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# Proximity Point Properties for Admitting Center Maps

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Abstract. In this work we investigate a class of admitting center maps on a metric space. We state and prove some fixed point and best proximity point theorems for them. We obtain some results and relevant examples. In particular, we show that if $X$ is a reflexive Banach space with the Opial condition and $T : C \to X$ is a continuous admiting center map, then $T$ has a fixed point in $X$. Also, we show that in some conditions, the set of all best proximity points is nonempty and compact.

## 1. Introduction

Finding fixed points for certain mappings is one of the important issues in the fixed point theory and plays an important role in nonlinear analysis and applied mathematical analysis; see [2, 11] and references therein.

The concept of center of a map was introduced and discussed on Banach spaces by García-Falset et al. in [8] and subsequently, this study was taken up in [4].

Let $C$ be a subset of a metric space $(X, d)$. We say that $y_0 \in X$ is a center for a map $T : C \to X$, if for all $x \in C$, we have

$$d(Tx, y_0) \leq d(x, y_0). \tag{1.1}$$

The map $T$ is called an admitting center map with center $y_0$. The point $y_0 \in X$ is called the strict center for the mapping $T : C \to X$ if for any

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$x \in C$ such that $x \neq T(x)$, we have

$$d(Tx, y_0) < d(x, y_0).$$

The set of all centers of $T$ is denoted by

$$Z(T) := \{y_0 \in X : \|Tx - y_0\| \leq \|x - y_0\|, \forall x \in C\}.$$

**Remark 1.1.** The inequality (1.1) may be satisfied even for a nonexpansive fixed point free mapping. For example (see [8]), consider the affine Beal's mapping. Let $c_0$ be the space of all real sequences converging to 0 with the supremum norm. Let $B$ be the unit ball of $c_0$ and $T : B \to c_0$ is defined by

$$T(x_1, x_2, \ldots) = (1, x_1, x_2, \ldots).$$

Take $y_0 = (2, 0, 0, \ldots)$. It can be easily seen that $T$ satisfies inequality (1.1). So for any $x = (x_1, x_2, \ldots) \in B$,

$$\begin{aligned} \|Tx - y_0\| &= \|(-1, x_1, x_2, \ldots)\| \\ &= 1 \\ &\leq 2 - x_1 \\ &= \|x - y_0\|. \end{aligned}$$

Although $y_0$ is not a fixed point of the nonexpansive map $T$, but $y_0 \in c_0$ is a center of $T$.

It may be pointed out that $T : C \to C$ is quasi nonexpansive provided that $T$ has at least one fixed point in $C$ and every fixed point is a center for $T$. It turns out that the class of all admitting center maps contains all contraction maps defined on closed subsets of Banach spaces and even all the so-called quasi nonexpansive mappings introduced by Tricomi for real functions and further studied by Diaz and Metcalf [**?** ] and Dotson [6] for mappings on Banach spaces. It is not hard to see that the class of quasi nonexpansive mappings properly contains the class of nonexpansive maps having fixed points, although there exists a continuous admitting center map that is not quasi nonexpansive.

Our purpose is to investigate the class of all mappings admitting a center. This class contains nonexpansive mappings having fixed points, although there are nonquasi nonexpansive maps that admit a center (see [8]).

**Remark 1.2** ([8])**.** If $T : C \to X$ has a center $y_0 \in C$, then trivially $T(y_0) = y_0$. Thus fixed point results for mappings admitting centers are nontrivial provided they have a center $y_0 \in X \setminus C$.

In the following, we obtain a Lipschitzian mapping with unique fixed point, which has not a center.

