lambda_\mathcal{G}\Lambda_\mathcal{F}}\| < 1$. We denote $S_{\Lambda_\mathcal{G}\Lambda_\mathcal{F}}$ by $S_{\mathcal{G}\mathcal{F}}$.

Since $\|Id_E - S_{\Lambda\Gamma}\| < 1$, we obtain that $S_{\Lambda\Gamma}$ is invertible with

$$S_{\Lambda\Gamma}{}^{-1} = \sum_{n=0}^{\infty} (Id_E - S_{\Lambda\Gamma})^n.$$

Now for each $f \in E$, we have the following reconstruction formulas:

$$f = \sum_{n=0}^{\infty} S_{\Lambda\Gamma}(Id_E - S_{\Lambda\Gamma})^n f, \qquad f = \sum_{n=0}^{\infty} (Id_E - S_{\Lambda\Gamma})^n S_{\Lambda\Gamma} f.$$

It is also obtained from Theorem 3.2 in [13] that if Λ and Γ are approximately dual g-frames, then Λ and Γ are standard g-frames. The similar results hold for Bessel sequences and Bessel fusion sequences.

Recall that $\{f_i\}_{i\in I}, \{g_i\}_{i\in I} \subseteq E$ are called biorthogonal if $\langle f_i, g_j\rangle = 0$, for $i \neq j$ and $\langle f_i, g_i\rangle = 1_{\mathfrak{A}}$.

In the following theorem $\mathcal{F}_i = \{f_{ij}\}_{j\in J_i}$, $\mathcal{F}_i{}' = \{f'_{ij}\}_{j\in J_i}$ and $\mathcal{G}_i = \{g_{ij}\}_{j\in J_i}$, $\mathcal{G}_i{}' = \{g'_{ij}\}_{j\in J_i}$ are standard Bessel sequences for E_i such that the sequence of their upper bounds is bounded above.

Theorem 3.2. *Assume that $\mathcal{F}_i{}'$ and $\mathcal{G}_i{}'$ are duals of $\mathcal{F}_i$ and $\mathcal{G}_i$, respectively such that $\mathcal{F}'_i$ and $\mathcal{G}'_i$ are biorthogonal for each $i \in I$. Then Γ is a g-dual (an approximate g-dual) of Λ if and only if $\{\Gamma_i^* g_{ij}\}_{i\in I, j\in J_i}$ is a dual (resp. an approximate dual) of $\{\Lambda_i^* f_{ij}\}_{i\in I, j\in J_i}$.*

Proof. Let $B = \sup_{i\in I}\{B_{\mathcal{F}_i}\}$. Then for each $f \in E$, we have

$$\begin{aligned}\left\|\sum_{i\in I}\sum_{j\in J_i}\langle f, \Lambda_i^* f_{ij}\rangle \langle \Lambda_i^* f_{ij}, f\rangle\right\| &= \left\|\sum_{i\in I}\sum_{j\in J_i}\langle \Lambda_i f, f_{ij}\rangle \langle f_{ij}, \Lambda_i f\rangle\right\| \\ &\leq \left\|\sum_{i\in I} B_{\mathcal{F}_i}\langle \Lambda_i f, \Lambda_i f\rangle\right\| \\ &\leq BB_\Lambda \|f\|^2.\end{aligned}$$

Hence, Theorem 2.6 in [1] implies that $\mathcal{F} = \{\Lambda_i^* f_{ij}\}_{i\in I, j\in J_i}$ is a standard Bessel sequence. Similarly, $\mathcal{G} = \{\Gamma_i^* g_{ij}\}_{i\in I, j\in J}$ is a standard Bessel sequence. Now we have

$$\begin{aligned} S_{\Gamma\Lambda} f &= \sum_{i\in I}\Gamma_i^*\left(\sum_{j\in J_i}\langle \Lambda_i f, f_{ij}\rangle f'_{ij}\right) \\ &= \sum_{i\in I}\sum_{j\in J_i}\langle \Lambda_i f, f_{ij}\rangle \Gamma_i^* f'_{ij} \\ &= \sum_{i\in I}\sum_{j\in J_i}\langle \Lambda_i f, f_{ij}\rangle \Gamma_i^*\left(\sum_{k\in J_i}\langle f'_{ij}, g'_{ik}\rangle g_{ik}\right)\end{aligned}$$

$$= \sum_{i\in I}\sum_{j\in J_i}\sum_{k\in J_i} \langle \Lambda_i f, f_{ij}\rangle \left\langle f'_{ij}, g'_{ik}\right\rangle \Gamma_i^* g_{ik}$$
$$= \sum_{i\in I}\sum_{j\in J_i} \langle f, \Lambda_i^* f_{ij}\rangle \Gamma_i^* g_{ij}$$
$$= S_{\mathcal{G}\mathcal{F}} f.$$

