Sahand Communications in Mathematical Analysis (SCMA) Vol. 15 No. 1 (2019), 169-187
http://scma.maragheh.ac.ir
DOI: 10.22130/scma.2018.85866.432

Some Properties of Continuous K-frames in Hilbert Spaces

Gholamreza Rahimlou[1], Reza Ahmadi[2*], Mohammad Ali Jafarizadeh[3], and Susan Nami[4]

Abstract. The theory of continuous frames in Hilbert spaces is extended, by using the concepts of measure spaces, in order to get the results of a new application of operator theory. The K-frames were introduced by Găvruta (2012) for Hilbert spaces to study atomic systems with respect to a bounded linear operator. Due to the structure of K-frames, there are many differences between K-frames and standard frames. K-frames, which are a generalization of frames, allow us in a stable way, to reconstruct elements from the range of a bounded linear operator in a Hilbert space. In this paper, we get some new results on the continuous K-frames or briefly cK-frames, namely some operators preserving and some identities for cK-frames. Also, the stability of these frames are discussed.

1. Introduction

Nowadays, frames are used in some various branches of science and engineering. Among them are signal processing, image processing, data compression and sampling in sampling theory (see [2, 3, 5, 10]). Frames were introduced by Duffin and Schaeffer in the context of Non-harmonic Fourier series [7]. They were intended as an alternative to the orthonormal or Riesz bases in Hilbert spaces. Much of the abstract theory of frames is elegantly laid out in that paper. A frame is a family of elements in a separable Hilbert space which allows for a stable, not necessarily unique, decomposition of an arbitrary element into an expansion of the frame elements (see [4, 8, 9, 11, 14]).

2010 *Mathematics Subject Classification.* Primary 42C15; Secondary 42C40, 41A58.
Key words and phrases. K-frame, c-frame, cK-frame, Local cK-atoms.
Received: 09 May 2018, Accepted: 15 October 2018.
* Corresponding author.

The theory of continuous frames in Hilbert spaces, using the concepts of measurment spaces, in order to get the results of a new application of operator theory is extended. The concept of a generalization of frames to an indexed family by some locally compact spaces endowed with a Radon measure was proposed by G. Kaiser [10] and independently by Ali, Antoine and Gazeau [1]. These frames are known as the continuous frames. In continuous K-frames, the lower bound of the frame is replaced by the norm of a bounded operator on a Hilbert space. This changes the overall structure of the frame and gives new results in terms of combining operators and frame perturbation.

This paper consists of four sections. We review the foundation for the theory of continuous frames in Hilbert spaces in Section 1. The necessary tools to construct a contiuous frame will be provided. Also the structure of continuous K-frames is expressed. In Section 2, the operators that preserve continuous K-frames are discussed. In Section 3, we present some useful identities and inequalities for those frames. Finally, we study the perturbation of continuous K-frames and the lower bound of frames by using a new technique for getting perturbation of continuous K-frames in Section 4.

Throughout this paper, H, H_0, H_1 and H_2 are Hilbert spaces, $(H)_1$ is the closed unit ball in H. (X, μ) is a σ-finite measure space, $\mathcal{L}(H_0, H)$ is the set of all linear mappings of H_0 to H and $\mathcal{B}(H_0, H)$ is the set of all bounded linear mappings. Instead of $\mathcal{B}(H, H)$, we simply write $\mathcal{B}(H)$. Also for brevity, continuous K-frame is denoted by cK-frame.

Definition 1.1. Let $\{f_n\} \subseteq H$. We say that the sequence $\{f_n\}$ is a frame for H if there exist constants $A, B > 0$ such that

$$A \|h\|^2 \leq \sum_n |\langle h, f_n \rangle|^2 \leq B \|h\|^2, \quad h \in H.$$

Definition 1.2. Let $F : X \to H$ be a weakly measurable mapping (i.e., for all $h \in H$, the mapping $x \mapsto \langle F(x), h \rangle$ is measurable). Then F is called a c-frame for H if there exist $0 \leq A \leq B < \infty$ such that for all $h \in H$,

$$A \|h\|^2 \leq \int_X |\langle F(x), h \rangle|^2 \, d\mu \leq B \|h\|^2 .$$

The constants A and B are called c-frame bounds. If A, B can be chosen so that $A = B$, we call this c-frame an A-tight frame, and if $A = B = 1$ it is called a c-Parseval frame. If we only have the upper bound, we call f a c-Bessel mapping for H. The representation space employed in this setting is

$$L^2(X, H) = \{\varphi : X \to H | \varphi \text{ is measurable and } \|\varphi\|_2 < \infty\},$$

where $\|\varphi\|_2 = \left(\int_X \|\varphi(x)\|^2 \, d\mu\right)^{\frac{1}{2}}$. For each $F, G \in L^2(X, H)$, the mapping $x \to \langle F(x), G(x)\rangle$ of X to $\mathbb{C}$ is measurable, and it can be proved that $L^2(X, H)$ is a Hilbert spase with the inner product defined by

$$\langle F, G\rangle_{L^2} = \int_X \langle F(x), G(x)\rangle \, d\mu.$$

We shall write $L^2(X)$ when $H = \mathbb{C}$.

Theorem 1.3 ([8]). *Let $F : X \to H$ be a c-Bessel mapping for H, and $U \in \mathcal{B}(H, H_0)$. Then $UF : X \to H_0$ is a c-Bessel mapping for H_0 with*

$$UT_F = T_{UF}.$$

Theorem 1.4 ([6]). *Suppose the H, H_1 and H_2 are Hilbert spaces, $L_1 \in \mathcal{B}(H_1, H)$ and $L_2 \in \mathcal{B}(H_2, H)$. Then the following assertions are equivalent:*

(i) $\mathcal{R}(L_1) \subset \mathcal{R}(L_2)$,
(ii) $\exists\lambda \geq 0$, *such that* $L_1L_1^* \leq \lambda L_2L_2^*$,
(iii) *There exists* $X \in \mathcal{B}(H_1, H_2)$ *such that* $L_1 = L_2X$.

Definition 1.5. Let $K \in \mathcal{B}(H_0, H)$, and $\{f_n\} \subseteq H$. We say that the sequence $\{f_n\}$ is a K-frame for H with respect to H_0, if there exist constants $A, B > 0$ such that

$$A\|K^*h\|^2 \leq \sum_n |\langle h, f_n\rangle|^2 \leq B\|h\|^2, \quad h \in H.$$

Definition 1.6. Let $F : X \to H$ be weakly measurable. We define the map $\int_X \cdot F d\mu : L^2(X) \to H$ as follows:

$$\left\langle \int_X gF d\mu, h \right\rangle := \int_X g(x)\langle F(x), h\rangle \, d\mu, \quad h \in H, g \in L^2(X).$$

It is clear that, the vector valued integral $\int_X gF d\mu$ exists in H if for each $h \in H$, $\int_X g(x)\langle F(x), h\rangle \, d\mu$ exists.

