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A Proposed Preference Index For Ranking Fuzzy Numbers Based On α-Optimistic Values

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Abstract. In this paper, we propose a novel method for ranking a set of fuzzy numbers. In this method a preference index is proposed based on α-optimistic values of a fuzzy number. We propose a new ranking method by adopting a level of credit in the ordering procedure. Then, we investigate some desirable properties of the proposed ranking method.

1. Introduction

Ranking of fuzzy numbers is a critical task in linguistic decision making and some other fuzzy application systems such as decision-making, data analysis, artificial intelligence, socioeconomic systems, statistical procedures, and etc. The problem of ordering fuzzy sets has received considerable attention in the fuzzy set theory literature. For instance, many statistical hypothesis tests with fuzzy information (imprecise observations, fuzzy hypotheses, fuzzy significance level, and etc) are based on the comparison of a observed p-value and a given level of significance. On the other hand, in many statistical procedures, vague statistical data takes the form of fuzzy numbers that need to be ordered. Many fuzzy ranking indices have been proposed since 1976. Because of the nature of the measurement, there are many different strategies have been proposed for ranking of fuzzy numbers. These include methods based on the coefficient of variation, distance measure, centroid point and original point, and weighted mean value, preference degree, and so on. Ranking fuzzy numbers were first proposed by Jain [20] for decision

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making in fuzzy environment. Since then, various procedures to rank fuzzy quantities are proposed by various researchers. Bortolan and Degani [5] considered the problem of ranking n fuzzy sets. They reviewed some methods suggested in the literature [2, 4, 5, 15, 46–48] on a group of selected examples. Chen [7] presented ranking fuzzy numbers with maximizing set and minimizing set to decide the ordering value of each fuzzy number and uses these values to determine the order of the n fuzzy numbers. They also gave a method for calculating the ordering value of each fuzzy number with triangular, trapezoidal, and a typical shaped membership functions. Dubois and Prade [15] developed the most known indices for ranking fuzzy numbers; the Pos index and its dual Nec index. Lee and Li [24] compared fuzzy numbers based on the probability measure of fuzzy events. Delgado et al. [13] proposed a procedure for ranking fuzzy numbers. Campos and Muñoz [12] presented a subjective approach for ranking fuzzy numbers. Kim and Park [21] introduced a method for comparing fuzzy numbers based on the combination of maximizing possibility and minimizing possibility using an index of optimism in $[0, 1]$ reflecting the decision makers risk taking attitude. Yuan [50] proposed a criterion for evaluating fuzzy numbers. He also investigated the proposed method based on some criteria for evaluating fuzzy ranking methods: fuzzy preference representation, rationality of fuzzy ordering, distinguishably, and robustness. Saade and Schwarzlander [33] presented the ordering of fuzzy sets over the real line from the point of view of ordering intervals. Liou and Wang [26] compared fuzzy numbers with integral value. They also used an index of optimism to reflect the decision makers optimistic attitude in their proposed ranking method. Chang and Lee [6] formalized the concept of existence for the ranking of fuzzy sets. Since then several methods have been proposed by various researchers which includes ranking fuzzy numbers using area compensation, distance method, maximizing and minimizing set, decomposition principle, and signed distance [10, 17, 32, 49]. Wang and Kerre [39, 40] classified all the above ranking procedures into three classes. The first class consists of ranking procedures based on fuzzy mean and spread [2, 12, 17, 26, 46–48], and the second class consists ranking procedures based on fuzzy scoring [7, 21, 38], whereas the third class consists methods based on preference relations [4, 13, 15, 22, 30, 33, 50]. They concluded that the ordering procedures associated with the first class are relatively reasonable for the ordering of fuzzy numbers. Specifically, the ranking procedure presented by Adamo [2] satisfies all the reasonable properties for the ordering of fuzzy quantities. Moreover, the methods presented in the second class are not doing well while the methods [22, 30, 33, 50] which belong to class three are reasonable.

