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Coefficient Estimates for Some Subclasses of Analytic and Bi-Univalent Functions Associated with Conic Domain

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Abstract. The main objective of this investigation is to introduce certain new subclasses of the class Σ of bi-univalent functions by using concept of conic domain. Furthermore, we find non-sharp estimates on the first two Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$ for functions in these new subclasses. We consider various corollaries and consequences of our main results. We also point out relevant connections to some of the earlier known developments.

1. Introduction

Let $\mathcal{A}$ be the class of all functions f, that are analytic in the open unit disk

$$\mathbb{U} := \{z \in \mathbb{C} \quad \text{and} \quad |z| < 1\},$$

where $\mathbb{C}$ is, as usual, the complex plane and normalized by

$$f(0) = 0 = f'(0) - 1.$$

In other words, the function, f in $\mathcal{A}$ have the power series representation

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad (z \in \mathbb{U}). \tag{1.1}$$

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Furthermore, by $\mathcal{S} \subset \mathcal{A}$ we shall denote the class of all functions which are univalent in $\mathbb{U}$. Given functions f and $g \in \mathcal{A}$, f is said to be subordinate to g in $\mathbb{U}$, denoted by

$$f(z) \prec g(z), \quad (z \in \mathbb{U}),$$

if there exists a function

$$w \in \mathcal{B}_0,$$

where

$$\mathcal{B}_0 = \{w \in \mathcal{A} : \ w(0) = 0, \ |w(z)| < 1, \quad (z \in \mathbb{U})\},$$

such that

$$f(z) = g(w(z)), \quad (z \in \mathbb{U}).$$

If g is univalent in $\mathbb{U}$, then it follows that

$$f(z) \prec g(z), \quad (z \in \mathbb{U}) \quad \Rightarrow \quad f(0) = g(0) \text{ and } f(\mathbb{U}) \subset g(\mathbb{U}).$$

A function $f \in \mathcal{A}$ is said to be strongly starlike of order α $(0 < \alpha < 1)$, if and only if

$$\frac{zf'(z)}{f(z)} \prec \left(\frac{1+z}{1-z}\right)^{\alpha}. \tag{1.2}$$

One can find that (see [22])

$$\frac{zf'(z)}{f(z)} \prec \left(\frac{1+z}{1-z}\right)^{\alpha} \qquad \left(\Longleftrightarrow \left(\frac{zf'(z)}{f(z)}\right)^{\frac{1}{\alpha}} \prec \frac{1+z}{1-z}\right),$$

or equivalently the condition (1.2) can be written as follows

$$\left|arg\left(\frac{zf'(z)}{f(z)}\right)\right| < \frac{\alpha\pi}{2}.$$

We denote the class of strongly starlike functions of order α by $\mathcal{SS}^*(\alpha)$. If $\alpha = 1$ then $\mathcal{SS}^*(1) = \mathcal{S}^*$ is the well-known class of starlike functions in $\mathbb{U}$ (see for example [5]).

We next denote by $\mathcal{P}$ the class of analytic functions p which are normalized by

$$p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n, \tag{1.3}$$

such that

$$\Re(p(z)) > 0.$$

Furthermore, it is well-known that every univalent function f has an inverse f^{-1}, defined by

$$f^{-1}(f(z)) = z, \quad (z \in \mathbb{U}),$$

and

$$f(f^{-1}(w)) = w, \quad \left(|w| < r_0(f), r_0(f) \geq \frac{1}{4}\right),$$

where

$$g(w) = w - a_2 w^2 + \left(2a_2^2 - a_3\right) w^3 - \left(5a_2^3 - 5a_2 a_3 + a_4\right) w^4 + \cdots . \tag{1.4}$$

A function $f \in \mathcal{A}$ is said to be bi-univalent in $\mathbb{U}$ if both f and f^{-1} are univalent in $\mathbb{U}$. We denote the class of all such functions by Σ. In the recent years the work of Srivastava et al. [25] fundamentally stimulated the investigation of different subclasses of the bi-univalent function. In a considerably large number of subsequent to the work of Srivastava *et al.* [25], several distinct subclasses of the bi-univalent function class were presented and examined similarly by the many authors (see, for example, [6, 7, 26, 29–31, 33]).

