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Coefficient Bounds for Analytic bi-Bazilevič Functions Related to Shell-like Curves Connected with Fibonacci Numbers

Hatun Özlem GÜNEY

Abstract. In this paper, we define and investigate a new class of bi-Bazilevic functions related to shell-like curves connected with Fibonacci numbers. Furthermore, we find estimates of first two coefficients of functions belonging to this class. Also, we give the Fekete-Szegö inequality for this function class.

1. Introduction

Let $\mathbb{U} = \{z : |z| < 1\}$ denote the unit disc in the complex plane. The class of all analytic functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

in the open unit disc $\mathbb{U}$ with normalization $f(0) = f'(0) - 1 = 0$ is denoted by $\mathcal{A}$ and the class $\mathcal{S} \subset \mathcal{A}$ is the class which consists of univalent functions in $\mathbb{U}$.

A function f is subordinate to F in $\mathbb{U}$, written as $f \prec F$, if and only if $f(z) = F(w(z))$ for some analytic function w such that $|w(z)| \leq |z|$ for all $z \in \mathbb{U}$. We recall important subclasses of $\mathcal{S}$ in geometric function theory such that if $f \in \mathcal{A}$ and

$$\mathcal{S}^*[p(z)] = \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec p(z); z \in \mathbb{U} \right\}, \tag{1.2}$$

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and

$$\mathcal{C}[p(z)] = \left\{ f \in \mathcal{A} : 1 + \frac{zf''(z)}{f'(z)} \prec p(z); z \in \mathbb{U} \right\}, \tag{1.3}$$

where $p(z) = \frac{1+(1-2\alpha)z}{1-z}, 0 \le \alpha < 1$, then $p(\mathbb{U})$ is the half plane $\operatorname{Re}(w) > \alpha$, and the sets (1.2) and (1.3) become the classes starlike of order α and convex of order α, respectively. These functions form known classes denoted by $\mathcal{S}^*(\alpha)$ and $\mathcal{C}(\alpha)$, respectively. Especially, it is known that $\mathcal{S}^*(0) = \mathcal{S}^*$ and $\mathcal{C}(0) = \mathcal{C}$ (see for details [4]).

For $0 \le \alpha < 1$, $0 \le \beta < 1$, a function $f \in \mathcal{S}$ is said to be Bazilevič [1] of order α and type β, denoted by $\mathcal{B}(\alpha, \beta)$, if

$$Re \left\{ \frac{zf'(z)f(z)^{\beta-1}}{z^{\beta}} \right\} > \alpha, \quad z \in \mathbb{U}. \tag{1.4}$$

The Koebe one quarter theorem [4] guarantees that the image of $\mathbb{U}$ under every univalent function $f \in \mathcal{A}$ contains a disk of radius $\frac{1}{4}$. So, every univalent function f has an inverse f^{-1} satisfying

$$f^{-1}(f(z)) = z, \ (z \in \mathbb{U}), \qquad f(f^{-1}(w)) = w \ (|w| < r_0(f), r_0(f) \ge \frac{1}{4}).$$

A function $f \in \mathcal{A}$ is said to be bi-univalent in $\mathbb{U}$ if both the function f and its inverse function f^{-1} are univalent in $\mathbb{U}$. Let Σ denote the class of bi-univalent functions defined in the unit disk $\mathbb{U}$. Since $f \in \Sigma$ has the Maclaurian series given by (1.1), a computation shows that its inverse $g = f^{-1}$ has the expansion

$$g(w) = f^{-1}(w) = w - a_2 w^2 + \left(2a_2^2 - a_3\right) w^3 + \cdots . \tag{1.5}$$

In addition, a function is said to be bi-Bazilevič in $\mathbb{U}$ if both the function and its inverse are Bazilevič in $\mathbb{U}$.