Later on, ranking fuzzy numbers has been developed using numerous different techniques including preference ratio [29], left and right dominance [9], fuzzy distance measure [35], area between the centroid point and original point [11], preference weighting function expectations [28], sign distance [1], fuzzy simulation analysis method [34], an area method using radius of gyration [14], distance minimization [3], and fuzzy risk analysis based on the ranking of generalized trapezoidal fuzzy numbers [8]. More development in ranking fuzzy numbers can also be found in [16, 25, 36, 37, 42–45].

The purpose of this paper is to provide a new method of ranking of fuzzy numbers based on α-optimistic values by adopting a level of credit in the ordering procedure.

The rest of the paper is organized as follows: In Section 2, we recall some necessary concepts related to fuzzy numbers and some properties of α-optimistic values. Section 3 presents a new method for ranking fuzzy numbers. In addition, we investigate some basic properties of the proposed method. Concluding remarks are finally made in Section 4.

2. Preliminaries

A fuzzy set $\tilde{A}$ of the universal set $\mathbb{X}$ is defined by its membership function $\mu_{\tilde{A}} : \mathbb{X} \to [0,1]$. In this paper, we consider $\mathbb{R}$ (the real line) as the universal set. We denote by $\tilde{A}[\alpha] = \left\{x \in \mathbb{R} : \mu_{\tilde{A}}(x) \geq \alpha\right\}$ the α-level set (α-cut) of the fuzzy set $\tilde{A}$, for every $\alpha \in (0,1]$, and for $\alpha = 0, \tilde{A}[0]$ is the closure of the set $\left\{x \in \mathbb{R} : \mu_{\tilde{A}}(x) > 0\right\}$. A fuzzy set $\tilde{A}$ of $\mathbb{R}$ is called a fuzzy number if for every $\alpha \in [0,1]$, the set $\tilde{A}[\alpha]$ is a non-empty compact interval and there exists unique $x^* \in \mathbb{R}$ such that $\mu_{\tilde{A}}(x^*) = 1$. Such an interval will be denoted by $\tilde{A}[\alpha] = [\tilde{A}^L_\alpha, \tilde{A}^U_\alpha]$, where $\tilde{A}^L_\alpha = \inf\left\{x : x \in \tilde{A}[\alpha]\right\}$ and $\tilde{A}^U_\alpha = \sup\left\{x : x \in \tilde{A}[\alpha]\right\}$. A fuzzy number $\tilde{A}$ is said to be positive whenever $\tilde{A}[0] \subseteq (0, \infty)$. We denote by $\mathcal{F}(\mathbb{R})$, the set of all fuzzy numbers of $\mathbb{R}$.

The imprecision or vagueness can be treated by means of a particular kind (family) of fuzzy numbers, the LR-fuzzy numbers since they can be characterized by means of three real numbers: the center, the left spread, and the right spread. The term LR is due to the left (L) and the right (R) shape of the membership function referred to the fuzzy set [23].

Definition 2.1. A LR-fuzzy number $\tilde{A} = \left(a, a^l, a^r\right)_{LR}, a^l, a^r \geq 0$, is defined as follows

$$\mu_{\tilde{A}}(x) = \begin{cases} L\left(\frac{a-x}{a^l}\right), & a - a^l \leq x \leq a, \\ \\ R\left(\frac{a-x}{a^r}\right), & a < x \leq a + a^r, \end{cases}$$

where a^l and a^r are the left-hand and the right-hand spreads, respectively. L and R are continuous and strictly decreasing functions with $L(0) = R(0) = 1$ and $L(1) = R(1) = 0$.