**Example 1.3.** Let $T : \left[\frac{1}{2}, 2\right] \to \mathbb{R}$ be a mapping given by $T(x) = \frac{1}{x}$. Since the derivative $T'(x) = \frac{-1}{x^2}$ is bounded on $C := \left[\frac{1}{2}, 2\right]$ , the mapping $T$ is a Lipschitzian map on $C$. It is obvious that $x_0 = 1$ is the an unique fixed point of $T$, but $x_0$ is not a center for $T$ because

$$\begin{aligned}\left|T\left(\frac{1}{2}\right) - 1\right| &= 1 \\ &> \left|\frac{1}{2} - 1\right| \\ &= \frac{1}{2}.\end{aligned}$$

Furthermore, if $y_0 > 2$, then $|T(2) - y_0| = |1/2 - y_0| > |2 - y_0|$, and thus $y_0$ is not a center for $T$. If $y_0 < 1/2$, then $|T(1/2) - y_0| = |2 - y_0| > |1/2 - y_0|$. Finally, if $y_0 \in C$, then we must have $|T(y_0) - y_0| \leq |y_0 - y_0|$ and so $y_0 = 1$, which is not a center of $T$. Therefore, $Z(T) = \emptyset$.

Let $A$ and $B$ be nonempty subsets of normed space $(X, \|.\|)$. Put

$$\begin{aligned}d(A, B) &= \inf\{\|x - y\| : x \in A, y \in B\}, \\ A_0 &= \{x \in A : d(x, y) = d(A, B), \text{ for some } y \in B\}, \\ B_0 &= \{y \in B : d(x, y) = d(A, B), \text{ for some } x \in A\}.\end{aligned}$$

We can find the best proximity points of the set $A$, by considering a map $T : A \to B$. We say that $x \in A$ is a best proximity point of the pair $(A, B)$, if $d(A, B) = d(x, Tx)$ and the set of all best proximity points of $(A, B)$ denoted by $P_T(A)$, i.e.

$$P_T(A) := \{x \in A : d(x, Tx) = d(A, B)\}.$$

The notion of the best proximity point is an important tool for solving some optimization equations and many authors have been worked on it [9, 10, 13–15].

Our main results are in two sections. In Section 2, we give some fixed point theorems for the admitting center maps. In section 3, we introduce a center for a mapping $T : A \to B$ and proximity point property for $(A, B)$, a pair of two nonempty subsets of the Banach space $X$, and show that if $(A, B)$ has a proximity point property for continuous mappings admitting a center, then the set of all best proximity points is nonempty and compact.

## 2. Fixed Point Theorems for Admitting Center Maps

In this section, we state and prove some fixed point theorems for admitting center maps. Let $C$ be a nonempty subset of a Banach space

$X$. For $x \in C$, the inward set of $x$ relative to $C$ is the set

$$I_C(x) = \{x + t(y - x) : y \in C \text{ and } t \geq 0\}.$$

Let $C$ be a nonempty subset of a Banach space $X$ and $T : C \to X$ a mapping. Then $T$ is said to be a weakly inward map, if $Tx \in \overline{I_C(x)}$, for all $x \in C$.

Let $C$ be a nonempty subset of a metric space $(X, d)$. A mapping $T : C \to X$ is said to satisfy the Lipschitz admitting center condition on $C$, if there exists a constant $L > 0$ and $y_0 \in X$ such that

$$d(Tx, y_0) \leq Ld(x, y_0), \forall x \in C.$$

If $L$ is the smollest number for which the Lipschitz admitting center condition holds, then $L$ is called the Lipschitz admitting center constant. In this case, we say that $T$ is an $L$-Lipschitz admitting center mapping or simply a Lipschitzian admitting center map with the Lipschitz constant $L$. An $L$-Lipschitz admitting center map $T$ is said to be a contraction admitting center if $L < 1$ and an admitting center if $L = 1$. The mapping $T$ is said to be a strict admitting center if

$$d(Tx, y_0) < Ld(x, y_0), \forall x \in C.$$

**Lemma 2.1.** *Let $C$ be a nonempty closed convex subset of a Banach space $X$ and mapping $T : C \to X$ admits a center $y_0 \in X$, then there exists a sequence $\{x_n\} \subseteq C$ such that $x_n - Tx_n \to 0$ as $n \to \infty$.*

*Proof.* Let $t \in (0, 1)$. The mapping $T_t : C \to X$ defined by $T_t x = (1 - t)y_0 + tTx$ is a contraction and has a fixed point $x_t$ in $C$. Now the result follows by Theorems 4.1.3 and 5.1.2 in [3] and Propositions 5.1.1 and 5.2.1 in [3]. □

We need the following definition of [3].