The work of Srivastava et al. [18] essentially revived the investigation of various subclasses of the bi-univalent function class in recent years. In a considerably large number of sequels to the aforementioned work of Srivastava et al. [18], several different subclasses of the bi-univalent function class Σ were introduced and studied analogously by many authors (see, for example, [2, 3, 8, 9, 13–22]), but only non-sharp estimates on the initial coefficients $|a_2|$ and $|a_3|$ in the Taylor-Maclaurin expansion (1.1) were obtained in these recent papers.

The object of the present work is to introduce a new subclass of the function class Σ and find estimates on the coefficients $|a_2|$ and $|a_3|$ for functions in this new subclass of the function class Σ using the technique of Srivastava et al. [18]

In [12], Sokół familiarized the class $\mathcal{SL}$ of shell-like functions as the set of functions $f \in \mathcal{A}$ which is defined in the following definition:

Definition 1.1. The function $f \in \mathcal{A}$ belongs to the class $\mathcal{SL}$ if it satisfies the condition that

$$\frac{zf'(z)}{f(z)} \prec \tilde{p}(z), \tag{1.6}$$

with

$$\tilde{p}(z) = \frac{1+\tau^2 z^2}{1-\tau z-\tau^2 z^2}, \tag{1.7}$$

where $\tau = (1-\sqrt{5})/2 \approx -0.618$.

It should be observed that $\mathcal{SL}$ is a subclass of the starlike functions $\mathcal{S}^*$.

The function $\tilde{p}$ is not univalent in $\mathbb{U}$, but this function is univalent in the disc $|z| < (3-\sqrt{5})/2 \approx 0.38$. Indeed, $\tilde{p}(0) = \tilde{p}(-1/2\tau) = 1$ and $\tilde{p}(e^{\mp i \arccos(1/4)}) = \sqrt{5}/5$, and also it may be realized that

$$\frac{1}{|\tau|} = \frac{|\tau|}{1-|\tau|},$$

which shows that the number $|\tau|$ divides $[0,1]$ such that it fulfils the golden section. The image of the unit circle $|z| = 1$ under $\tilde{p}$ is a curve described by the equation given by

$$(10x-\sqrt{5})y^2 = (\sqrt{5}-2x)(\sqrt{5}x-1)^2,$$

which is translated and revolved trisectrix of Maclaurin given in the equation

$$x^3 + 3ax^2 + (x-a)y^2 = 0,$$

with

$$a = \left(\frac{1-2\tau}{10}\right) = \frac{\sqrt{5}}{10}.$$

The curve $\tilde{p}(re^{it})$ is a closed curve without any loops for $0 < r \le r_0 = (3-\sqrt{5})/2 \approx 0.38$. For $r_0 < r < 1$, it has a loop, and for $r = 1$, it has a vertical asymptote. It is easy to observe that

$$\mathrm{Re}[\tilde{p}(z)] \to a,$$

and

$$\mathrm{Im}[\tilde{p}(z)] \to \infty,$$

when

$$z \to -1^+.$$

Thus, if $f \in \mathcal{SL}$, then

$$\mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > a,$$

for $z \in \mathbb{U}, a = \left(\frac{1-2\tau}{10}\right) = \frac{\sqrt{5}}{10}$, which leads to the following corollary.

Corollary 1.2 ([5]). *Let $\mathcal{SL}$ and $\mathcal{S}^*(a)$ be defined as above. Then*

$$\mathcal{SL} \subset \mathcal{S}^*(a) = \left\{f \in \mathcal{A} : \mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > a, z \in \mathbb{U}\right\}, \tag{1.8}$$

where $a = \left(\frac{1-2\tau}{10}\right) = \frac{\sqrt{5}}{10}$, which means that if $f \in \mathcal{SL}$, then it is starlike of order a. Thus it is univalent in the unit disc $\mathbb{U}$.