A special type of LR-fuzzy numbers is the so-called triangular fuzzy numbers with the shape functions $L(x) = R(x) = \max\{0, 1 - |x|\}$, $x \in \mathbb{R}$. The membership function of triangular fuzzy number $\tilde{A} = \left(a, a^l, a^r\right)_T$ is given by

$$\mu_{\tilde{A}}(x) = \begin{cases} 0, & x < a - a^l, \\ \\ \frac{x-\left(a-a^l\right)}{a^l}, & a - a^l \leq x \leq a, \\ \\ \frac{a+a^r-x}{a^r}, & a \leq x \leq a + a^r, \\ \\ 0, & x > a + a^r. \end{cases}$$

Here, we recall an index introduced by Liu [27] for degree of belonging a fuzzy number $\tilde{A} \in \mathcal{F}(\mathbb{R})$ to a set of real numbers. We will apply this index to introduce a method of ranking of fuzzy numbers.

Definition 2.2. Let $\tilde{A} \in \mathcal{F}(\mathbb{R})$ and $B \subseteq \mathbb{R}$. The index

$$C : \mathcal{F}(\mathbb{R}) \times \mathbb{R} \longrightarrow [0, 1],$$

which is defined by

$$C\left(\tilde{A} \in B\right) = \frac{\sup_{y \in B} \mu_{\tilde{A}}(y) + 1 - \sup_{y \notin B} \mu_{\tilde{A}}(y)}{2}, \tag{2.1}$$

shows the credibility degree that "$\tilde{A}$ belongs to B ".

Remark 2.3. It is readily seen that if a fuzzy number $\tilde{A}$ reduces to the crisp number a, then $C(\tilde{A} \in B) = I[a \in B]$, where I is the indicator function,

$$I(\rho) = \begin{cases} 1, & \text{if } \rho \text{ is true}, \\ \\ 0, & \text{if } \rho \text{ is false}. \end{cases}$$

Definition 2.4 ([27])**.** Let $\tilde{A} \in \mathcal{F}(\mathbb{R})$ and $\alpha \in [0, 1]$, then

$$\tilde{A}_\alpha = \sup\left\{x \in \tilde{A}[0] : C(\tilde{A} \in [x, \infty)) \geq \alpha\right\}, \tag{2.2}$$

is called the α-optimistic value of $\tilde{A}$. It is clear that $\tilde{A}_\alpha$ is a non-increasing function of $\alpha \in [0,1]$.

Example 2.5. For a given LR-fuzzy number $\tilde{A} = \left(a, a^l, a^r\right)_{LR}$, it is easy to verify that

$$\tilde{A}_\alpha = \begin{cases} a + a^r R^{-1}(2\alpha), & \text{for } 0.0 \le \alpha \le 0.5, \\ \\ a - a^l L^{-1}(2(1-\alpha)), & \text{for } 0.5 \le \alpha \le 1.0. \end{cases}$$

Especially, if $\tilde{A} = \left(a, a^l, a^r\right)_T$ is a triangular fuzzy number, then

$$\tilde{A}_\alpha = \begin{cases} a^r - 2\alpha(a^r - a), & \text{for } 0.0 \le \alpha \le 0.5, \\ \\ 2a - a^l - 2\alpha(a - a^l), & \text{for } 0.5 \le \alpha \le 1.0. \end{cases}$$

For instance, the α-optimistic values of $\tilde{A} = (-2, 0, 1)_T$ are obtained as

$$\tilde{A}_\alpha = \begin{cases} 1 - 2\alpha, & \text{for } 0.0 \le \alpha \le 0.5, \\ \\ 2 - 4\alpha, & \text{for } 0.5 \le \alpha \le 1.0. \end{cases}$$

3. The Proposed Approach to Raking Fuzzy Numbers

This section proposes a new method for ranking fuzzy numbers based on a novel preference index.