Definition 1.7. Let $H_0 \subseteq H$. Suppose that $F : X \to H$ is weakly measurable and $K \in \mathcal{B}(H_0, H)$. Then F is called a family of local cK-atoms for H_0 if the following conditions are satisfied:

(i) For each $g \in L^2(X)$ the vector valued integral $\int_X gF \, d\mu$ exists in H.
(ii) There exist some $a > 0$ and $\ell : X \to \mathcal{L}(H_0, \mathbb{C})$ such that for each $h \in H_0$, $\ell(\cdot)(h) \in L^2(X)$ and also

$$\|\ell(\cdot)(h)\|_2 \leq a\|h\|, \qquad Kh = \int_X \ell(\cdot)(h) F \, d\mu.$$

If K is the identity function on H_0 then F is called a family of local atoms for H_0.

Definition 1.8. Let $K \in \mathcal{B}(H_0, H)$ and $F : X \to H$ be weakly measurable. Then the map F is called a cK-frame with respect to H_0, if there exist constants $A, B > 0$ such that for each $h \in H$,

$$A \|K^* h\|^2 \leq \int_X |\langle F(x), h\rangle|^2 \, d\mu \leq B \|h\|^2 .$$

A cK-frame F is called a Parseval cK-frame, whenever for every $h \in H$,

$$\int_X |\langle F(x), h\rangle|^2 \, d\mu = \|K^* h\|^2 .$$

Lemma 1.9 ([13])**.** *Let $F : X \to H$ be weakly measurable. For each $\varphi \in L^2(X)$, the value of $\int_X \varphi F \, d\mu$ exists in H if and only if for each $h \in H$, $\langle F, h\rangle \in L^2(X)$.*

Lemma 1.10 ([12])**.** *Let $F : X \to H$ be weakly measurable. Then F is a c-Bessel mapping for H if and only if for each $\varphi \in L^2(X)$, $\int_X \varphi F d\mu$ exists in H.*

Remark 1.11. Let $F : X \to H$ be a c-Bessel mapping for H. The synthesis operator is defined by

$$T_F : L^2(X) \to H, \qquad T_F(\varphi) = \int_X \varphi F \, d\mu.$$

Hence, for each $\varphi \in L^2(X)$ and $h \in H$,

$$\left\langle \int_X \varphi F \, d\mu, h \right\rangle = \int_X \varphi(x) \langle F(x), h\rangle \, d\mu.$$

The analysis operator is defined by

$$T_F^* : H \to L^2(X), \qquad T_F^*(h) = \langle h, F\rangle .$$

So, for the frame operator $S_F := T_F T_F^*$ we have

$$S_F(h) = \int_X \langle h, F\rangle F \, d\mu, \quad h \in H.$$

Theorem 1.12 ([12])**.** *Let $H_0 \subseteq H$. Let $F : X \to H$ be weakly measurable, and $K \in \mathcal{B}(H_0, H)$. Then the following assertions are equivalent:*

(i) *F is a family of local cK-atoms for H_0.*
(ii) *F is a cK-frame for H with respect to H_0.*
(iii) *F is a c-Bessel mapping for H, and there exists $G \in \mathcal{B}(H_0, L^2(X))$ such that*

$$Kh = \int_X G(h) F \, d\mu, \quad h \in H_0.$$

Theorem 1.13 ([12]). *Let $K \in \mathcal{B}(H_0, H)$, and $F : X \to H$ be a cK-frame for H with respect to H_0, with bounds A,B. If K is closed range then S_F is invertibale on $\mathcal{R}(K)$, and for each $h \in \mathcal{R}(K)$*

$$A \left\| K^{\dagger} \right\|^{-2} \|h\|^2 \leq \langle S_F(h), h \rangle \leq B \|h\|^2 . \tag{1.1}$$

2. Operators Preserving ck-Frames

Theorem 2.1. *Suppose that $F : X \to H$ is a cK-frame for H and $U \in \mathcal{B}(H)$ with $\mathcal{R}(U) \subseteq \mathcal{R}(K)$. Then F is a cU-frame for H.*

Proof. Let F be a cK-frame for H with bounds A and B. Since $\mathcal{R}(U) \subseteq \mathcal{R}(K)$, by Theorem 1.4 there exists $\alpha > 0$ such that $UU^* \leq \alpha^2 KK^*$. By the definition of cK-frames, for each $h \in H$ we have

$$\begin{aligned} A\alpha^{-2} \|U^*(h)\|^2 &\leq A \|K^*(h)\|^2 \\ &\leq \int_X |\langle h, F(x) \rangle|^2 \, d\mu. \end{aligned}$$

Hence, F is a cU-frame for H. □

Theorem 2.2. *Let $K \in \mathcal{B}(H)$ with dense range, $F : X \to H$ be a cK-frame and $U \in \mathcal{B}(H)$ be closed range. If UF is a cK-frame for H then U is surjective.*

Proof. suppose UF is a cK-frame for H with frame bounds A and B. Then for any $h \in H$ we have

$$A \|K^* h\|^2 \leq \int_X |\langle h, UF(x) \rangle|^2 \, d\mu \leq B \|h\|^2 . \tag{2.1}$$

Since K is with dense range, K^* is injective. By (2.1), $\mathcal{N}(U^*) \subset \mathcal{N}(K^*)$, then U^* is injective. Moreover $\mathcal{R}(U) = \mathcal{N}(U^*)^{\perp} = H$. Thus, U is surjective. □

Theorem 2.3. *Suppose $K \in \mathcal{B}(H)$ and let $F : X \to H$ be a cK-frame for H. If $U \in \mathcal{B}(H)$ has closed range with $UK = KU$, then $UF : X \to H$ is a cK-frame for $\mathcal{R}(U)$.*

Proof. Since U has closed range, then it has the pseudo-inverse $U^{\dagger}$ such that $UU^{\dagger} = I$. Now $I = I^* = \left(U^{\dagger}\right)^* U^*$. Then for each $h \in \mathcal{R}(U)$, $K^* h = \left(U^{\dagger}\right)^* U^* K^* h$. So we have

$$\begin{aligned} \|K^* h\| &= \left\| (U^{\dagger})^* U^* K^* h \right\| \\ &\leq \left\| (U^{\dagger})^* \right\| \|U^* K^* h\| . \end{aligned}$$

Therefore, $\left\|(U^{\dagger})^{*}\right\|^{-1}\|K^{*}h\| \leq \|U^{*}K^{*}h\|$. Now for each $h \in \mathcal{R}(U)$,

$$\begin{aligned}\int_X |\langle h, UF(x)\rangle|^2 \, d\mu &= \int_X |\langle U^{*}h, F(x)\rangle|^2 \, d\mu \\ &\geq A\|K^{*}U^{*}h\|^2 \\ &= A\|U^{*}K^{*}h\|^2 \\ &\geq A\left\|(U^{\dagger})^{*}\right\|^{-2}\|K^{*}h\|^2 .\end{aligned}$$

Since F is a c-Bessel mapping with bound B, we have

$$\begin{aligned}\int_X |\langle h, UF(x)\rangle|^2 \, d\mu &= \int_X |\langle U^{*}h, F(x)\rangle|^2 \, d\mu \\ &\leq B\|U^{*}h\|^2 \\ &\leq B\|U\|^2\|h\|^2 .\end{aligned}$$

Therefore, UF is a cK-frame for $\mathcal{R}(U)$. □

Remark 2.4. From Theorems 2.2 and 2.3 we conclude the following: Let $K \in \mathcal{B}(H)$ be with dense range. Let F be a cK-frame for H and $U \in \mathcal{B}(H)$ has closed range with $UK = KU$. Then UF is a cK-frame for H if and only if U is surjective.