Considering (1.7), we understand that

$$\tilde{p}(1) = \tilde{p}\left(\tau^4 =\right) = 5a, \quad \tilde{p}\left(e^{\pm arccos(1/4)}\right) = 2a, \quad \tilde{p}(0) = \tilde{p}\left(-\frac{1}{2\tau}\right) = 1.$$

Since τ satisfies the equation $\tau^2 = 1 + \tau$, this expression can be used to obtain higher powers τ^n as a linear function of lower powers, which in turn can be decomposed all the way down to a linear combination of τ and 1. The resulting recurrence relationships yield the Fibonacci numbers u_n:

$$\tau^n = u_n\tau + u_{n-1}.$$

In [11], taking $\tau z = t$, Raina and Sokół showed that

$$\begin{aligned}\tilde{p}(z) &= \frac{1+\tau^2z^2}{1-\tau z-\tau^2z^2} \\ &= \left(t+\frac{1}{t}\right)\frac{t}{1-t-t^2} \\ &= \frac{1}{\sqrt{5}}\left(t+\frac{1}{t}\right)\left(\frac{1}{1-(1-\tau)t}-\frac{1}{1-\tau t}\right) \\ &= \left(t+\frac{1}{t}\right)\sum_{n=1}^{\infty}\frac{(1-\tau)^n-\tau^n}{\sqrt{5}}t^n \\ &= \left(t+\frac{1}{t}\right)\sum_{n=1}^{\infty}u_nt^n = 1+\sum_{n=1}^{\infty}(u_{n-1}+u_{n+1})\tau^nz^n,\end{aligned} \tag{1.9}$$

where

$$u_n = \frac{(1-\tau)^n-\tau^n}{\sqrt{5}}, \qquad \tau = \frac{1-\sqrt{5}}{2}, \quad (n = 1, 2, \ldots). \tag{1.10}$$

This shows that the relevant connection of $\tilde{p}$ with the sequence of Fibonacci numbers u_n, such that $u_0 = 0$, $u_1 = 1$, $u_{n+2} = u_n + u_{n+1}$ for

$n = 0, 1, 2, \ldots$. And they got

$$\tilde{p}(z) = 1 + \sum_{n=1}^{\infty} \tilde{p}_n z^n = 1 + (u_0 + u_2)\tau z + (u_1 + u_3)\tau^2 z^2 + \sum_{n=3}^{\infty}(u_{n-3} + u_{n-2} + u_{n-1} + u_n)\tau^n z^n = 1 + \tau z + 3\tau^2 z^2 + 4\tau^3 z^3 + 7\tau^4 z^4 + 11\tau^5 z^5 + \cdots . \tag{1.11}$$

Let $\mathcal{P}(\alpha)$, $0 \le \alpha < 1$, denote the class of analytic functions p in $\mathbb{U}$ with $p(0) = 1$ and $\mathrm{Re}\{p(z)\} > \alpha$. Especially, we will use $\mathcal{P}$ instead of $\mathcal{P}(0)$.

Theorem 1.3 ([6]). *The function* $\tilde{p}(z) = \frac{1+\tau^2 z^2}{1-\tau z-\tau^2 z^2}$ *belongs to the class* $\mathcal{P}(a)$ *with* $a = \sqrt{5}/10 \approx 0.2236$.

Now we give the following lemma which will be used in sequel.

Lemma 1.4 ([10]). *Let* $p \in \mathcal{P}$ *with* $p(z) = 1 + c_1 z + c_2 z^2 + \cdots$, *then*

$$|c_n| \le 2, \text{ for } n \ge 1. \tag{1.12}$$

In the present work, we introduce a new subclass of Σ associated with shell-like functions connected with the Fibonacci numbers and obtain the initial Taylor coefficients $|a_2|$ and $|a_3|$ for this function class. Further, Fekete and Szegö [7] introduced the generalized functional $a_3 - \mu a_2^2$, where μ is some real number. Also, we give a bound for the Fekete-Szegö functional $|a_3 - \mu a_2^2|$ in this function class.