In general, for a proposed ranking method (based on a given preference index P), we expect the following axioms:

- We expect that the proposed ranking method saves the classical properties of the ordinary ordering of real numbers on the space of fuzzy numbers.
- Let $\tilde{A}, \tilde{B} \in \mathcal{F}(\mathbb{R})$, where $\tilde{A}_\alpha \ge \tilde{B}_{1-\alpha}$ for any $\alpha \in [0,1]$. Then, it is natural to say that $\tilde{A}$ is absolutely "preferred to" $\tilde{B}$.
- Let $\tilde{A}^*$ be a estimation of $\tilde{A}$. If $\tilde{A}$ is preferred to $\tilde{B}$, then we expect that $\tilde{A}^*$ is also nearly preferred to $\tilde{B}$.
- If $\tilde{A}^*$ is preferred to $\tilde{B}$, then $\tilde{B}$ can not be preferred to $\tilde{A}$.
- A preference index should not be very sensitive to a minor estimation error in the estimation of fuzzy membership functions, i.e., the degree of the preference between two fuzzy numbers should not be changed if the change of the membership function is sufficiently small.

So, we need a suitable relation to investigate the degree to which $\tilde{A}$ is "preferred to" $\tilde{B}$. In the following, we propose a preference relation for such cases in the space of fuzzy numbers. In this method, we consider decision makers attitude by considering a level of credit.

Definition 3.1. For two fuzzy numbers $\tilde{A}$ and $\tilde{B}$, the preference index $P : \mathcal{F}(\mathbb{R}) \times \mathcal{F}(\mathbb{R}) \longrightarrow \mathbb{R}$ is defined as

$$P\left(\tilde{A} \succ \tilde{B}\right) = \begin{cases} \sup E\left(\tilde{A}, \tilde{B}\right), & \text{if } E\left(\tilde{A}, \tilde{B}\right) \neq \emptyset, \\ 0, & \text{if } E\left(\tilde{A}, \tilde{B}\right) = \emptyset, \end{cases} \tag{3.1}$$

where $E\left(\tilde{A}, \tilde{B}\right) = \left\{\alpha \in [0,1] : \tilde{A}_\alpha \geq \tilde{B}_{1-\alpha}\right\}$. $P\left(\tilde{A} \succ \tilde{B}\right)$ represents the degree of preference of $\tilde{A}$ over $\tilde{B}$.

Remark 3.2. It is worth noting that $P\left(\tilde{A} \succ \tilde{B}\right) = 1$ if and only if $\tilde{B}_{1-\alpha} \leq \tilde{A}_\alpha$, for every $\alpha \in [0,1]$.

Definition 3.3. For two fuzzy numbers $\tilde{A}$ and $\tilde{B}$, we say that $\tilde{A}$ is preferred to $\tilde{B}$ at credit level of $\gamma \in [0.5, 1]$ (denoting by $\tilde{A} \succ_P^\gamma \tilde{B}$ if $P\left(\tilde{A} \succ \tilde{B}\right) \geq \gamma$.

It is mentioned that the level of credit $\gamma \in [0.5, 1]$ represents a decision maker's attitude. A larger γ indicates a higher degree of credit. The calculation of $P\left(\tilde{A} \succ \tilde{B}\right)$ is graphically illustrated in Fig. 1.

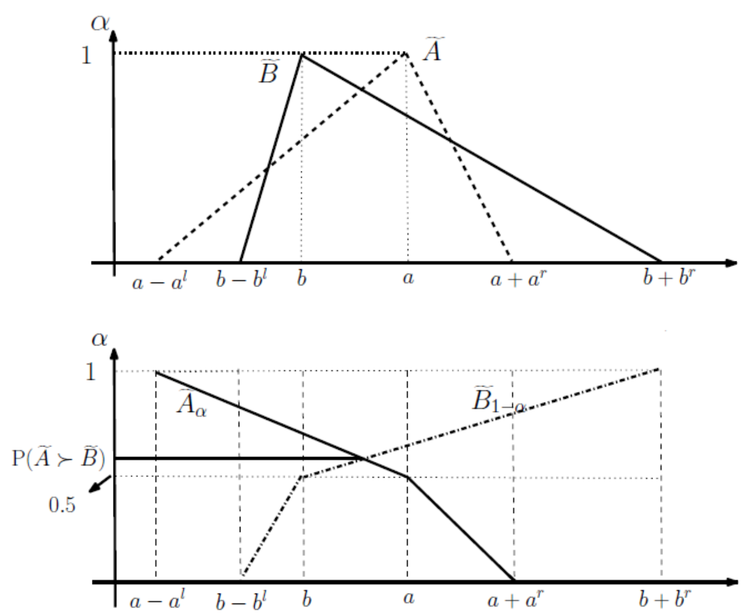

FIGURE 1. An illustration of calculating $P\left(\tilde{A} \succ \tilde{B}\right)$ for two typical triangular fuzzy numbers $\tilde{A}$ and $\tilde{B}$.