Theorem 2.5. *Suppose $K \in \mathcal{B}(H)$ has dense range, F is a cK-frame and $U \in \mathcal{B}(H)$ has closed range. If UF and $U^{*}F$ are cK-frames for H, then U is invertible.*

Proof. Suppose UF is a cK-frame for H with frame bounds A_1 and B_1. Then for any $h \in H$

$$A_1\|K^{*}h\|^2 \leq \int_X |\langle h, UF(x)\rangle|^2 \, d\mu \leq B_1\|h\|^2 . \tag{2.2}$$

Since K has dense range, then K^{*} is injective. By (2.2) we have $\mathcal{N}(U^{*}) \subset \mathcal{N}(K^{*})$, therefore U^{*} is injective. Moreover $\mathcal{R}(U)$=$\mathcal{N}(U^{*})^{\perp} = H$, then U is surjective. Suppose A_2 and B_2 are frame bounds for $U^{*}F$, then for any $h \in H$,

$$A_2\|K^{*}h\|^2 \leq \int_X |\langle h, U^{*}F(x)\rangle|^2 \, d\mu \leq B_2\|h\|^2 . \tag{2.3}$$

As K has dense range, K^{*} is injective. Then, by (2.3) we get $\mathcal{N}(U) \subset \mathcal{N}(K^{*})$, so U is injective. Thus U is bijective. Now, by the Bounded Inverse Theorem, U is invertible. □

Theorem 2.6. *Let $K \in \mathcal{B}(H)$ and F be a cK-frame for H and $U \in \mathcal{B}(H)$ be a co-isometry with $UK = KU$. Then UF is a cK-frame for H.*

Proof. Let F be a cK-frame for H. Since U is a co-isometry, we have for each $h \in H$

$$\begin{aligned}\int_X |\langle h, UF(x)\rangle|^2 \, d\mu &= \int_X |\langle U^*h, F(x)\rangle|^2 \, d\mu \\ &\geq A \|K^*u^*h\|^2 \\ &= A \|U^*K^*h\|^2 \\ &= A \|K^*h\|^2 .\end{aligned}$$

It is clear that UF is a c-Bessel mapping. Since $F : X \to H$ is a c-Bessel mapping, then for each $h \in H$

$$\begin{aligned}\int_X |\langle h, UF(x)\rangle|^2 \, d\mu &= \int_X |\langle U^*h, F(x)\rangle|^2 \, d\mu \\ &\leq B \|U\|^2 \|h\|^2 .\end{aligned}$$

Therefore, UF is a cK-frame for H. □

Theorem 2.7. *Let $F : X \to H$ be a c-Bessel mapping for H. Then $F : X \to H$ is a cK-frame for H if and only if there exists $A > 0$ such that $S_F \geq AKK^*$, where S_F is the frame operator for F.*

Proof. $F : X \to H$ is a cK-frame for H with frame bounds A, B and frame operator S_F, if and only if,

$$\begin{aligned}A \|K^*h\|^2 &\leq \int_X |\langle h, F(X)\rangle|^2 \, d\mu \\ &= \langle S_F(h), h\rangle \\ &\leq B \|h\|^2 , \quad \forall h \in H,\end{aligned}$$

if and only if,

$$\langle AKK^*h, h\rangle \leq \langle S_F(h), h\rangle \leq \langle Bh, h\rangle , \quad \forall h \in H,$$

if and only if,

$$S_F \geq AKK^*.$$

□

Theorem 2.8. *Let $F : X \to H$ be a c-frame for H. Then $KF : X \to H$ and $K \in \mathcal{B}(H)$ is a cK-frame for H.*

Proof. By the definition of c-frame we have

$$\begin{aligned}\int_X |\langle h, KF(x)\rangle|^2 \, d\mu &= \int_X |\langle K^*h, F(x)\rangle|^2 \, d\mu \\ &\leq B \|K^*h\|^2 \\ &\leq B \|K^*\|^2 \|h\|^2 .\end{aligned}$$

So, KF is a Bessel mapping. By theorem 1.12 it is sufficient to show that KF is an atomic system for H. For each $h \in H$ we have $\langle h, KF \rangle \in L^2(X)$, so $\int_X g\,(KF)\,d\mu \in H$ for each $g \in L^2(X)$. By Theorem 3.5 in [8], for each $h \in H$ we have $h = T_F(\left\langle S_F^{-1}(h), F \right\rangle)$; therefore

$$\begin{aligned} Kh &= KT_F(\left\langle S_F^{-1}(h), F \right\rangle) \\ &= T_{KF}(\left\langle S_F^{-1}(h), F \right\rangle) \\ &= \int_X \left\langle S_F^{-1}(h), F(x) \right\rangle KF(x)\,d\mu. \end{aligned}$$

So, for all $h_1 \in H$

$$\begin{aligned} \langle Kh, h_1 \rangle &= \left\langle \int_X \left\langle S_F^{-1}(h), F(x) \right\rangle KF(x)\,d\mu, h_1 \right\rangle \\ &= \int_X \left\langle S_F^{-1}(h), F(x) \right\rangle \langle h_1, KF(x) \rangle \; d\mu. \end{aligned}$$

Let

$$\begin{aligned} &\ell : X \to \mathcal{L}(H, \mathbb{C}), \\ &\ell(x)(h) = \left\langle S_F^{-1}(h), F(x) \right\rangle, \quad h \in H, x \in X. \end{aligned}$$

So, for each $h \in H$ and $x \in X$, we get $\ell(x)(h) \in L^2\,(X)$ and

$$\begin{aligned} \|\ell(x)(h)\|_2 &= \left(\int_X \left| \left\langle S_F^{-1}(h), F(x) \right\rangle \right|^2 \, d\mu \right)^{\frac{1}{2}} \\ &\leq \left(B \left\| S_F^{-1}(h) \right\|^2 \right)^{\frac{1}{2}} \\ &\leq \sqrt{B} \left\| S_F^{-1} \right\| \|h\| \,. \end{aligned}$$