2. Bi-Bazilevič Function Class $\mathcal{B}_\Sigma(\alpha, \beta)$

Firstly, let $p(z) = 1 + p_1 z + p_2 z^2 + \cdots$, and $p \prec \tilde{p}$. Then there exists an analytic function u such that $|u(z)| < 1$ in $\mathbb{U}$ and $p(z) = \tilde{p}(u(z))$. Therefore, the function

$$h(z) = \frac{1 + u(z)}{1 - u(z)} = 1 + c_1 z + c_2 z^2 + \cdots , \tag{2.1}$$

is in the class $\mathcal{P}(0)$. It follows that

$$u(z) = \frac{c_1 z}{2} + \left(c_2 - \frac{c_1^2}{2}\right)\frac{z^2}{2} + \left(c_3 - c_1 c_2 + \frac{c_1^3}{4}\right)\frac{z^3}{2} + \cdots , \tag{2.2}$$

and

(2.3)

$$\tilde{p}(u(z)) = 1 + \tilde{p}_1 \left\{ \frac{c_1 z}{2} + \left(c_2 - \frac{c_1^2}{2}\right)\frac{z^2}{2} + \left(c_3 - c_1 c_2 + \frac{c_1^3}{4}\right)\frac{z^3}{2} + \cdots \right\}$$

$$+\tilde{p}_2\left\{\frac{c_1z}{2}+\left(c_2-\frac{c_1^2}{2}\right)\frac{z^2}{2}+\left(c_3-c_1c_2+\frac{c_1^3}{4}\right)\frac{z^3}{2}+\cdots\right\}^2$$
$$+\tilde{p}_3\left\{\frac{c_1z}{2}+\left(c_2-\frac{c_1^2}{2}\right)\frac{z^2}{2}+\left(c_3-c_1c_2+\frac{c_1^3}{4}\right)\frac{z^3}{2}+\cdots\right\}^3+\cdots$$
$$=1+\frac{\tilde{p}_1c_1z}{2}+\left\{\frac{1}{2}\left(c_2-\frac{c_1^2}{2}\right)\tilde{p}_1+\frac{c_1^2}{4}\tilde{p}_2\right\}z^2$$
$$+\left\{\frac{1}{2}\left(c_3-c_1c_2+\frac{c_1^3}{4}\right)\tilde{p}_1+\frac{1}{2}c_1\left(c_2-\frac{c_1^2}{2}\right)\tilde{p}_2+\frac{c_1^3}{8}\tilde{p}_3\right\}z^3+\cdots.$$

And similarly, there exists an analytic function v such that $|v(w)|<1$ in $\mathbb{U}$ and $p(w)=\tilde{p}(v(w))$. Therefore, the function

$$k(w)=\frac{1+v(w)}{1-v(w)}=1+d_1w+d_2w^2+\ldots, \tag{2.4}$$

is in the class $\mathcal{P}(0)$. It follows that

$$v(w)=\frac{d_1w}{2}+\left(d_2-\frac{d_1^2}{2}\right)\frac{w^2}{2}+\left(d_3-d_1d_2+\frac{d_1^3}{4}\right)\frac{w^3}{2}+\cdots, \tag{2.5}$$

and

(2.6)
$$\tilde{p}(v(w))=1+\frac{\tilde{p}_1d_1w}{2}+\left\{\frac{1}{2}\left(d_2-\frac{d_1^2}{2}\right)\tilde{p}_1+\frac{d_1^2}{4}\tilde{p}_2\right\}w^2$$
$$+\left\{\frac{1}{2}\left(d_3-d_1d_2+\frac{d_1^3}{4}\right)\tilde{p}_1+\frac{1}{2}d_1\left(d_2-\frac{d_1^2}{2}\right)\tilde{p}_2+\frac{d_1^3}{8}\tilde{p}_3\right\}w^3+\cdots.$$

Definition 2.1. For $(\frac{1-2\tau}{10})\le\alpha<1$, $0\le\beta<1$, a function $f\in\Sigma$ of the form (1.1) is said to be in the class $\mathcal{B}_\Sigma(\alpha,\beta)$, the class of bi-Bazilevic functions of order α and type β, if and only if

$$\operatorname{Re}\left\{\frac{zf'(z)f(z)^{\beta-1}}{z^\beta}\right\}>\alpha, \tag{2.7}$$

and

$$\operatorname{Re}\left\{\frac{wg'(w)g(w)^{\beta-1}}{w^\beta}\right\}>\alpha, \tag{2.8}$$

where $z,w\in\mathbb{U}$; g and τ are given by (1.5) and (1.10), respectively.