Theorem 3.4. *Let $\tilde{A}$, $\tilde{B}$ and $\tilde{C}$ be three fuzzy numbers. Then*

1) $P\left(\tilde{A} \succ \tilde{B}\right) = 1 - P\left(\tilde{B} \succ \tilde{A}\right)$.
2) *At a given level of credit $\gamma \in [0.5, 1]$, P is transitive, i.e., $\tilde{A} \succ_P^\gamma \tilde{B}$ and $\tilde{B} \succ_P^\gamma \tilde{C}$ imply $\tilde{A} \succ_P^\gamma \tilde{C}$.*

Proof. For two fuzzy numbers $\tilde{A}$ and $\tilde{B}$, we have

$$\begin{aligned} 1 - P\left(\tilde{A} \succ \tilde{B}\right) &= 1 - \sup\left\{\alpha : \tilde{A}_\alpha \geq \tilde{B}_{1-\alpha}\right\} \\ &= 1 - \inf\left\{\alpha : \tilde{B}_{1-\alpha} \geq \tilde{A}_\alpha\right\} \\ &= \sup\left\{1 - \alpha : \tilde{B}_{1-\alpha} \geq \tilde{A}_{1-(1-\alpha)}\right\} \\ &= \sup\left\{\alpha : \tilde{B}_\alpha \geq \tilde{A}_{1-\alpha}\right\} \\ &= P\left(\tilde{B} \succ \tilde{A}\right). \end{aligned}$$

So item 1) is held. To prove the second item (transitivity property), assume that $\tilde{A} \succ_P^\gamma \tilde{B}$ and $\tilde{B} \succ_P^\gamma \tilde{C}$. It is easy to verify that $\tilde{A} \succ_P^\gamma \tilde{B}$ if and only if $\tilde{A}_\gamma \geq \tilde{B}_{1-\gamma}$ or $d_\gamma\left(\tilde{A}, \tilde{B}\right) = \tilde{A}_\gamma - \tilde{B}_{1-\gamma} \geq 0$. So we have $d_\gamma\left(\tilde{A}, \tilde{B}\right) \geq 0$ and $d_\gamma\left(\tilde{B}, \tilde{C}\right) \geq 0$. Now, since $\tilde{B}_\gamma$ is decreasing with respect to $\gamma \in [0.5, 1]$, it is concluded that $\tilde{B}_{1-\gamma} \geq \tilde{B}_\gamma$ and therefore $d_\gamma\left(\tilde{A}, \tilde{C}\right) \geq 0$ or $\tilde{A} \succ_P^\gamma \tilde{C}$, which is the desired result. □

Remark 3.5 ([50])**.** The linguistic interpretation can be defined as a fuzzy set on the calculated degree of preference, and the membership of the interpretation set indicates the truth level of the interpretation. An illustration is shown in Fig. 2. The nine cases are; $P_1 : \tilde{B}$ is absolutely preferred to $\tilde{A}$. $P_2 : \tilde{B}$ is strongly preferred to $\tilde{A}$. $P_3 : \tilde{B}$ is moderately preferred to $\tilde{A}$. $P_4 : \tilde{B}$ is weakly preferred to $\tilde{A}$. $P_5 : \tilde{A}$ and $\tilde{B}$ are equally preferred to each other. $P_6 : \tilde{A}$ is weakly preferred to $\tilde{B}$. $P_7 : \tilde{A}$ is moderately preferred to $\tilde{B}$. $P_8 : \tilde{A}$ is strongly preferred to $\tilde{B}$. $P_9 : \tilde{A}$ is absolutely preferred to $\tilde{B}$. For example, if $P\left(\tilde{A} \succ \tilde{B}\right) = 0.54$, we may conclude that $\tilde{A}$ is weakly preferred to $\tilde{B}$ with truth level 0.80 and $\tilde{B}$ and $\tilde{A}$ are preferred to each other with truth level 0.20.