Now, if $a := \sqrt{B} \left\| S_F^{-1} \right\|$, by Definition 1.7 the proof is completed. □

3. Some Identities and Inequalities for cK-Frames

In this section, we introduce some useful identities and inequalities by frame operators. Let $K \in \mathcal{B}(H_0, H)$, $F : X \to H$ be c-Bessel mappings for H and $G : X \to H_0$ be a c-Bessel mapping for H_0. We say that F, G is a cK-dual pair, if

$$Kh_0 = T_F\left(\langle h_0, G \rangle\right),$$

for any $h \in H$ and $h_0 \in H_0$. In this case, we know that F is a cK-frame for H with respect to H_0 and G is a cK^*-frame for H_0 with respect to H (for more details, we refer to [12]). Now, we define

$$M_{X_1} h := \int_{X_1} \langle h, G(x) \rangle \, F(x)\,d\mu,$$

for each $h \in H$. So, $M_{X_1}h$ is well-dfined and bounded. Indeed, if $h \in H$ then

$$\begin{aligned}
\|M_{X_1}h\|^2 &= \left(\sup_{\|h'\|=1} \left|\left\langle M_{X_1}h, h'\right\rangle\right| \right)^2 \\
&= \left(\sup_{\|h'\|=1} \left|\left\langle \int_{X_1} \langle h, G(x)\rangle F(x)\, d\mu, h' \right\rangle\right| \right)^2 \\
&\leq \int_{X_1} |\langle h, G(x)\rangle|^2 \, d\mu . \sup_{\|h'\|=1} \int_{X_1} \left|\left\langle F(x), h'\right\rangle\right|^2 \, d\mu \\
&\leq BB' \|h\|^2 ,
\end{aligned}$$

where, B, B' are uper bounds for F, G, respectively. It is easy to check that $M_{X_1} + M_{X_1^c} = K$ where, X_1^c is the complement of X_1.

Theorem 3.1. *Let F be a cK-frame for H with the dual G. Then for each measureable subspace $X_1 \subseteq X$ and $h \in H$,*

$$\int_{X_1} \langle h, G(x)\rangle \overline{\langle Kh, F(x)\rangle} d\mu - \|M_{X_1}h\|^2 = \int_{X_1^c} \overline{\langle h, G(x)\rangle} \langle Kh, F(x)\rangle \, d\mu - \left\|M_{X_1^c}h\right\|^2 .$$

Proof. Suppose that $h \in H$ and $X_1 \subseteq X$. We have

$$\begin{aligned}
&\int_{X_1} \langle h, G(x)\rangle \overline{\langle Kh, F(x)\rangle} d\mu - \|M_{X_1}h\|^2 \\
&= \langle M_{X_1}h, Kh\rangle - \langle M_{X_1}h, M_{X_1}h\rangle \\
&= \langle K^* M_{X_1}h, h\rangle - \left\langle M_{X_1}^* M_{X_1}h, h\right\rangle \\
&= \left\langle (K^* - M_{X_1}^*) M_{X_1}h, h\right\rangle \\
&= \left\langle M_{X_1^c}^* (K - M_{X_1^c})h, h\right\rangle \\
&= \left\langle M_{X_1^c}^* Kh, h\right\rangle - \left\langle M_{X_1^c}^* M_{X_1^c}h, h\right\rangle \\
&= \left\langle h, K^* M_{X_1^c}h\right\rangle - \left\|M_{X_1^c}h\right\|^2 \\
&= \int_{X_1^c} \overline{\langle h, G(x)\rangle} \langle Kh, F(x)\rangle \, d\mu - \left\|M_{X_1^c}h\right\|^2 .
\end{aligned}$$

□

Theorem 3.2. *Let $F : X \to H$ be a Parseval cK-frame for H. For every $h \in H$, $X_1 \subseteq X$ and $E \subseteq X_1^c$ we have*

$$\left\| \int_{X_1 \cup E} \langle h, F(x)\rangle F(x)\, d\mu \right\|^2 - \left\| \int_{X_1^c | E} \langle h, F(x)\rangle F(x)\, d\mu \right\|^2$$

$$= \left\| \int_{X_1} \langle h, F(x) \rangle F(x)\, d\mu \right\|^2 - \left\| \int_{X_1^c} \langle h, F(x) \rangle\, F(x)\, d\mu \right\|^2 + 2\mathrm{Re} \int_E \langle h, F(x) \rangle\, \overline{\langle KK^* h, F(x) \rangle}\, d\mu.$$

Proof. For each measurable subspace $X_1 \subseteq X$, we define

$$S_{X_1} h = \int_{X_1} \langle h, F(x) \rangle\, F(x)\, d\mu.$$

We have $S_{X_1} + S_{X_1^c} = KK^*$. Therefore,

$$\begin{aligned} S_{X_1}^2 - S_{X_1^c}^2 &= S_{X_1}^2 - (KK^* - S_{X_1})^2 \\ &= KK^* S_{X_1} + S_{X_1} KK^* - (KK^*)^2 \\ &= KK^* S_{X_1} - (KK^* - S_{X_1})\, KK^* \\ &= KK^* S_{X_1} - S_{X_1}^c KK^*. \end{aligned}$$

Hence, for every $h \in H$ we obtain

$$\begin{aligned} &\left\| \int_{X_1 \cup E} \langle h, F(x) \rangle\, F(x)\, d\mu \right\|^2 - \left\| \int_{X_1^c | E} \langle h, F(x) \rangle\, F(x)\, d\mu \right\|^2 \\ &= \left\langle KK^* S_{X_1 \bigcup E} h, h \right\rangle - \left\langle S_{X_1^c | E} KK^* h, h \right\rangle \\ &= \left\langle S_{X_1 \bigcup E} h, KK^* h \right\rangle - \left\langle KK^* h, S_{X_1^c | E} h \right\rangle \\ &= \left\langle \int_{X_1 \bigcup E} \langle h, F(x) \rangle\, F(x)\, d\mu, KK^* h \right\rangle - \overline{\left\langle S_{X_1^c | E} h, KK^* h \right\rangle} \\ &= \int_{X_1 \bigcup E} \langle h, F(x) \rangle \langle F(x), KK^* h \rangle\, d\mu \\ &\quad - \overline{\int_{X_1^c | E} \langle h, F(x) \rangle\, \langle F(x), KK^* h \rangle\, d\mu} \\ &= \int_{X_1} \langle h, F(x) \rangle\, \langle F(x), KK^* h \rangle\, d\,\mu \\ &\quad + \int_E \langle h, F(x) \rangle\, \langle F(x), KK^* h \rangle\, d\mu \\ &\quad - \overline{\int_{X_1^c} \langle h, F(x) \rangle\, \langle F(x), KK^* h \rangle\, d\mu} \\ &\quad + \overline{\int_E \langle h, F(x) \rangle\, \langle F(x), KK^* h \rangle\, d\mu} \\ &= \left\| \int_{X_1} \langle h, F(x) \rangle\, F(x) d\mu \right\|^2 - \left\| \int_{X_1^c} \langle h, F(x) \rangle\, F(x)\, d\mu \right\|^2 \end{aligned}$$

$$+ 2\mathrm{Re} \int_E \langle h, F(x) \rangle \overline{\langle KK^*h, F(x) \rangle} \, d\mu.$$