Historically speaking, the conic domain Ω_k, $k \geq 0$, was first introduced by Kanas and Wiśniowska (see [8, 9]) as

$$\Omega_k = \left\{ u + iv : u > k\sqrt{(u-1)^2 + v^2} \right\}.$$

Moreover for fixed k, this domain represents the right half plane ($k = 0$), a parabola ($k = 1$), the right branch of hyperbola ($0 < k < 1$) and an ellipse ($k > 1$), (see also [15, 16, 32] and recently [11, 14, 17]). Indeed the extremal functions for these conic regions are

$$p_k(z) = \begin{cases} \frac{1+z}{1-z}, & k = 0, \\ \frac{1}{1-k^2} \cosh\left\{ \left(\frac{2}{\pi} \arccos k\right) \log \frac{1+\sqrt{z}}{1-\sqrt{z}} \right\} - \frac{k^2}{1-k^2}, & 0 \leq k < 1, \\ 1 + \frac{2}{\pi^2} \left(\log \frac{1+\sqrt{z}}{1-\sqrt{z}} \right)^2, & k = 1, \\ \frac{1}{k^2-1} \sin\left(\frac{\pi}{2K(\kappa)} \int_0^{\frac{u(z)}{\sqrt{\kappa}}} \frac{dt}{\sqrt{1-t^2}\sqrt{1-\kappa^2 t^2}} \right) + \frac{k^2}{k^2-1}, & k > 1, \end{cases} \tag{1.5}$$

where

$$u(z) = \frac{z - \sqrt{\kappa}}{1 - \sqrt{\kappa} z}, \quad (\forall\ z \in \mathbb{E}),$$

and $\kappa \in (0,1)$ is chosen such that $k = \cosh\left(\pi K'(\kappa)/(4K(\kappa))\right)$. Here $K(\kappa)$ is Legendre's complete elliptic integral of first kind and $K'(\kappa) = K(\sqrt{1-\kappa^2})$, i.e. $K'(\kappa)$ is the complementary integral of $K(\kappa)$, [1, 2].

If $p_k(z) = 1 + \delta_k z + \cdots$, then it is showed in [10] that from (1.5) one can have

$$\delta_k = \begin{cases} \frac{8(\arccos k)^2}{\pi(1-k^2)}, & 0 \leq k < 1 \\ \frac{8}{\pi^2}, & k = 1, \\ \frac{\pi^2}{4k^2(1+\kappa)\sqrt{\kappa}R^2(\kappa)}, & k > 1. \end{cases}$$

In the recent years, several interesting subclasses of analytic and bi-univalent functions have been introduced and investigated, see for example [3, 4, 12, 13, 18–21, 23, 27, 28]. Motivated by the above mentioned works, we here introduce two new subclasses of the function class Σ and find estimates on the coefficients $|a_2|$ and $|a_3|$ for functions in these new subclasses of the function class Σ.

2. A set of Lemmas

In order to derive our main results, the following lemmas will be required.

Lemma 2.1 ([5]). *Let the function p given by*

$$p(z) = 1 + p_1 z + p_2 z^2 + \cdots,$$

be in the class $\mathcal{P}$ of functions with positive real part. Then

$$|p_n| \leq 2, \quad (n \in \mathbb{N} = \{1, 2, 3, \ldots\}).$$

This last inequality is sharp.