Conditions (2.7) and (2.8) in the above theorem can be rewritten as follows:

$$\frac{zf'(z)f(z)^{\beta-1}}{z^\beta}=\alpha+(1-\alpha)p(z), \tag{2.9}$$

and

$$\frac{wg'(w)g(w)^{\beta-1}}{w^{\beta}} = \alpha + (1-\alpha)p(w), \tag{2.10}$$

where $p(z)$ and $p(w) \in \mathcal{P}$ have the forms (2.3) and (2.6), respectively.

Specializing the parameter $\beta = 0$ we have the following:

Definition 2.2. A function $f \in \Sigma$ of the form (1.1) is said to be in the class $\mathcal{S}^*{}_{\Sigma}(\alpha)$ if and only if

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha, \tag{2.11}$$

and

$$\operatorname{Re}\left\{\frac{wg'(w)}{g(w)}\right\} > \alpha, \tag{2.12}$$

where $z, w \in \mathbb{U}$, $\alpha = (\frac{1-2\tau}{10}) = \frac{\sqrt{5}}{10}$, and g is given by (1.5).

In the following theorem we determine the initial Taylor coefficients $|a_2|$ and $|a_3|$ for the function class $\mathcal{B}_{\Sigma}(\alpha, \beta)$.

Theorem 2.3. *Let f given by* (1.1) *be in the class $\mathcal{B}_{\Sigma}(\alpha, \beta)$. Then*

$$|a_2| \leq \frac{\sqrt{2}(1-\alpha)|\tau|}{\sqrt{2(1+\beta) - (1-\alpha)(4+5\beta)\tau}}, \tag{2.13}$$

and

$$|a_3| \leq \frac{(1-\alpha)|\tau|\,[2(1+\beta) - (1-\alpha)(8+7\beta)\tau]}{(2+\beta)\,[2(1+\beta) - (1-\alpha)(4+5\beta)\tau]}. \tag{2.14}$$

Proof. Let $f \in \mathcal{B}_{\Sigma}(\alpha, \beta)$ and $g = f^{-1}$. Considering (2.3) and (2.6), we have

$$\frac{zf'(z)f(z)^{\beta-1}}{z^{\beta}} = \alpha + (1-\alpha)\tilde{p}(u(z)), \tag{2.15}$$

and

$$\frac{wg'(w)g(w)^{\beta-1}}{w^{\beta}} = \alpha + (1-\alpha)\tilde{p}(v(w)), \tag{2.16}$$

where $z, w \in \mathbb{U}$ and g is given by (1.5). Since

(2.17)

$$\begin{aligned}\frac{zf'(z)f(z)^{\beta-1}}{z^{\beta}} &= \left(\frac{f(z)}{z}\right)^{\beta} \frac{zf'(z)}{f(z)} \\ &= 1 + (\beta+1)a_2 z + \left\{\frac{(\beta-1)(\beta+2)}{2}a_2^2 + (\beta+2)a_3\right\} z^2 + \cdots \\ &= \alpha + (1-\alpha)\tilde{p}(u(z)),\end{aligned}$$

and

(2.18)

$$\frac{wg'(w)g(w)^{\beta-1}}{w^{\beta}} = \left(\frac{g(w)}{w}\right)^{\beta} \frac{wg'(w)}{g(w)}$$
$$= 1-(\beta+1)a_2w+\left\{\frac{(\beta+2)(\beta+3)}{2}a_2^2-(\beta+2)a_3\right\}w^2+\cdots$$
$$= \alpha+(1-\alpha)\tilde{p}(v(w)),$$