□

Theorem 3.3. *Let $F : X \to H$ be a Parseval cK-frame for H. For every $h \in H$ and $X_1 \subseteq X$ we have,*

$$\begin{aligned}\mathrm{Re} \left(\int_{X_1^c} \langle h, F(x) \rangle \overline{\langle KK^*h, F(x) \rangle} \, d\mu \right) &+ \left\| \int_{X_1} \langle h, F(x) \rangle F(x) \, d\mu \right\|^2 \\ &= \mathrm{Re} \left(\int_{X_1} \langle h, F(x) \rangle \overline{\langle KK^*h, F(x) \rangle} \, d\mu \right) \\ &\quad + \left\| \int_{X_1^c} \langle h, F(x) \rangle F(x) \, d\mu \right\|^2 \\ &\geq \frac{3}{4} \left\| KK^*h \right\|^2.\end{aligned}$$

Proof. Since $S_{X_1}^2 - S_{X_1^c}^2 = KK^*S_{X_1} - S_{X_1^c}KK^*$ and $S_{X_1} + S_{X_1^c} = KK^*$, we can write

$$\begin{aligned}S_{X_1}^2 + S_{X_1^c}^2 &= 2 \left(\frac{KK^*}{2} - S_{X_1} \right)^2 + \frac{(KK^*)^2}{2} \\ &\geq \frac{(KK^*)^2}{2}.\end{aligned}$$

Consequently

$$\begin{aligned}KK^*S_{X_1} + S_{X1^c}^2 + \left(KK^*S_{X_1} + S_{X_1^c}^2 \right)^* &= KK^*S_{X_1} + S_{X_1^c}^2 + S_{X_1}KK^* + S_{X_1^c}^2 \\ &= KK^* \left(S_{X_1} + S_{X_1^c} \right) + S_{X_1}^2 + S_{X_1^c}^2 \\ &= (S_{X_1} + S_{X_1^c})KK^* + S_{X_1}^2 + S_{X_1^c}^2 \\ &\geq \frac{3}{2} \left(KK^* \right)^2.\end{aligned}$$

Thus, we obtain

$$\begin{aligned}&\mathrm{Re} \left(\int_{X_1^c} \langle h, F(x) \rangle \overline{\langle KK^*h, F(x) \rangle} \, d\mu \right) + \left\| \int_{X_1} \langle h, F(x) \rangle F(x) \, d\mu \right\|^2 \\ &\quad = \mathrm{Re} \left(\int_{X_1} \langle h, F(x) \rangle \overline{\langle KK^*h, F(x) \rangle} \, d\mu \right) + \left\| \int_{X_1^c} \langle h, F(x) \rangle F(x) \, d\mu \right\|^2 \\ &\quad = \frac{1}{2} \left(\langle KK^*S_{X_1}h, h \rangle + \left\langle S_{X_1^c}^2 h, h \right\rangle + \langle h, KK^*S_{X_1}h \rangle + \left\langle h, S_{X_1^c}^2 h \right\rangle \right) \\ &\quad \geq \frac{3}{4} \left\| KK^*h \right\|^2.\end{aligned}$$

□

Theorem 3.4. *Let K be a closed operator and $F : X \to H$ be a cK-frame for H with the optimal lower bound A. Then,*

(I) *For each $h \in H$,*

$$\left\|\int_X \langle h, F(x)\rangle F(x)\, d\mu\right\|^2 \le \|S_F\| \int_X |\langle h, F(x)\rangle|^2\, d\,\mu.$$

(II) *For any $h \in \mathcal{R}(K)$,*

$$\int_X |\langle h, F(x)\rangle|^2\ d\mu \le \frac{1}{A}\left\|K^\dagger\right\|^2 \left\|\int_X \langle h, F(x)\rangle F(x)\, d\mu\right\|^2.$$

Proof. (I) For any $h \in H$, we can write

$$\begin{aligned}\left\|\int_X \langle h, F(x)\rangle F(x)\, d\mu\right\|^2 &= |\langle S_F(h), S_F(h)\rangle| \\ &\le \|S_F\|\, |\langle S_F(h), h\rangle| \\ &= \|S_F\| \int_X |\langle h, F(x)\rangle|^2\ d\mu.\end{aligned}$$

(II) For each $h \in \mathcal{R}(K)$,

$$\begin{aligned}\left(\int_X |\langle h, F(x)\rangle|^2\ d\mu\right)^2 &\le |\langle S_F(h), h\rangle|^2 \\ &\le \|S_F(h)\|^2\, \|h\|^2 \\ &\le \|S_F(h)\|^2 \left\|\left(K^\dagger\right)^* K^* h\right\|^2 \\ &\le \|S_F(h)\|^2 \left\|(K^\dagger)\right\|^2 \|K^* h\|^2 \\ &\le \frac{1}{A}\|S_F(h)\|^2 \left\|(K^\dagger)\right\|^2 \int_X |\langle h, F(x)\rangle|^2\ d\mu,\end{aligned}$$

and the proof is completed. □

Some applications of above theorems, we present the following interesting assertions. Let $F : X \to H$ be a cK-Parseval frame for H. We consider

$$\upsilon_+(F, K, X_1) := \sup_{h\neq 0} \frac{\operatorname{Re}\left(\int_{X_1^c} \overline{\langle h, F(x)\rangle}\, \langle KK^* h, F(x)\rangle\ d\mu\right) + \left\|\int_{X_1} \langle h, F(x)\rangle F(x)\, d\mu\right\|^2}{\|KK^* h\|^2},$$

$$\upsilon_-(F, K, X_1) := \inf_{h\neq 0} \frac{\operatorname{Re}\left(\int_{X_1^c} \overline{\langle h, F(x)\rangle}\, \langle KK^* h, F(x)\rangle\ d\mu\right) + \left\|\int_{X_1} \langle h, F(x)\rangle F(x)\, d\mu\right\|^2}{\|KK^* h\|^2}.$$

Theorem 3.5. *Suppose that $F : X \to H$ is a cK-Parseval frame for H. The following assertions hold:*

(I) $\frac{3}{4} \leq \upsilon_- (F, K, X_1) \leq \upsilon_+ (F, K, X_1) \leq \|K\| \left\|K^\dagger\right\| \left(1 + \|K\| \left\|K^\dagger\right\|\right)$.

(II) $\upsilon_+ (F, K, X_1) = \upsilon_+ (F, K, X_1^c)$ *and* $\upsilon_-(F, K, X_1) = \upsilon_- (F, K, X_1^c)$.