**Definition 2.2.** Let $C$ be a nonempty subset of a Banach space $X$ and $T : C \to X$ be a map. Then $T$ is said to be demiclosed at $v \in X$, if for any sequence $\{x_n\}$ in $C$ the following implication holds:

$$x_n \rightharpoonup u \in X \text{ and } Tx_n \to v \text{ imply } Tu = v.$$

A normed space $X$ is said to be satisfied the Opial's condition if for any sequence $\{x_n\} \subseteq X$ weakly convergent to $x \in X$, the inequality

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|, \tag{2.1}$$

holds, for all $y \in X$, not equal to $x$.

**Theorem 2.3.** *Let $X$ be a Banach space satisfies the Opial condition, $C$ a nonempty weakly compact subset of $X$, and $T : C \to X$ a mapping admitting a center $y_0 \in X$. Then the mapping $I - T$ is demiclosed at zero.*

*Proof.* Let $\{x_n\}$ be a sequence in $Z(T) \neq \emptyset$ such that $x_n \rightharpoonup x \in X$ and $(I-T)x_n \to 0$. We show that $(I-T)x = 0$.
Let $x \neq Tx$. The Opial condition implies that

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - Tx\|.$$

Since $x_n \in Z(T)$, we have

$$\|x_n - Tx\| \leq \|x_n - x\|,$$

which is a contradiction. Therefore, $(I-T)x = 0$. □

**Theorem 2.4.** *Let $X$ be a reflexive Banach space with the Opial condition. Let $C$ be a nonempty closed convex bounded subset of $X$ and $T : C \to X$ a continuous mapping admitting a center $y_0 \in X$. Then $T$ has a fixed point in $X$.*

*Proof.* By Lemma 2.1, there exists a sequence $\{x_n\}$ in $C$ such that $\|x_n - Tx_n\| \to 0$. By the reflexivity of $X$, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \rightharpoonup x \in X$. By Theorem 2.3, $I - T$ is demiclosed at zero, i.e., $x_{n_k} \rightharpoonup x \in X$ and $x_{n_k} - Tx_{n_k} \to 0$ imply $x - Tx = 0$. Therefore, $x$ is a fixed point of $T$. □

In the following, we give nonexpansive and non-Lipschitzian, $J$-type mappings.

**Example 2.5.** For $n \in \mathbb{N}$, the mappings $T_n : [0,1] \to [0,1]$ given by $T_n(x) = x^n$ admits the point $y_0 \leq 0$ as a center. Of course, for $n \geq 2$, $T_n$ is not nonexpansive on $[0,1]$. Note that any $T_n$ has two fixed points in $[0,1]$, namely $y_1 = 1$ and $y_2 = 0$. While $y_2$ is a center for $T_n$, $y_1$ is not a center for $T_n$ and hence such mappings can not be quasi nonexpansive. On the other hand, the non-Lipschitzian mapping $T : [0,1] \to [0,1]$ given by $T(x) = x^{\frac{1}{2}}$ admits the fixed point $y_1 = 1$ as a center.

**Example 2.6.** The mapping $T : \mathbb{R} \to \mathbb{R}$ defined by

$$T(x) = \begin{cases} x - \frac{x^2-1}{x^4+2}, & |x| \geq 1, \\ x, & \text{otherwise}, \end{cases}$$

is not nonexpansive on $\mathbb{R}$ (see $x = 2, y = 3$). But $T$ has $x = \pm 1$ as fixed points in $|x| \leq 1$, and admits at $x = 0$ as a center.