Therefore $S_{\Gamma\Lambda} = Id_E$ if and only if $S_{\mathcal{G}\mathcal{F}} = Id_E$ and $\|S_{\Gamma\Lambda} - Id_E\| < 1$ if and only if $\|S_{\mathcal{G}\mathcal{F}} - Id_E\| < 1$. □

In the following corollary, which is a generalization of Proposition 3.2 in [15] to Hilbert C^*−modules, $\mathcal{F}_i = \{f_{ij}\}_{j\in J_i}$,${\mathcal{F}_i}' = \{f'_{ij}\}_{j\in J_i}$ and $\mathcal{G}_i = \{g_{ij}\}_{j\in J_i}$, ${\mathcal{G}_i}' = \{g'_{ij}\}_{j\in J_i}$ are standard Bessel sequences for W_i and V_i, respectively such that the sequence of their upper bounds is bounded above.

Corollary 3.3. *Assume that ${\mathcal{F}_i}'$ and ${\mathcal{G}_i}'$ are duals of $\mathcal{F}_i$ and $\mathcal{G}_i$, respectively such that $\mathcal{F}'_i$ and $\mathcal{G}'_i$ are biorthogonal for each $i \in I$. Then $\mathcal{V}$ is an approximate dual (resp. a dual) of $\mathcal{W}$ if and only if $\{\upsilon_i g_{ij}\}_{i\in I, j\in J_i}$ is an approximate dual (resp. a dual) of $\{\omega_i f_{ij}\}_{i\in I, j\in J_i}$.*

Proof. It is enough to consider in Theorem 3.2, $\Gamma = \{\Gamma_i\}_{i\in I}$ with $\Gamma_i = \upsilon_i \pi_{V_i}$ and $\Lambda = \{\Lambda_i\}_{i\in I}$ with $\Lambda_i = \omega_i \pi_{W_i}$. □

Modular Riesz bases in Hilbert C^*–modules were introduced in [10].

Definition 3.4. A standard frame $\{f_i\}_{i\in I}$ for E is a modular Riesz basis if it has the following property:
if an $\mathfrak{A}$-linear combination $\sum_{i\in K} a_i f_i$ with coefficients $\{a_i : i \in K\} \subseteq \mathfrak{A}$ and $K \subseteq I$ is equal to zero, then $a_i = 0$, for each $i \in K$.

Proposition 3.5. *Suppose that $\mathcal{F}_i = \{f_{ij}\}_{j\in J_i}$ is a modular Riesz basis for E_i with $\sup_{i\in I}\{B_{\mathcal{F}_i}\} < \infty$. Then Γ is a g-dual (resp. an approximate g-dual) of Λ if and only if $\{\Gamma_i^* \tilde{f_{ij}}\}_{i\in I, j\in J_i}$ is a dual (resp. an approximate dual) of $\{\Lambda_i^* f_{ij}\}_{i\in I, j\in J_i}$.*

Proof. First we show that $\{f_{ij}\}_{j\in J_i}$ and $\{\tilde{f_{ij}}\}_{j\in J_i}$ are biorthogonal, for each $i \in I$. Let $i \in I$ and $j_0 \in J_i$. Then we have

$$f_{ij_0} = \sum_{j\in J_i} \left\langle f_{ij_0}, S_{\mathcal{F}_i}^{-1} f_{ij} \right\rangle f_{ij},$$

so

$$\sum_{j\in J_i} a_{ij} f_{ij} = 0,$$

where $a_{ij} = \left\langle f_{ij_0}, S_{\mathcal{F}_i}^{-1} f_{ij} \right\rangle$, for each $j \neq j_0$ and $a_{ij_0} = \left\langle f_{ij_0}, S_{\mathcal{F}_i}^{-1} f_{ij_0} \right\rangle - 1_{\mathfrak{A}}$. Since $\mathcal{F}_i$ is a modular Riesz basis, the equality

$$\sum_{j \in J_i} a_{ij} f_{ij} = 0,$$

implies that $\left\langle f_{ij_0}, S_{\mathcal{F}_i}^{-1} f_{ij} \right\rangle = 0$, for every $j \neq j_0$ and $\left\langle f_{ij_0}, S_{\mathcal{F}_i}^{-1} f_{ij_0} \right\rangle = 1_{\mathfrak{A}}$. This means that $\mathcal{F}_i$ and $\widetilde{\mathcal{F}_i}$ are biorthogonal. Now the result follows from Theorem 3.2 by considering $\mathcal{F}_i = \{f_{ij}\}_{j \in J_i} = \mathcal{G}_i{}'$ and $\mathcal{G}_i = \{\widetilde{f_{ij}}\}_{j \in J_i} = \mathcal{F}_i{}'$. □