Proof.
(I). It is enough to prove the upper inequality. By Theorem 3.4, (I), we get

$$\begin{aligned}
\left\| \int_{X_1} \langle h, F(x)\rangle F(x)\, d\mu \right\|^2 &\leq \|S_{X_1}\| \int_{X_1} |\langle h, F(x)\rangle|^2 \, d\mu \\
&\leq \|S_{X_1}\| \int_{X} |\langle h, F(x)\rangle|^2 \, d\mu \\
&\leq \|K\|^2 \|K^* h\|^2 \\
&= \|K\|^2 \left\| K K^\dagger K^* h \right\|^2 \\
&\leq \|K\|^2 \left\| K^\dagger \right\|^2 \|K K^* h\|^2 .
\end{aligned}$$

Moreover,

$$\begin{aligned}
&\operatorname{Re}\left(\int_{X_1^c} \overline{\langle h, F(x)\rangle} \langle K K^* h, F(x)\rangle \, d\mu \right) \\
&\qquad \leq \left(\int_X |\langle h, F(x)\rangle|^2 \, d\mu \right)^{\frac{1}{2}} \left(\int_X |\langle K K^* h, F(x)\rangle|^2 \, d\mu \right)^{\frac{1}{2}} \\
&\qquad = \|K^* h\| \, \|K^* K K^* h\| \\
&\qquad = \left\| K K^\dagger K^* h \right\| \|K^* K K^* h\| \\
&\qquad \leq \|K\| \left\| K^\dagger \right\| \|K K^* h\|^2 .
\end{aligned}$$

Therefore,

$$\upsilon_- (F, K, X_1) \leq \upsilon_+ (F, K, X_1) \leq \|K\| \left\| K^\dagger \right\| \left(1 + \|K\| \left\| K^\dagger \right\| \right).$$

(II). By the proof of Theorem 3.2, we have

$$S_{X_1}^2 + S_{X_!^c} K K^* = K K^* S_{X_1} + S_{X_!^c}^2 .$$

Hence, for any $h \in H$,

$$\left\langle S_{X_1}^2 h, h \right\rangle + \left\langle S_{X_1^c} K K^* h, h \right\rangle = \left\langle S_{X_1^c}^2 h, h \right\rangle + \langle K K^* S_{X_1} h, h \rangle .$$

So,

$$\begin{aligned}
&\left\| \int_{X_1} \langle h, F(x)\rangle F(x)\, d\mu \right\|^2 + \overline{\int_{X_1^c} \langle h, F(x)\rangle \overline{\langle K K^* h, F(x)\rangle}\, d\mu} \\
&\qquad = \left\| \int_{X_1^c} \langle h, F(x)\rangle F(x)\, d\mu \right\|^2 + \int_{X_1} \langle h, F(x)\rangle \overline{\langle K K^* h, F(x)\rangle}\, d\mu,
\end{aligned}$$

and this results
(II). □

4. Perturbation of cK-Frames

Throughout this section, the orthogonal projection of H onto a closed subspace $V \subseteq H$ is denoted by Π_V.

Theorem 4.1. *Let $F : X \to H$ be a cK frame for H with bounds A, B, and μ be a σ-finite measure. Let $G : X \to H$ be weakly measurable and assume that there exist constants $\lambda_1, \lambda_2, \gamma \geq 0$ such that $\max\left\{\lambda_1 + \frac{\gamma}{\sqrt{A}}\left\|K^\dagger\right\|, \lambda_2\right\} < 1$ and*

(4.1)
$$\left|\int_X \varphi(x)\left\langle F(x) - G(x), h\right\rangle\, d\mu\right| \leq \lambda_1\left|\int_X \varphi(x)\left\langle F(x), h\right\rangle\, d\mu\right| + +\lambda_2\left|\int_X \varphi(x)\left\langle G(x), h\right\rangle\, d\mu\right| + \gamma\left\|\varphi\right\|_2,$$

for each $\varphi \in L^2(X)$ and $h \in (H)_1$. Then $G : X \to H$ is a continuous $\Pi_{Q(R(k))}K$- frame for H with bounds

$$\frac{\left[\sqrt{A}\left\|K^\dagger\right\|^{-1}(1-\lambda_1) - \gamma\right]^2}{(1+\lambda_2)^2\left\|K\right\|^2}, \qquad \frac{\left[\sqrt{B}(1+\lambda_1) + \gamma\right]^2}{(1-\lambda_2)^2},$$

where $Q = U_G T_F^*$ and T_F, U_G are synthesis operators for F and G, respectively.

Proof. The condition (4.1) implies that for all $\varphi \in L^2(X)$ and $h \in (H)_1$

$$\begin{aligned}\left|\int_X \varphi(x)\left\langle G(x), h\right\rangle\, d\mu\right| &\leq \left|\int_X \varphi(x)\left\langle F(x) - G(x), h\right\rangle\, d\mu\right| + \left|\int_X \varphi(x)\left\langle F(x), h\right\rangle\, d\mu\right| \\ &\leq (1+\lambda_1)\left|\int_X \varphi(x)\left\langle F(x), h\right\rangle\, d\mu\right| + \lambda_2\left|\int_X \varphi(x)\left\langle G(x), h\right\rangle\, d\mu\right| + \gamma\left\|\varphi\right\|_2.\end{aligned}$$

So

$$\begin{aligned}\left|\int_X \varphi(x)\left\langle G(x), h\right\rangle\, d\mu\right| &\leq \frac{1+\lambda_1}{1-\lambda_2}\left|\int_X \varphi(x)\left\langle F(x), h\right\rangle\, d\mu\right| + \frac{\gamma}{1-\lambda_2}\left\|\varphi\right\|_2 \\ &\leq \frac{1+\lambda_1}{1-\lambda_2}\left[\left(\int_X \left|\varphi(x)\right|^2\, d\mu\right)^{\frac{1}{2}}\left(\int_X \left|\left\langle F(x), h\right\rangle\right|^2\, d\mu\right)^{\frac{1}{2}}\right] + \frac{\gamma}{1-\lambda_2}\left\|\varphi\right\|_2\end{aligned}$$

$$\leq \left(\frac{1+\lambda_1}{1-\lambda_2}\sqrt{B}+\frac{\gamma}{1-\lambda_2}\right)\|\varphi\|_2 .$$

Let

$$U_G : L^2(x) \to H,$$
$$\langle (U_G)\varphi, h\rangle = \int_X \varphi(x)\langle G(x), h\rangle \, d\mu,$$

for all $h \in H$ and $\varphi \in L^2(X)$. Then

$$\begin{aligned}\|(U_G)\varphi\| &= \sup_{h\in(H)_1} |\langle (U_G)\varphi, h\rangle| \\ &= \sup_{h\in(H)_1} \left|\int_X \varphi(x)\langle G(x), h\rangle \, d\mu\right| \\ &\leq \left(\frac{1+\lambda_1}{1-\lambda_2}\sqrt{B}+\frac{\gamma}{1-\lambda_2}\right)\|\varphi\|_2 .\end{aligned}$$