**Theorem 2.7.** *Let $C$ be a nonempty closed subset of a compact metric space $(X, d)$ and $T : C \to C$ be a continuous map admitting a strict center at $y_0 \in X$. Then $T$ has a fixed point $v$ in $C$.*

*Proof.* For each $x \in C$, define $\varphi : C \to \mathbb{R}^+ \cup \{0\}$ by $\varphi(x) = d(y_0, Tx)$. Then $\varphi$ is continuous on $C$ and by compactness of $C$, $\varphi$ attains its minimum on $C$. Let $\varphi(v) = \min_{x\in C} \varphi(x)$. If $v \neq Tv$, then

$$\varphi(Tv) = d(y_0, TTv)$$

$$< d(y_0, Tv)$$
$$= \varphi(v),$$

which contradicts the minimality of $\varphi(v)$. Hence $v = Tv$. □

## 3. Best Proximity Point for Admitting Center Maps

We start with a new definition of center for a map and then we discuss the proximity point property for admitting center maps.

**Definition 3.1.** Let $(A, B)$ be a pair of two nonempty bounded closed convex subsets of a normed space $(X, \|.\|)$. A pair $(a, b) \in A \times B$ is said to be a center for a mapping $T : A \to B$, if for each $x \in A$, we have

$$\|Tx - b\| \leq \|x - a\|.$$

**Definition 3.2.** Let $(A, B)$ be the pair of two nonempty subsets of $X$. We say that $(A, B)$ has a proximity point property if for every continuous admitting center map $T : A \to B$, the pair $(A, B)$ has a best proximity point.

Let $M$ be a subset of a normed space $(X, \|.\|)$. We remember that a point $g_0 \in M$ is said to be a best coapproximation of $x \in X$, if $\|g_0 - g\| \leq \|x - g\|$, for all $g \in M$. Let

$$R_M(x) = \{g_0 \in M : \|g_0 - g\| \leq \|x - g\|, \forall g \in M\},$$

be the set of all best coapproximations of $x \in X$. The set $M$ is called coproximinal in $X$ if $R_M(x)$ is nonempty for any $x \in X$. If $R_M(x)$ is singleton for any $x \in X$, then $M$ is the called co-Chebyshev (see [12]).

In the following we give an important theorem that is generalization of Theorem 3.3, [4] we give some conditions on $T$ and $A$ so that $P_T(A)$ be a nonempty compact set.

**Theorem 3.3.** *Let $X$ be a Banach space and $A, B$ be nonempty closed, bounded and convex subsets of $X$ such that $A_0$ is co-Chebyshev. If $(A, B)$ has the proximity point property for continuous admitting center maps $T : A \to B$, then $P_T(A)$ is nonempty and compact.*

*Proof.* On the contrary, suppose that there exists $B \subseteq X$ such that either $P_T(A)$ is noncompact or $P_T(A) = \emptyset$. In the first case, there exists a nonexpansive map $S : A_0 \to B_0$ without any best proximity points. Since $A_0$ is a co-Chebyshev set, there exists a continuous mapping $r : A \to A_0$ such that $r(x) = x$, for all $x \in A_0$. Define $T : A \to B_0$ by $T(x) = S(r(x))$. Clearly $T$ is a continuous map. Moreover,

$$\|T(x) - S(y)\| = \|S(r(x)) - S(y)\|$$
$$\leq \|r(x) - y\|$$

$$\leq \|x-y\|,$$

i.e., $(y, S(y))$ is a center for $T$. Therefore, $T$ has a best proximity point $x \in P_T(A) \subseteq A_0$. Hence

$$\begin{aligned} d(A, B_0) &= \|x - T(x)\| \\ &= \|x - S(r(x))\| \\ &= \|x - S(x)\|, \end{aligned}$$

which contradicts the fact that $S$ has no best proximity point.