Lemma 2.2 ([24]). *Let the function p given by*

$$p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n,$$

be subordinate to the function H given by

$$H(z) = 1 + \sum_{n=1}^{\infty} C_n z^n.$$

If the function H is univalent in $\mathbb{U}$ and $H(\mathbb{U})$ is convex, then

$$|p_n| \leq |C_1|, \quad (n \in \mathbb{N}).$$

Throughout this paper, we assume that

$$0 < \beta \leq 1, \quad 0 < \alpha \leq 1, \quad k \geq 0 \quad \text{and} \quad \lambda \geq 0.$$

3. Coefficient bounds for the function class $\mathcal{H}_\Sigma(\alpha,\lambda,k)$

Definition 3.1. A function $f\in\mathcal{A}$ of the form given by (1.1) be in the function class $\mathcal{H}_\Sigma(\alpha,\lambda,k)$, if the following conditions are satisfied $f\in\Sigma$ and

$$f'(z)+z\lambda f''(z)\prec(p_k(z))^\alpha,\quad(z\in\mathbb{U})$$

$$\Longleftrightarrow\quad\left(f'(z)+z\lambda f''(z)\right)^{\frac{1}{\alpha}}\prec p_k(z),\quad(z\in\mathbb{U}),\tag{3.1}$$

and

$$g'(w)+w\lambda g''(w)\prec(p_k(w))^\alpha,\quad(w\in\mathbb{U})$$

$$\Longleftrightarrow\quad\left(g'(w)+w\lambda g''(w)\right)^{\frac{1}{\alpha}}\prec p_k(w),\quad(z\in\mathbb{U}),\tag{3.2}$$

where the function g is given by (1.4).

Remark 3.2. First of all it is easily seen that

$$\mathcal{H}_\Sigma(\alpha,\lambda,0)=\mathcal{H}_\Sigma(\alpha,\lambda),\quad(\lambda>0),$$

where $\mathcal{H}_\Sigma(\alpha,\lambda)$, is the function class introduced and studied by Frasin (see [6]). Secondly, we have

$$\mathcal{H}_\Sigma(\alpha,0,0)=\mathcal{H}_\Sigma(\alpha)=\mathcal{H}_\Sigma^\alpha,$$

where $\mathcal{H}_\Sigma^\alpha$, is the function class introduced and studied by Srivastava et al. (see [26]).

Theorem 3.3. *Let the function $f\in\mathcal{A}$ of the form given by* (1.1) *be in the function class $\mathcal{H}_\Sigma(\alpha,\lambda,k)$. Then:*

$$|a_2|\le\alpha\sqrt{\frac{\delta_k}{(\alpha+2)+2\lambda(\alpha+\lambda+2-\alpha\lambda)}},\tag{3.3}$$

and

$$|a_3|\le\frac{(\alpha\delta_k)^2}{4(1+\lambda)^2}+\frac{\alpha\delta_k}{3(1+2\lambda)}.\tag{3.4}$$

Proof. First of all it can be seen from the conditions (3.1) and (3.2), that

$$f'(z)+z\lambda f''(z)=[p(z)]^\alpha,\tag{3.5}$$

where

$$p(z)\prec p_k(z),$$

$$1+p_1z+p_2z^2+\cdots\prec1+\delta_kz+\cdots,\tag{3.6}$$

and

$$g'(w)+w\lambda g''(w)=[q(w)]^\alpha,\tag{3.7}$$

where

$$q(w) \prec p_k(w),$$

$$1 + q_1 w + q_2 w^2 + \cdots \prec 1 + \delta_k w + \cdots . \tag{3.8}$$

in virtue of (3.6), (3.8) and Lemma 2.2 we have

$$|p_n| < |\delta_k| \quad \text{and} \quad |q_n| < |\delta_k| . \tag{3.9}$$

Now equating the coefficients in (3.5) and (3.7), we have

$$2(1+\lambda) a_2 = \alpha p_1, \tag{3.10}$$

$$3(1+2\lambda) a_3 = \alpha p_2 + \frac{\alpha(\alpha-1)}{2} p_1^2, \tag{3.11}$$

$$-2(1+\lambda) a_2 = \alpha q_1, \tag{3.12}$$

and

$$3(1+2\lambda)\left(2a_2^2 - a_3\right) = \alpha q_2 + \frac{\alpha(\alpha-1)}{2} q_1^2. \tag{3.13}$$

From (3.10) and (3.12), we have

$$p_1 = -q_1, \tag{3.14}$$

and

$$8(1+\lambda)^2 a_2^2 = \alpha^2\left(p_1^2 + q_1^2\right). \tag{3.15}$$

Also from (3.11), (3.13) and (3.15), we find after some simplification that

$$a_2^2 = \frac{\alpha^2(p_2+q_2)}{2(\alpha+2) + 4\lambda(\alpha+\lambda+2-\alpha\lambda)}. \tag{3.16}$$

Finally, by using (3.9) in conjunction with (3.16), we obtain the desired estimate on the coefficient $|a_2|$ as stated in (3.3).