we have

$$(1+\beta)a_2 = \frac{(1-\alpha)c_1}{2}\tau, \tag{2.19}$$

$$(\beta+2)a_3+\frac{(\beta-1)(\beta+2)}{2}a_2^2 = \frac{(1-\alpha)}{2}\left(c_2-\frac{c_1^2}{2}\right)\tau+\frac{3(1-\alpha)}{4}c_1^2\tau^2, \tag{2.20}$$

and

$$-(1+\beta)a_2 = \frac{(1-\alpha)d_1}{2}\tau, \tag{2.21}$$

$$-(\beta+2)a_3+\frac{(\beta+2)(\beta+3)}{2}a_2^2 = \frac{(1-\alpha)}{2}\left(d_2-\frac{d_1^2}{2}\right)\tau+\frac{3(1-\alpha)}{4}d_1^2\tau^2. \tag{2.22}$$

From (2.19) and (2.21), we have

$$c_1 = -d_1, \tag{2.23}$$

and

$$2(1+\beta)^2a_2^2 = \frac{(1-\alpha)^2}{4}(c_1^2+d_1^2)\tau^2. \tag{2.24}$$

Now, by summing (2.20) and (2.22), we obtain

$$(\beta+1)(\beta+2)a_2^2 = \frac{1-\alpha}{2}(c_2+d_2)\tau-\frac{1-\alpha}{4}(c_1^2+d_1^2)\tau+\frac{3(1-\alpha)}{4}(c_1^2+d_1^2)\tau^2. \tag{2.25}$$

By putting (2.24) in (2.25), we have

$$(\beta+1)^2\left[(-4-5\beta)(1-\alpha)\tau+2(\beta+1)\right]a_2^2 = \frac{(1-\alpha)^2(\beta+1)}{2}(c_2+d_2)\tau^2. \tag{2.26}$$

Therefore, using Lemma (1.4) we obtain

$$|a_2| \leq \frac{\sqrt{2}(1-\alpha)|\tau|}{\sqrt{2(1+\beta)-(1-\alpha)(4+5\beta)\tau}}. \tag{2.27}$$

Now, so as to find the bound on $|a_3|$, let's subtract from (2.20) and (2.22). So, we find

$$2(\beta+2)a_3 - 2(\beta+2)a_2^2 = \frac{1-\alpha}{2}\,(c_2 - d_2)\,\tau. \tag{2.28}$$

Hence, we get

$$|a_3| \leq \frac{1-\alpha}{\beta+2}\tau + |a_2|^2. \tag{2.29}$$

Then, in view of (2.27), we obtain

$$|a_3| \leq \frac{(1-\alpha)|\tau|\,[2(1+\beta)-(1-\alpha)(8+7\beta)\tau]}{(2+\beta)\,[2(1+\beta)-(1-\alpha)(4+5\beta)\tau]}. \tag{2.30}$$

If we take the parameter $\beta = 0$ in the above theorem, we have the following initial Taylor coefficients $|a_2|$ and $|a_3|$ for the function classes $\mathcal{S}^*{}_\Sigma(\alpha)$. □

Corollary 2.4. *Let f given by* (1.1) *be in the class $\mathcal{S}^*{}_\Sigma(\alpha)$. Then*

$$|a_2| \leq \frac{(1-\alpha)|\tau|}{\sqrt{1-2(1-\alpha)\tau}}, \tag{2.31}$$

and

$$|a_3| \leq \frac{(1-\alpha)|\tau|\,[1-4(1-\alpha)\tau]}{[2-4(1-\alpha)\tau]}. \tag{2.32}$$

3. Fekete-Szegö Inequality for the Function Class $\mathcal{B}_\Sigma(\alpha,\beta)$

Due to Zaprawa [23], the following theorem is the solution of the Fekete-Szegö problem in $\mathcal{B}_\Sigma(\alpha,\beta)$.