Definition 3.6. Let H be a Hilbert space. We say that $\{f_i\}_{i \in I}$ is a Riesz basis for H, if it is complete in H and there exist two constants $0 < A \leq B < \infty$, such that

$$A \sum_{i \in F} |c_i|^2 \leq \left\| \sum_{i \in F} c_i f_i \right\|^2 \leq B \sum_{i \in F} |c_i|^2,$$

for each sequence of scalars $\{c_i\}_{i \in F}$, where F is a finite subset of I.

It is easy to see that modular Riesz bases coincide with Riesz bases in Hilbert spaces, so using the above proposition, we get the following result.

Corollary 3.7. *Suppose that H is a Hilbert space and $\mathcal{F}_i = \{f_{ij}\}_{j \in J_i}$ is a Riesz basis for a Hilbert space H_i with $\sup_{i \in I}\{B_{\mathcal{F}_i}\} < \infty$. Then $\{\Gamma_i \in \mathfrak{L}(H, H_i)\}_{i \in I}$ is a g-dual (resp. an approximate g-dual) of $\{\Lambda_i \in \mathfrak{L}(H, H_i) : i \in I\}$ if and only if $\{\Gamma_i^* \tilde{f}_{ij}\}_{i \in I, j \in J_i}$ is a dual (resp. an approximate dual) of $\{\Lambda_i^* f_{ij}\}_{i \in I, j \in J_i}$.*

Also, Proposition 3.5 implies the following result which is a generalization of Corollary 3.1 in [15] to Hilbert C^*-modules.

Corollary 3.8. *Suppose that $\{f_{ij}\}_{j \in J_i}$ is a modular Riesz basis for W_i with upper bound B_i and $\sup_{i \in I}\{B_i\} < \infty$. Then $\mathcal{W}$ is an approximate dual (resp. a dual) of itself if and only if $\{\omega_i f_{ij}\}_{i \in I, j \in J_i}$ is an approximate dual (resp. a dual) of $\{\omega_i \widetilde{f_{ij}}\}_{i \in I, j \in J_i}$.*

Proof. It is enough to consider in Proposition 3.5, $\Gamma = \{\Gamma_i\}_{i \in I}$ and $\Lambda = \{\Lambda_i\}_{i \in I}$ with $\Gamma_i = \Lambda_i = \omega_i \pi_{W_i}$. □

4. $Q-$approximate Duals of g-frames and Fusion Frames in Hilbert C^*-modules

$Q-$approximate duals of g-frames and fusion frames in Hilbert spaces were introduced in [15]. In this section, we generalize the concept of $Q-$approximate duality of g-frames and fusion frames to Hilbert C^*-modules.

Definition 4.1. Let Λ and Γ be standard g-Bessel sequences for E.

(i) If there exists an operator $Q \in \mathfrak{L}(\oplus_{i\in I} E_i)$ such that $T_\Lambda Q T_\Gamma^* = Id_E$, then Λ is called a $Q-$dual of Γ.

(ii) If there exists an operator $Q \in \mathfrak{L}(\oplus_{i\in I} E_i)$ such that $\|T_\Lambda Q T_\Gamma^* - Id_E\| < 1$, then Λ is called a $Q-$approximate dual of Γ.

Note that if Λ is an approximate g-dual (resp. a g-dual) of Γ, then Λ is a $Q-$approximate dual (resp. $Q-$dual) of Γ with $Q = Id_{(\oplus_{i\in I} E_i)}$.

The following result is a generalization of Theorem 3.1 in [15] to Hilbert C^*-modules.

Theorem 4.2. *Let Λ and Γ be standard g-Bessel sequences for E. If Λ is a $Q-$approximate dual of Γ, then*

(i) $\|T_\Gamma Q^* T_\Lambda^* - Id_E\| < 1$.

(ii) T_Γ^* *is injective and* $T_\Lambda Q$ *is surjective.*

(iii) T_Λ^* *is injective and* $T_\Gamma Q^*$ *is surjective.*

(iv) Λ *and* Γ *are standard g-frames.*

Proof. (i) We have

$$\begin{aligned} \|T_\Gamma Q^* T_\Lambda^* - Id_E\| &= \|(T_\Lambda Q T_\Gamma^* - Id_E)^*\| \\ &= \|T_\Lambda Q T_\Gamma^* - Id_E\| \\ &< 1. \end{aligned}$$

(ii) Since $\|T_\Lambda Q T_\Gamma^* - Id_E\| < 1$, by the Newmann algorithm $T_\Lambda Q T_\Gamma^*$ is invertible. Hence T_Γ^* is injective and $T_\Lambda Q$ is surjective.