Thus, U_G is bounded, so G is a c-Bessel mapping for H with bound $\dfrac{\left[\sqrt{B}(1+\lambda_1)+\gamma\right]^2}{(1-\lambda_2)^2}$. Now, we prove that G has a lower cK-frame bound. By Remark 1.11 we can define the following operators for all $\varphi \in L^2(X)$

$$\begin{array}{ll} T_F : L^2(X) \to H, & U_G : L^2(X) \to H, \\ T_F(\varphi) = \int \varphi F d\mu, & U_G(\varphi) = \int \varphi G d\mu. \end{array}$$

By (4.1) we obtain

$$|\langle T_F(\varphi) - U_G(\varphi), h\rangle| \leq \lambda_1 |\langle T_F(\varphi), h\rangle| + \lambda_2 |\langle U_G(\varphi), h\rangle| + \gamma \|\varphi\|_2 . \tag{4.2}$$

Now, let $T_F^*(h') := \varphi$, $h' \in R(K)$. By (4.2) we have

(4.3)
$$|\langle S_F(h') - U_G T_F^*(h'), h\rangle| \leq \lambda_1 |\langle S_F(h'), h\rangle| + \lambda_2 |\langle U_G T_F^*(h'), h\rangle| + \gamma \|T_F^*(h')\|_{l^2} ,$$

for any $h' \in R(K)$. By (1.1), we get

$$\begin{aligned}\left\|T_F^*(h')\right\|^2 &= \left\langle S_F(h'), h'\right\rangle \\ &\leq \left\|S_F(h')\right\| \left\|h'\right\| \\ &\leq A^{-1}\left\|K^\dagger\right\|^2 \left\|S_F(h')\right\|^2 ,\end{aligned}$$

thus

$$\left\|T_F^*(h')\right\|^2 \leq A^{-1}\left\|K^\dagger\right\|^2 \left\|S_F(h')\right\|^2 , \tag{4.4}$$

for each $h' \in R(K)$. By (4.3) and (4.4), for any $h' \in R(K)$, we have

$$\left|\left\langle S_F(h') - U_G T_F^*(h'), h)\right\rangle\right|$$

$$\leq \lambda_1 \left|\left\langle S_F(h'), h\right\rangle\right| + \lambda_2 \left|\left\langle U_G T_F^*(h'), h\right\rangle\right| + \frac{\gamma}{\sqrt{A}} \left\|K^\dagger\right\| \; \left\|S_F(h')\right\|$$
$$\leq \lambda_1 \left\|S_F(h')\right\| \|h\| + \lambda_2 \left\|U_G T_F^*(h')\right\| \; \|h\| + \frac{\gamma}{\sqrt{A}} \left\|K^\dagger\right\| \left\|S_F(h')\right\|$$
$$= \left(\lambda_1 \|h\| + \frac{\gamma}{\sqrt{A}} \left\|K^\dagger\right\|\right) \left\|S_F(h')\right\| + \lambda_2 \left\|U_G T_F^*(h')\right\| \; \|h\| .$$

So,

$$\left\|S_F(h') - U_G T_F^*(h')\right\| = \sup_{h \in (H)_1} \left|\left\langle S_F(h') - U_G T_F^*(h'), h\right\rangle\right|$$
$$\leq \left(\lambda_1 + \frac{\gamma}{\sqrt{A}} \left\|K^\dagger\right\|\right) \left\|S_F(h')\right\| + \lambda_2 \left\|U_G T_F^*(h')\right\| .$$

Therefore, we can write

$$\frac{1 - \left(\lambda_1 + \frac{\gamma}{\sqrt{A}} \left\|K^\dagger\right\|\right)}{1 + \lambda_2} \left\|S_F(h')\right\| \leq \left\|U_G T_F^*(h')\right\| , \tag{4.5}$$

and

$$\left\|U_G T_F^*(h')\right\| \leq \frac{1 + \lambda_1 + \frac{\gamma}{\sqrt{A}} \left\|K^\dagger\right\|}{1 - \lambda_2} \left\|S_F(h')\right\| . \tag{4.6}$$

Combinig (1.1), (4.5) and (4.6), we have

$$\frac{\left[1 - \left(\lambda_1 + \frac{\gamma}{\sqrt{A}} \left\|K^\dagger\right\|\right)\right] A \left\|K^\dagger\right\|^{-2}}{1 + \lambda_2} \|h'\| \leq \|U_G T_F^*(h')\| \tag{4.7}$$
$$\leq \frac{\left[1 + \lambda_1 + \frac{\gamma}{\sqrt{A}} \left\|K^\dagger\right\|\right] B}{1 - \lambda_2} \|h'\| ,$$

for each $h' \in R(K)$.

Let $Q := U_G T_F^*$. Now, we prove that $R(Q)$ is closed. In fact, for each $\{y_n\}_{n=1}^{\infty} \subset R(Q)$ with

$$\lim_{n \to \infty} y_n = y, \quad y \in H,$$

there exists $x_n \in R(K)$ such that

$$y_n = Q(x_n) . \tag{4.8}$$

By (4.7) and (4.8) we have

$$\|x_n - x_m\| \leq D^{-1} \|Q(x_n - x_m)\|$$
$$\leq D^{-1} \|y_m - y_n\| ,$$

where

$$D=\frac{\left[1-\left(\lambda_1+\frac{\gamma}{\sqrt{A}}\left\|K^{\dagger}\right\|\right)\right]A\left\|K^{\dagger}\right\|^{-2}}{1+\lambda_2}.$$

It follows that $\{x_n\}_{n=1}^{\infty}$ is a cauchy sequence, so there exists $x\in R(K)$ such that $\lim_{n\to\infty}\ x_n=x$. By the continuity of Q we have

$$y=\lim_{n\to\infty}y_n=\lim_{n\to\infty}Q(x_n)=Q(x)\in R(Q),$$

which implies that $R(Q)$ is closed. By (4.7), we know that Q is injective on $R(K)$. Then, we conclude that $Q:R(K)\to R(Q)$ is invertible. Thus, combining with (4.5) and (4.7) we obtain that, for all $y\in Q(R(K))$

$$\left\|S_FQ^{-1}(y)\right\|\leq\frac{1+\lambda_2}{1-\left(\lambda_1+\frac{\gamma}{\sqrt{A}}\left\|K^{\dagger}\right\|\right)}\|y\|, \tag{4.9}$$

$$\left\|Q^{-1}(y)\right\|\leq\frac{1+\lambda_2}{\left[1-\left(\lambda_1+\frac{\gamma}{\sqrt{A}}\left\|K^{\dagger}\right\|\right)\right]A\left\|K^{\dagger}\right\|^{-2}}\|y\|. \tag{4.10}$$