Now for the case, $P_T(A) = \emptyset$, we proceed as follows. Let $d := d(A, B) > 0$. We take $a > 0$ such that

$$a + d < \sup\{\|x-y\| : x \in A, y \in B\}.$$

For each $m \in \mathbb{N}$, we consider the following nonempty sets:

$$B_m = \mathcal{B}\left[A, d + \frac{a}{m}\right] \cap B, \qquad A_m = \mathcal{B}\left[B, d + \frac{a}{m}\right] \cap A,$$

where $\mathcal{B}[B, r] := \{x \in X : \inf_{y \in B} \|y - x\| < r\}$. Set

$$B'_m := B_m \setminus B_{m+1}, \qquad A'_m := A_m \setminus A_{m+1},$$

$$S_m := \left\{x \in B : \inf_{y \in A} \|x - y\| = d + \frac{a}{m}\right\}.$$

Since $A_0 = B_0 = \emptyset$, we have

$$A_1 = \bigcup_{m=1}^{\infty} A'_m, \qquad B_1 = \bigcup_{m=1}^{\infty} B'_m.$$

Fix an arbitrary $y_1 \in S_1$ and by induction, define a sequence $\{y_m\}$ such that $y_m \in S_m$ and the segment $(y_{m+1}, y_m]$ does not meet $B_{m+1}$.

For $x \in A_1$ there exists a unique positive integer $n$ such that $x \in A'_n$. Also there exists an unique $y \in B_1$ such that $d(x, B) = d(y, A)$, $\|x - y\| = 2d(x, B) - d(A, B)$. In this case, we define

$$S(x) = \frac{d(y, A) - (d + \frac{a}{m+1})}{\frac{a}{m(m+1)}} y_{m+1} + \left(1 - \frac{d(y, A) - (d + \frac{a}{m+1})}{\frac{a}{m(m+1)}}\right) y_{m+2}.$$

It is routine to check that $S$ is a continuous mapping from $A_1$ to $B_1$. Furthermore, $S(A'_m) \subset (y_{m+2}, y_{m+1}] \subset B'_{m+1}$, for any $m \geq 1$.

Let $r$ be a continuous retraction from $A$ into the closed convex subset $A_1$. We can define $T : A \to B$ by $T(x) = S(r(x))$. Hence $T$ has a center without any best proximity point. □

**Theorem 3.4.** *Let $A, B$ be weakly compact convex subsets of a Banach space $X$. Then $A_0$ is a nonempty weakly compact set.*

*Proof.* Since $A, B$ are weakly compact convex sets and $(x, y) \mapsto \|y-x\|$ is a continuous function on $A \times B$, hence $A_0$ is nonempty and closed. Now we show that $A_0$ is a convex set. Let $x_1, x_2 \in A_0$ and $\lambda \in (0,1)$. So there exist $y_1, y_2 \in B$ such that $\|x_1 - y_1\| = d(A, B) = \|x_2 - y_2\|$. Since $A$ and $B$ are convex sets, we have $\lambda x_1 + (1-\lambda)x_2 \in A$ and $\lambda y_1 + (1-\lambda)y_2 \in B$. Thus

$$\begin{aligned} d(A,B) &\leq \|\lambda x_1 + (1-\lambda)x_2 - (\lambda y_1 + (1-\lambda)y_2)\| \\ &= \|\lambda(x_1 - y_1) + (1-\lambda)(x_2 - y_2)\| \\ &\leq \lambda\|x_1 - y_1\| + (1-\lambda)\|x_2 - y_2\| \\ &= d(A,B). \end{aligned}$$

That is $\|\lambda x_1 + (1-\lambda)x_2 - (\lambda y_1 + (1-\lambda)y_2)\| = d(A,B)$. Thus $A_0$ is a convex set. Therefore $A_0$ is weakly compact. □

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