(iii) We can obtain the result similar to (ii) using part (i).

(iv) Let $S_{\Lambda Q\Gamma} = T_\Lambda Q T_\Gamma^*$. Then $S_{\Lambda Q\Gamma}^* = S_{\Gamma Q^*\Lambda}$ and since $\|S_{\Lambda Q\Gamma} - Id_E\| < 1$, $S_{\Lambda Q\Gamma}$ and $S_{\Gamma Q^*\Lambda}$ are invertible. Now for each $f \in E$, we have

$$\begin{aligned} \|f\| &= \left\| S_{\Gamma Q^*\Lambda}^{-1} S_{\Gamma Q^*\Lambda} f \right\| \\ &\leq \left\| S_{\Gamma Q^*\Lambda}^{-1} \right\| \|S_{\Gamma Q^*\Lambda} f\| \\ &= \left\| S_{\Gamma Q^*\Lambda}^{-1} \right\| \left(\sup_{\|g\|=1} \|\langle S_{\Gamma Q^*\Lambda} f, g\rangle\| \right) \\ &= \left\| S_{\Gamma Q^*\Lambda}^{-1} \right\| \left(\sup_{\|g\|=1} \|\langle Q^*(\{\Lambda_i f\}_{i\in I}), T_\Gamma^* g\rangle\| \right) \\ &\leq \left\| S_{\Gamma Q^*\Lambda}^{-1} \right\| \|Q^*\| \, \|\{\Lambda_i f\}_{i\in I}\|_2 \|T_\Gamma^*\| \end{aligned}$$

$$\leq \sqrt{B_\Gamma}\left\|S^{-1}_{\Gamma Q^*\Lambda}\right\|\|Q^*\|\left\|\sum_{i\in I}\langle\Lambda_i f,\Lambda_i f\rangle\right\|^{\frac{1}{2}}.$$

Thus

$$\frac{1}{B_\Gamma\|S^{-1}_{\Gamma Q^*\Lambda}\|^2\|Q^*\|^2}\|f\|^2 \leq \left\|\sum_{i\in I}\langle\Lambda_i f,\Lambda_i f\rangle\right\|.$$

Now, Theorem 3.1 in [17] yields that Λ is a standard g-frame. Similarly, we can see that Γ is a standard g-frame. □

Now we introduce $Q-$approximate duality for standard Bessel fusion sequences in Hilbert C^*-modules.

Definition 4.3. Let $\mathcal{W}$ and $\mathcal{V}$ be standard Bessel fusion sequences for E.

(i) If there exists an operator $Q \in \mathfrak{L}(\ell^2(I,E))$ such that$T_{\Lambda_\mathcal{W}}QT^*_{\Lambda_\mathcal{V}} = Id_E$, then $\mathcal{W}$ is called a $Q-$dual of $\mathcal{V}$.

(ii) If there exists an operator $Q \in \mathfrak{L}(\ell^2(I,E))$ such that $\|T_{\Lambda_\mathcal{W}}QT^*_{\Lambda_\mathcal{V}} - Id_E\| < 1$, then $\mathcal{W}$ is called a $Q-$approximate dual of $\mathcal{V}$.

As a consequence of Theorem 4.2, we get the following result which is a generalization of Lemma 3.2 in [8] and Theorem 3.2 in [15] to the approximate duality of fusion frames in Hilbert C^*-modules.

Theorem 4.4. *Let $\mathcal{W}$ and $\mathcal{V}$ be standard Bessel fusion sequences for E. If $\mathcal{W}$ is a $Q-$approximate dual of $\mathcal{V}$, then*

(i) $\|T_{\Lambda_\mathcal{V}}Q^*T^*_{\Lambda_\mathcal{W}} - Id_E\| < 1$.

(ii) $T^*_{\Lambda_\mathcal{V}}$ *is injective and* $T_{\Lambda_\mathcal{W}}Q$ *is surjective.*

(iii) $T^*_{\Lambda_\mathcal{W}}$ *is injective and* $T_{\Lambda_\mathcal{V}}Q^*$ *is surjective.*

(iv) $\mathcal{W}$ *and* $\mathcal{V}$ *are standard fusion frames.*

In the following result, $\mathcal{F}_i = \{f_{ij}\}_{j\in J_i}$ and $\mathcal{G}_i = \{g_{ij}\}_{j\in J_i}$ are standard Bessel sequences for E_i such that the sequence of their upper bounds is bounded from above.