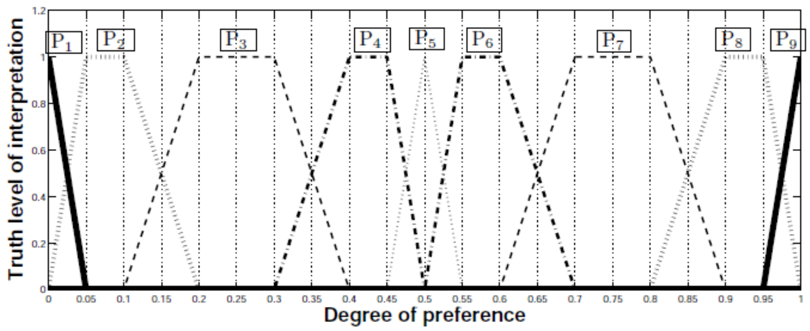

FIGURE 2. An illustration of linguistic interpretation of $P\left(\tilde{A} \succ \tilde{B}\right)$.

Remark 3.6. From Definition 2.4, note that the α-cuts of a fuzzy number $\tilde{A}$ can be rewritten as $\tilde{A}[\alpha] = [\tilde{A}_{1-\alpha/2}, \tilde{A}_{\alpha/2}], \alpha \in [0,1]$. Thus, the interval arithmetic operators on fuzzy numbers (e.g. see [23]) reduce as follows

- Addition: $\tilde{A}[\alpha] \oplus \tilde{B}[\alpha] = \left[\tilde{A}_{1-\alpha/2} + \tilde{B}_{1-\alpha/2}, \tilde{A}_{\alpha/2} + \tilde{B}_{\alpha/2}\right]$.
- Subtraction: $\tilde{A}[\alpha] \ominus \tilde{B}[\alpha] = \left[\tilde{A}_{1-\alpha/2} - \tilde{B}_{\alpha/2}, \tilde{A}_{\alpha/2} - \tilde{B}_{1-\alpha/2}\right]$.
- Multiplication:

$$\tilde{A}[\alpha] \otimes \tilde{B}[\alpha] = \left[\min\left\{\tilde{A}_{1-\alpha/2}\tilde{B}_{\alpha/2}, \tilde{A}_{1-\alpha/2}\tilde{B}_{1-\alpha/2}, \tilde{A}_{\alpha/2}\tilde{B}_{1-\alpha/2}, \tilde{A}_{\alpha/2}\tilde{B}_{\alpha/2}\right\}, \max\left\{\tilde{A}_{1-\alpha/2}\tilde{B}_{\alpha/2}, \tilde{A}_{1-\alpha/2}\tilde{B}_{1-\alpha/2}, \tilde{A}_{\alpha/2}\tilde{B}_{1-\alpha/2}, \tilde{A}_{\alpha/2}\tilde{B}_{\alpha/2}\right\}\right].$$

- Division:

$$\tilde{A}[\alpha] \oslash \tilde{B}[\alpha] = \left[\min\left\{\tilde{A}_{1-\alpha/2}\ /\tilde{B}_{\alpha/2}, \tilde{A}_{1-\alpha/2}\ /\tilde{B}_{1-\alpha/2}, \tilde{A}_{\alpha/2}\ /\tilde{B}_{1-\alpha/2}, \tilde{A}_{\alpha/2}\ /\tilde{B}_{\alpha/2}\right\}, \max\left\{\tilde{A}_{1-\alpha/2}\ /\tilde{B}_{\alpha/2}, \tilde{A}_{1-\alpha/2}\ /\tilde{B}_{1-\alpha/2}, \tilde{A}_{\alpha/2}\ /\tilde{B}_{1-\alpha/2}, \tilde{A}_{\alpha/2}\ /\tilde{B}_{\alpha/2}\right\}\right],$$

excluding the case $\tilde{B}_{1-\alpha/2} = 0$ or $\tilde{B}_{\alpha/2} = 0$.