Next, in order to prove (3.4), we subtract (3.13) from (3.11). Indeed, we find that

$$6(1+2\lambda) a_3 - 6(1+2\lambda) a_2^2 = \alpha(p_2 - q_2) + \frac{\alpha(\alpha-1)}{2}\left(p_1^2 - q_1^2\right). \tag{3.17}$$

It follows from (3.14), (3.15) and (3.17) that

$$a_3 = \frac{\alpha^2\left(p_1^2+q_1^2\right)}{8(1+\lambda)^2} + \frac{\alpha(p_2-q_2)}{6(1+2\lambda)}. \tag{3.18}$$

Finally, by using (3.9) and (3.18), we find the desired estimate on the coefficient $|a_3|$ as stated in (3.4). □

We readily observe that, in Theorem 3.3, if we put $\lambda = 1$ and $\lambda = 0$, we respectively get the following corollaries.

Corollary 3.4. *Let the function $f \in \mathcal{A}$ of the form given by* (1.1) *be in the function class $\mathcal{H}_{\Sigma}(\alpha, 1, k)$. Then:*

$$|a_2| \leq \alpha \sqrt{\frac{\delta_k}{(\alpha+2)+6}},$$

and

$$|a_3| \leq \frac{(\alpha\delta_k)^2}{16} + \frac{\alpha\delta_k}{9}.$$

Corollary 3.5. *Let the function $f \in \mathcal{A}$ of the form given by* (1.1) *be in the function class $\mathcal{H}_{\Sigma}(\alpha, 0, k)$. Then:*

$$|a_2| \leq \alpha \sqrt{\frac{\delta_k}{(\alpha+2)}},$$

and

$$|a_3| \leq \frac{(\alpha\delta_k)^2}{4} + \frac{\alpha\delta_k}{3}.$$

In Theorem 3.3, if we let $\lambda > 0$ and $k = 0$, then $\delta_k = 2$. We have the following known result, stated and proved by Frasin.

Corollary 3.6 ([6, Theorem 2.2])**.** *Let $f(z)$ given by* (1.1) *be in the class $\mathcal{H}_{\Sigma}(\alpha, \lambda)$ where $\lambda > 0$, $0 < \alpha < 1$. Then*

$$|a_2| \leq \frac{2\alpha}{\sqrt{2(\alpha+2) + 4\lambda(\alpha+\lambda+2-\alpha\lambda)}},$$

and

$$|a_3| \leq \frac{\alpha^2}{(1+\lambda)^2} + \frac{2\alpha}{3(1+2\lambda)}.$$

Moreover, in Theorem 3.3, if we put $k = 0$ then $\delta_k = 2$ and $\lambda = 1$. Indeed we have the following known result.

Corollary 3.7 ([6, Corollary 2.3])**.** *Let $f(z)$ given by* (1.1) *be in the class $\mathcal{H}_{\Sigma}(\alpha, 1)$ where $0 < \alpha < 1$. Then*

$$|a_2| \leq \frac{2\alpha}{\sqrt{2(\alpha+2)+12}},$$

and

$$|a_3| \leq \frac{9\alpha^2 + 8\alpha}{36}.$$

By putting $\lambda = 0 = k$ in Theorem 3.3, we have the following known result.