Theorem 3.1. *Let f given by (1.1) be in the class $\mathcal{B}_\Sigma(\alpha,\beta)$ and $\mu\in\mathbb{R}$. Then we have*

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{(1-\alpha)}{\beta+2}|\tau|, & |\mu-1| \leq \frac{(\beta+1)[2(\beta+1)-(4+5\beta)(1-\alpha)\tau]}{2(\beta+2)(1-\alpha)|\tau|}, \\ \frac{2|1-\mu|(1-\alpha)^2\tau^2}{(\beta+1)[2(\beta+1)-(4+5\beta)(1-\alpha)\tau]}, & |\mu-1| \geq \frac{(\beta+1)[2(\beta+1)-(4+5\beta)(1-\alpha)\tau]}{2(\beta+2)(1-\alpha)|\tau|}. \end{cases}$$

Proof. From (2.26) and (2.28) we obtain

(3.1)

$$\begin{aligned} a_3 - \mu a_2^2 &= (1-\mu)\frac{(1-\alpha)^2\tau^2(c_2+d_2)}{2(\beta+1)\,[2(\beta+1)-(4+5\beta)(1-\alpha)\tau]} + \frac{(1-\alpha)\tau(c_2-d_2)}{4(\beta+2)} \\ &= \left(\frac{(1-\mu)(1-\alpha)^2\tau^2}{2(\beta+1)\,[2(\beta+1)-(4+5\beta)(1-\alpha)\tau]} + \frac{(1-\alpha)\tau}{4(\beta+2)}\right) c_2 \\ &\quad + \left(\frac{(1-\mu)(1-\alpha)^2\tau^2}{2(\beta+1)\,[2(\beta+1)-(4+5\beta)(1-\alpha)\tau]} - \frac{(1-\alpha)\tau}{4(\beta+2)}\right) d_2. \end{aligned}$$

So we have

$$a_3 - \mu a_2^2 = \left(h(\mu) + \frac{(1-\alpha)\tau}{4(\beta+2)} \right) c_2 + \left(h(\mu) - \frac{(1-\alpha)\tau}{4(\beta+2)} \right) d_2, \tag{3.2}$$

where

$$h(\mu) = \frac{(1-\mu)(1-\alpha)^2\tau^2}{2(\beta+1)\left[2(\beta+1) - (4+5\beta)(1-\alpha)\tau\right]}. \tag{3.3}$$

Then, by taking modulus of (3.2), we conclude that

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{1-\alpha}{\beta+2}|\tau|, & 0 \leq |h(\mu)| \leq \frac{1-\alpha}{4(\beta+2)}|\tau|, \\ 4|h(\mu)|, & |h(\mu)| \geq \frac{1-\alpha}{4(\beta+2)}|\tau|. \end{cases}$$

□

Taking $\mu = 1$, we have the following corollary.

Corollary 3.2. *If $f \in \mathcal{B}_\Sigma(\alpha, \beta)$, then*

$$|a_3 - a_2^2| \leq \frac{1-\alpha}{\beta+2}|\tau|. \tag{3.4}$$

If we take the parameter $\beta = 0$ in the above theorem, we have the following Fekete-Szegö inequalities for the function class $\mathcal{S}^*{}_\Sigma(\alpha)$.

Corollary 3.3. *Let f given by (1.1) be in the class $\mathcal{S}^*{}_\Sigma(\alpha)$ and $\mu \in \mathbb{R}$. Then we have*

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{(1-\alpha)|\tau|}{2}, & |\mu - 1| \leq \frac{1-2\tau}{2|\tau|}, \\ \frac{|1-\mu|(1-\alpha)^2\tau^2}{1-2(1-\alpha)\tau}, & |\mu - 1| \geq \frac{1-2\tau}{2|\tau|}. \end{cases}$$

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Dicle University, Department of Mathematics, Science Faculty, TR-21280 Diyarbakir, Turkey.

E-mail address: ozlemg@dicle.edu.tr