- Inverse interval:

$$\tilde{A}[\alpha]^{-1} = \left[\min\left\{1/\tilde{A}_{1-\alpha/2}, 1/\ \tilde{A}_{\alpha/2}\right\}, \max\left\{1/\tilde{A}_{1-\alpha/2}, \tilde{A}_{\alpha/2}\right\}\right],$$

excluding the case $\tilde{A}_{1-\alpha/2} = 0$ or $\tilde{A}_{\alpha/2} = 0$.

Therefore, using the extension principle, arithmetic operations on fuzzy numbers can be obtained by

$$\left(\tilde{A} \odot \tilde{B}\right)(z) = \sup_{x,y:x\circ y=z} \min\left\{\tilde{A}(x), \tilde{B}(y)\right\},$$

where $\odot$ is any kind of the extended arithmetic operations $\oplus, \ominus, \otimes$ and $\oslash$, and $\circ$ is any kind of the arithmetic operations $+, -, \times$, and $/$ (e.g. see [18]).

Theorem 3.7. *Assume $\tilde{A}, \tilde{B}, \tilde{C}$, and $\tilde{D}$ are four fuzzy numbers. Then the following relations are hold:*

1)

$$\tilde{A} \succ_P^\gamma \tilde{B} \text{ iff} \begin{cases} \lambda \otimes \tilde{A} \succ_P^\gamma \lambda \otimes \tilde{B}, & \text{for } \lambda > 0, \\ \\ \lambda \otimes \tilde{B} \succ_P^\gamma \lambda \otimes \tilde{B}, & \text{for } \lambda > 0. \end{cases}$$

2) $\tilde{A} \succ_P^\gamma \tilde{B}$ *iff* $\tilde{A} \oplus (\ominus)k \succ_P^\gamma \tilde{B} \oplus (\ominus)k, k \in \mathbb{R}$.
3) *if* $\tilde{A}$ *and* $\tilde{B}$ *are positive fuzzy numbers, then* $\tilde{A} \succ_P^\gamma \tilde{B}$ *iff* $\tilde{B}^{-1} \succ_P^\gamma \tilde{A}^{-1}$.
4) *if* $\tilde{A} \succ_P^\gamma \tilde{B}$ *and* $\tilde{C} \succ_P^\gamma \tilde{D}$ *then* $\tilde{A} \oplus \tilde{C} \succ_P^\gamma \tilde{B} \oplus \tilde{D}$.
5) *if* $\tilde{A}, \tilde{B}, \tilde{C}$, *and* $\tilde{D}$*are positive fuzzy numbers, then* $\tilde{A} \succ_P^\gamma \tilde{B}$ *and* $\tilde{C} \succ_P^\gamma \tilde{D}$ *imply* $\tilde{A} \otimes \tilde{C} \succ_P^\gamma \tilde{B} \otimes \tilde{D}$.
6) *if* $\tilde{A}, \tilde{B}, \tilde{C}$, *and* $\tilde{D}$ *are positive fuzzy numbers, then* $\tilde{A} \succ_P^\gamma \tilde{B}$ *and* $\tilde{C} \succ_P^\gamma \tilde{D}$ *imply* $\tilde{A} \oslash \tilde{D} \succ_P^\gamma \tilde{B} \oslash \tilde{C}$.