Corollary 3.8 ([26])**.** *Let the function* $f(z)$ *given by the Taylor-Maclaurin series expansion* (1.1) *be in the class* $\mathcal{H}_{\Sigma}^{\alpha}$ *where* $(0 \leq \alpha \leq 1)$. *Then*

$$|a_2| \leq \alpha\sqrt{\frac{2}{\alpha+2}} \quad \textit{and} \quad |a_3| \leq \frac{\alpha(3\alpha+2)}{3}.$$

4. Coefficient Bounds for the Function Class $\mathcal{H}_{\Sigma}(\beta,\lambda,k)$

Definition 4.1. A function $f \in \mathcal{A}$ of the form given by (1.1) is in the function class $\mathcal{H}_{\Sigma}(\beta,\lambda,k)$ $(0 \leq \beta \leq 1, \lambda > 0)$, if the following conditions are satisfied $f \in \Sigma$ and

$$\Re\left(f'(z) + z\lambda f''(z)\right) > k\left|f'(z) + z\lambda f''(z) - 1\right| + \beta, \quad (z \in \mathbb{U}),$$

or equivalently

$$\Re\left(f'(z) + z\lambda f''(z)\right) > \frac{\beta-k}{1-k}, \tag{4.1}$$

and

$$\Re\left(g'(w) + w\lambda g''(w)\right) > k\left|g'(w) + w\lambda g''(w) - 1\right| + \beta, \quad (w \in \mathbb{U}),$$

or equivalently

$$\Re\left(g'(w) + w\lambda g''(w)\right) > \frac{\beta-k}{1-k}, \tag{4.2}$$

where the function g is defined by (1.4).

Remark 4.2. First of all it is readily observe that

$$\mathcal{H}_{\Sigma}(\beta,\lambda,0) = \mathcal{H}_{\Sigma}(\beta,\lambda), \quad (\lambda > 0),$$

where $\mathcal{H}_{\Sigma}(\beta,\lambda)$, is the function class introduced and studied by Frasin (see [6]). Secondly, we have

$$\mathcal{H}_{\Sigma}(\beta,0,0) = \mathcal{H}_{\Sigma}(\beta),$$

where $\mathcal{H}_{\Sigma}(\beta)$, is the function class introduced and studied by Srivastava et al. (see [26]).

Theorem 4.3. *Let the function* $f \in \mathcal{A}$ *of the form given by* (1.1) *be in the function class* $\mathcal{H}_{\Sigma}(\beta,\lambda,k)$ *Then*

$$|a_2| \leq \sqrt{\frac{2(1-2k-\beta)}{3(1-k)(1+2\lambda)}}, \tag{4.3}$$

and

$$|a_3| \leq \left(\frac{(1-2k-\beta)}{(1-k)(1+\lambda)}\right)^2 + \left(\frac{2(1-2k-\beta)}{3(1-k)(1+2\lambda)}\right). \tag{4.4}$$

Proof. First of all it follows from conditions (4.1) and (4.2), that

$$f'(z) + z\lambda f''(z) = \frac{\beta - k}{1-k} + \left(\frac{1-2k-\beta}{1-k}\right) p(z), \quad (z \in \mathbb{U}), \tag{4.5}$$

and

$$g'(w) + w\lambda g''(w) = \frac{\beta - k}{1-k} + \left(\frac{1-2k-\beta}{1-k}\right) q(w), \quad (w \in \mathbb{U}), \tag{4.6}$$

where

$$p(z) = 1 + p_1 z + p_2 z^2 + \cdots, \qquad q(w) = 1 + q_1 w + q_2 w^2 + \cdots,$$

in $\mathcal{P}$. Now equating the coefficients in (4.5) and (4.6), we have

$$2(1+\lambda) a_2 = \left(\frac{1-2k-\beta}{1-k}\right) p_1, \tag{4.7}$$

$$3(1+2\lambda) a_3 = \left(\frac{1-2k-\beta}{1-k}\right) p_2, \tag{4.8}$$

$$-2(1+\lambda) a_2 = \left(\frac{1-2k-\beta}{1-k}\right) q_1, \tag{4.9}$$

and

$$3(1+2\lambda)\left(2a_2^2 - a_3\right) = \left(\frac{1-2k-\beta}{1-k}\right) q_2. \tag{4.10}$$

Now from (4.7) and (4.9), we have

$$p_1 = -q_1, \tag{4.11}$$

and

$$8(1+\lambda)^2 a_2^2 = \left(\frac{1-2k-\beta}{1-k}\right)^2 \left(p_1^2 + q_1^2\right). \tag{4.12}$$

Also from (4.8) and (4.10) we have

$$6(1+2\lambda) a_2^2 = \left(\frac{1-2k-\beta}{1-k}\right)(p_2 + q_2). \tag{4.13}$$

Finally, by applying Lemma 2.1 in conjunction with (4.13), we obtain the desired estimate on the coefficient $|a_2|$ as stated in (4.3).