Theorem 4.5. *If Q is defined by*

$$Q(\{h_i\}_{i\in I}) = \left\{\sum_{j\in J_i}\langle h_i, f_{ij}\rangle g_{ij}\right\}_{i\in I},$$

for each $\{h_i\}_{i\in I} \in \oplus_{i\in I}E_i$, then $Q \in \mathfrak{L}(\oplus_{i\in I}E_i)$ and the following statements are equivalent:

(i) Γ *is a $Q-$approximate g-dual (resp. $Q-g$-dual) of* Λ.

(ii) $\mathcal{G} = \{\Gamma_i^* g_{ij}\}_{i\in I,j\in J_i}$ *is an approximate dual (resp. a dual) of* $\mathcal{F} = \{\Lambda_i^* f_{ij}\}_{i\in I,j\in J_i}$.

Proof. For each $i \in I$, we have

$$S_{\mathcal{G}_i\mathcal{F}_i}h_i = \sum_{j\in J_i} \langle h_i, f_{ij}\rangle g_{ij} = T_{\mathcal{G}_i}T^*_{\mathcal{F}_i}h_i,$$

where $T_{\mathcal{G}_i}$ is the synthesis operator of $\mathcal{G}_i$. Hence

$$\|S_{\mathcal{G}_i\mathcal{F}_i}\| \leq \sqrt{B_{\mathcal{G}_i}B_{\mathcal{F}_i}} \leq \sqrt{BD},$$

where $B = \sup_{i\in I}\{B_{\mathcal{F}_i}\}$ and $D = \sup_{i\in I}\{B_{\mathcal{G}_i}\}$. Thus

$$\sum_{i\in I} \langle S_{\mathcal{G}_i\mathcal{F}_i}h_i, S_{\mathcal{G}_i\mathcal{F}_i}h_i\rangle \leq B_{\mathcal{F}_i}B_{\mathcal{G}_i}\sum_{i\in I}\langle h_i, h_i\rangle \leq BD\,\|\{h_i\}_{i\in I}\|_2^2 .$$

Therefore $Q(\{h_i\}_{i\in I}) = \{S_{\mathcal{G}_i\mathcal{F}_i}h_i\}_{i\in I}$ is well-defined with $\|Q\| \leq \sqrt{BD}$. Now for each $f \in H$

$$T_\Gamma Q T^*_\Lambda f = T_\Gamma\left(\sum_{j\in J_i}\langle f, \Lambda^*_i f_{ij}\rangle g_{ij}\right) = \sum_{i\in I}\sum_{j\in J_i}\langle f, \Lambda^*_i f_{ij}\rangle \Gamma^*_i g_{ij} = S_{\mathcal{G}\mathcal{F}}f.$$

Thus $\|T_\Gamma Q T^*_\Lambda - Id_E\| < 1$ (resp. $T_\Gamma Q T^*_\Lambda = Id_E$) if and only if $\|S_{\mathcal{G}\mathcal{F}} - Id_E\| < 1$ (resp. $S_{\mathcal{G}\mathcal{F}} = Id_E$). □

In the following corollary, which is a generalization of Theorem 3.12 in [8] and Proposition 3.3 in [15] to the approximate duality of fusion frames in Hilbert C^*-modules, $\mathcal{F}_i = \{f_{ij}\}_{j\in J_i}$ and $\mathcal{G}_i = \{g_{ij}\}_{j\in J_i}$ are standard Bessel sequences for W_i and V_i, respectively such that the sequence of their upper bounds is bounded above.

Corollary 4.6. *Suppose that Q is defined by*

$$Q(\{h_i\}_{i\in I}) = \left\{\sum_{j\in J_i}\langle h_i, f_{ij}\rangle g_{ij}\right\}_{i\in I},$$

for each $\{h_i\}_{i\in I} \in \ell^2(I,E)$. Then $Q \in \mathfrak{L}(\ell^2(I,E))$ and the following conditions are equivalent:

(i) $\{\upsilon_i g_{ij}\}_{i\in I, j\in J_i}$ *is an approximate dual of* $\{\omega_i f_{ij}\}_{i\in I, j\in J_i}$.
(ii) $\mathcal{V} = \{(V_i, \upsilon_i)\}_{i\in I}$ *is a $Q-$approximate dual of* $\mathcal{W} = \{(W_i, \omega_i)\}_{i\in I}$.

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