Now, for each $h_1\in H$, we get

$$\begin{aligned}\Pi_{Q(R(k))}Kh_1&=QQ^{-1}\Pi_{Q(R(K))}Kh_1\\&=U_G\left(T_F^*Q^{-1}\Pi_{Q(R(k))}Kh_1\right)\\&=\int_X\left(T_F^*Q^{-1}\Pi_{Q(R(k))}Kh_1\right)G\,d\mu\\&=\int_X\psi G\,d\mu,\end{aligned}$$

where $\psi:=T_F^*(Q^{-1}\Pi_{Q(R(K))}Kh_1)\in L^2(X)$. Hence, for any $h\in H$

$$\begin{aligned}&\left\|K^*(\Pi_{Q(R(K))})^*h\right\|\\&\quad=\sup_{h_1\in(H)_1}\left|\left\langle K^*(\Pi_{Q(R(K))})^*h,h_1\right\rangle\right|\\&\quad=\sup_{h_1\in(H)_1}\left|\left\langle h,(\Pi_{Q(R(K))})Kh_1\right\rangle\right|\\&\quad=\sup_{h_1\in(H)_1}\left|\left\langle(\Pi_{Q(R(K))})Kh_1,h\right\rangle\right|\\&\quad=\sup_{h_1\in(H)_1}\left|\left\langle\int_X\psi G\,d\mu,h\right\rangle\right|\\&\quad=\sup_{h_1\in(H)_1}\left|\int_X\psi(x)\left\langle G(x),h\right\rangle\,d\mu\right|\end{aligned}$$

$$\begin{aligned}
&\leq \sup_{h_1\in(H)_1}\left(\int_X |\psi(x)|^2\ d\mu\right)^{\frac{1}{2}}\left(\int_X |\langle G(x),h\rangle|^2\ d\mu\right)^{\frac{1}{2}}\\
&= \sup_{h_1\in(H)_1}\|\psi\|_2\left(\int_X |\langle G(x),h\rangle|^2\ d\mu\right)^{\frac{1}{2}}\\
&= \sup_{h_1\in(H)_1}\left\|T_F^*Q^{-1}\Pi_{Q(R(K))}Kh_1\right\|_2\left(\int_X |\langle G(x),h\rangle|^2\ d\mu\right)^{\frac{1}{2}}\\
&= \sup_{h_1\in(H)_1}\left(\left\langle S_FQ^{-1}\Pi_{Q(R(K))}Kh_1, Q^{-1}\Pi_{Q(R(K))}Kh_1\right\rangle\right)^{\frac{1}{2}}\left(\int_X |\langle G(x),h\rangle|^2\ d\mu\right)^{\frac{1}{2}}\\
&\leq \left\|S_FQ^{-1}\Pi_{Q(R(K))}K\right\|^{\frac{1}{2}}\left\|Q^{-1}\Pi_{Q(R(K))}K\right\|^{\frac{1}{2}}\left(\int_X |\langle G(x),h\rangle|^2\ d\mu\right)^{\frac{1}{2}}\\
&\leq \frac{(1+\lambda_2)}{\left[1-\left(\lambda_1+\frac{\gamma}{\sqrt{A}}\|K^\dagger\|\right)\sqrt{A}\right]\|K^\dagger\|^{-1}}\|K\|\left(\int_X |\langle G(x),h\rangle|^2\ d\mu\right)^{\frac{1}{2}}\\
&= \frac{(1+\lambda_2)\|K\|}{\left[\sqrt{A}\,\|K^\dagger\|^{-1}(1-\lambda_1)-\gamma\right]}\left(\int_X |\langle G(x),h\rangle|^2\ d\mu\right)^{\frac{1}{2}}.
\end{aligned}$$

Thus, for each $h \in H$

$$\frac{\left[\sqrt{A}\left\|K^\dagger\right\|^{-1}(1-\lambda_1)-\gamma\right]^2}{(1+\lambda_2)^2\|K\|^2}\left\|K^*(\Pi^*_{Q(R(K))})h\right\|^2 \leq \int_X |\langle G(x),h\rangle|^2\ d\mu.$$

□

References

1. S.T. Ali, J.P. Antoine, and J.P. Gazeau, *Continuous frames in Hilbert spaces*, Ann.Phys., 222 (1993), pp. 1-37.
2. H. Bolcskel, F. Hlawatsch, and H.G. Feichyinger, *Frame-Theoretic analysis of oversampled filter bank*, IEEE Trans. Signal Processing., 46 (1998), pp. 3256- 3268.
3. P.G. Casazza, G. Kutyniok, and S. Li, *Fusion frames and Distributed Processing*, Appl. Comput. Harmon. Anal., 25 (2008), pp. 114-132.
4. O. Christensen, *Introduction to frames and Riesz bases*, Birkhäuser, Boston, 2003.
5. I. Daubechies, A. Grossmann, and Y. Meyer, *Painless nonorthogonal Expansions*, J. Math. Phys., 27 (1986), pp. 1271-1283.
6. R.G. Douglas, *On majorization, factorization and range inclusion of operators on Hilbert spaces*, Proc. Amer. Math. Soc., 17 (1966), pp. 413-415.

7. R.J. Duffin and A.C. Schaeffer, *A class of nonharmonik Fourier series*, Trans. Amer. Math. Soc., 72 (1952), pp. 341-366.
8. M.H. Faroughi and E. Osgooei, *C-Frames and C-Bessel Mappings*, Bull. Iran. Math., 38 (2012), pp. 203-222.
9. L. Găvruţa, *Frames for operators*, Appl. Comput. Harmon. Anal., 32 (2012), pp. 139-144.
10. G. Kaiser, *A Friendly Guide to Wavelets*, Birkhäuser, Boston, 1994.
11. A. Rahimi, A. Najati, and Y.N. Dehgan, *Continuous frame in Hilbert space*, Methods of Functional Analysis and Topology., 12 (2006), pp. 170-182.
12. GH. Rahimlou, R. Ahmadi, M.A. Jafarizadeh, and S. Nami, *Continuous k-Frames and their duals*, (2018) Submitted.
13. W. Rudin, *Real and Complex Analysis*, New York, Tata Mc Graw-Hill Editions, 1987.
14. X. Xiao, Y. Zhu, and L. Găvruţa, *Some Properties of K-frames in Hilbert Spaces*, Results. Math., 63 (2012), pp. 1243-1255.

[1] Department of Mathematics, Shabestar Branch, Islamic Azad University, Shabestar, Iran.
E-mail address: grahimlou@gmail.com

[2] Institute of Fundamental Sciences, University of Tabriz, Tabriz, Iran.
E-mail address: rahmadi@tabrizu.ac.ir

[3] Faculty of Physic, University of Tabriz, Tabriz, Iran.
E-mail address: jafarizadeh@tabrizu.ac.ir

[4] Faculty of Physic, University of Tabriz, Tabriz, Iran.
E-mail address: S.Nami@tabrizu.ac.ir