Proof. The results are immediately followed from the arithmetic intervals of fuzzy numbers (Remark 3.6) and the fact that $\tilde{A} \succ_P^\gamma \tilde{B}$ if and only if $\tilde{A}_{1-\alpha_0/2 \geq \tilde{B}_{\alpha_0/2}}, \alpha_0 = 2(1-\gamma)$. □

Here we investigate the robustness property of the proposed index P. For this, to define the robustness mathematically, we propose a distance as the measurement of a membership estimation error.

Definition 3.8. Let $\tilde{A}, \tilde{B} \in \mathcal{F}(\mathbb{R})$. The maximum difference between $\tilde{A}$ and $\tilde{B}$ is defined as

$$D\left(\tilde{A}, \tilde{B}\right) = \max_{\alpha \in [0,1]} \left|\tilde{A}_\alpha - \tilde{B}_\alpha\right|.$$

Lemma 3.9. *The maximum difference* $D : \mathcal{F}(\mathbb{R}) \times \mathcal{F}(\mathbb{R}) \longrightarrow [0, \infty)$ *is a distance on the space of fuzzy numbers, i.e., for three fuzzy numbers* $\tilde{A}, \tilde{B}$ *and* $\tilde{C}$ *we have*

- $D\left(\tilde{A}, \tilde{B}\right) = 0$ *if and only if* $\tilde{A} = \tilde{B}$, *i.e.,* $\tilde{A}_\alpha = \tilde{B}_\alpha$ *for any* $\alpha \in [0, 1]$.

- $D\left(\tilde{A},\tilde{B}\right)=D\left(\tilde{B},\tilde{A}\right)$.
- $D\left(\tilde{A},\tilde{C}\right)\leq D\left(\tilde{A},\tilde{B}\right)+D\left(\tilde{B},\tilde{C}\right)$.

Theorem 3.10. *The preference index P is robust, i.e., for any given $\tilde{A},\tilde{B}\in\mathcal{F}(\mathbb{R})$, and $\epsilon>0$, there exists $\delta>0$ for which*

$$\left|P\left(\tilde{A}\succ\tilde{B}\right)-P\left(\tilde{A}^*\succ\tilde{B}\right)\right|<\epsilon,\quad \textit{for all } \tilde{A}^*\in\mathcal{F}\left(\mathbb{R}\right)\ \textit{s.t.}\ D\left(\tilde{A},\tilde{A}^*\right)<\delta.$$

Using the preference index P defined for each ordered pair of fuzzy numbers, it is easy to rank n fuzzy numbers $\tilde{A}_1,\tilde{A}_2,\ldots,\tilde{A}_n$. The procedure is as follows:

1) Calculate $P\left(\tilde{A}_i\succ\tilde{A}_j\right)$ for $i=1,\ldots,n, j=1,\ldots,n$ which consists of an symmetric matrix. From Theorem 3.4, note that we only need to calculate $n\times(n-1)/2$ preference values.
2) Sort $\tilde{A}_1,\tilde{A}_2,\ldots,\tilde{A}_n$ into $\tilde{A}_{k_1},\tilde{A}_{k_2},\ldots,\tilde{A}_{k_n}$ so that for any $i<j$, $P\left(\tilde{A}_i\succ\tilde{A}_j\right)\geq\gamma$. The feasibility of the sorting is guaranteed by Theorem 3.4. Based on this sorting, therefore $\tilde{A}_{k_1}$ is the most preferred choice, $\tilde{A}_{k_2}$ is the second, etc.

4. Conclusion

As a natural extension of the ordinary ordering, this paper proposed a new approach for comparing and ranking fuzzy numbers based on the α-optimistic values of a fuzzy number with taking into account the decision makers attitude. Some basic properties of fuzzy ranking are investigated: 1) robustness, 2) transitivity, 3) rationality i.e., based on our method, if $\tilde{A}$ is preferred to $\tilde{B}$, then $\tilde{B}$ can not be preferred to $\tilde{A}$, 4) it is matching our intuition as ordering in the space of real numbers. The method is specially tailored for LR-fuzzy numbers and an algorithm is presented to rank a set of fuzzy numbers.

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