Next, in order to prove (4.4), we subtract (4.10) from (4.8). We have

$$6(1+2\lambda) a_3 - 6(1+2\lambda) a_2^2 = \left(\frac{1-2k-\beta}{1-k}\right)(p_2 - q_2), \tag{4.14}$$

which, upon substitution of the value of a_2^2 from (4.12), yields

$$a_3 = \frac{(1-2k-\beta)^2 \left(p_1^2 + q_1^2\right)}{8(1-k)^2 (1+\lambda)^2} + \frac{(1-2k-\beta)(p_2 - q_2)}{6(1-k)(1+2\lambda)}. \tag{4.15}$$

Finally, by using Lemma 2.1 and (4.15), we obtain the desired estimate on the coefficient $|a_3|$ as stated in (4.4). □

We see that for $\lambda = 1$ and $\lambda = 0$, we have the following special cases of Theorem 4.3, respectively.

Corollary 4.4. *Let the function $f \in \mathcal{A}$ of the form given by* (1.1) *be in the function class $\mathcal{H}_{\Sigma}(\beta, 1, k)$. Then*

$$|a_2| \leq \sqrt{\frac{2(1-2k-\beta)}{9(1-k)}},$$

and

$$|a_3| \leq \left(\frac{(1-2k-\beta)}{2(1-k)}\right)^2 + \left(\frac{2(1-2k-\beta)}{9(1-k)}\right).$$

Corollary 4.5. *Let the function $f \in \mathcal{A}$ of the form given by* (1.1) *be in the function class $\mathcal{H}_{\Sigma}(\beta, 0, k)$. Then*

$$|a_2| \leq \sqrt{\frac{2(1-2k-\beta)}{3(1-k)}},$$

and

$$|a_3| \leq \left(\frac{(1-2k-\beta)}{(1-k)}\right)^2 + \left(\frac{2(1-2k-\beta)}{3(1-k)}\right).$$

In Theorem 4.3, taking $\lambda = 0 = k$, we obtain the following known result.

Corollary 4.6 ([26])**.** *Let the function $f(z)$ given by the Taylor-Maclaurin series expansion* (1.1) *be in the class $\mathcal{H}_{\Sigma}(\beta)$. Then*

$$|a_2| \leq \sqrt{\frac{2(1-\beta)}{3}} \qquad \text{and} \qquad |a_3| \leq \frac{2(1-\beta)(5-3\beta)}{3}.$$

Furthermore, in Theorem 4.3, if we put $k = 0$ and let $\lambda > 0$, we obtain the following corollary, proved by Frasin.

Corollary 4.7 ([6, Theorem 3.2])**.** *Let $f(z)$ given by* (1.1) *be in the class $\mathcal{H}_{\Sigma}(\beta, \lambda)$ where $0 < \beta < 1$. Then*

$$|a_2| \leq \sqrt{\frac{2(1-\beta)}{3(1+2\lambda)}}, \qquad |a_3| \leq \frac{(1-\beta)^2}{(1+\lambda)^2} + \frac{2(1-\beta)}{3(1+2\lambda)}.$$

Finally, in Theorem 4.3, if we put $k = 0$ and $\lambda = 1$ we readily obtain the following known result

Corollary 4.8 ([6, Corollary 3.3])**.** *Let $f(z)$ given by* (1.1) *be in the class $\mathcal{H}_{\Sigma}(\beta, 1)$ where $0 < \beta < 1$. Then*

$$|a_2| \leq \frac{1}{3}\sqrt{2(1-\beta)}, \qquad |a_3| \leq \frac{(1-\beta)(9(1-\beta)+8)}{36}.$$

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