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Generalized F-contractions in Partially Ordered Metric Spaces

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ABSTRACT. We discuss about the generalized F-contraction mappings in partially ordered metric spaces. For this, we first introduce the notion of ordered weakly F-contraction mapping. We also present some fixed point results about this type of mapping in partially ordered metric spaces. Next, we introduce the notion of Ćirić type generalized ordered weakly F-contraction mapping. We also prove some fixed point results about this notion in partially ordered metric spaces. We also provide an example to support our results. In fact, this example shows that our main theorem is a genuine generalization in the area of the generalized F-contraction mappings in partially ordered metric spaces.

1. Introduction

It is well known that fixed point theory is one of the fundamental tools for solving various problems in nonlinear analysis and applied mathematical analysis. The beginning of the fixed point theory on a complete metric space is the Banach Contraction Principle, published in 1922 [4]. This principle is one of the very powerful tools for existence and uniqueness of the solution of considerable problems arising in mathematics. Because of its importance for mathematical theory, many authors have extended this principle in many directions (see [5, 7, 10, 17]).

Recently, motivated by the ideas of Tarski's fixed point theorem on ordered sets [18] and Banach fixed point theorem on complete metric space, Ran and Reurings obtained a fixed point result on an ordered complete metric space [16].

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In [16], the usual contraction of Banach fixed point principle is weakened when the operator is monotone. Then many mathematicians such as Abbas et al. [1], Agarwal et al. [3], Ćirić et al. [6], Kumam et al. [9], Nashine and Altun [12], and O'Regan and Petrusel [15] generalized this interesting result and obtained a lot of variant results. For example they studied the common fixed points for weak contraction, coincidence points for weakly increasing mappings on an ordered metric space. Also, as an application they presented some fixed point results for solving Fredholm and Voltera type integral equations. For another example, by taking the regularity of the space instead of continuity of T, Nieto [13] obtained a parallel result with [16]. Also, there are several applications of these theorems in this direction in linear and nonlinear analysis (See [14, 16]). Also, Agarwal generalized the results in [1, 2].

Recently, there has been a trend to weaken the requirement on the contraction by considering metric spaces endowed with a partial order. In 2012, one of the most popular fixed point theorems on a complete metric space is given by Wardowski [19]. For this, he introduced the concept of F-contraction, which is a proper generalization of the ordinary contraction.

In this paper, which is split in two parts, our aim is to give some fixed point results in generalized F-contraction in partially ordered metric spaces. In the first part, we recall the notion of F-contraction and present some examples and facts about this notion. In the second part of this paper, we first introduce the concept of ordered weakly F-contraction, then a fixed point theorem is proved for this concept in partially ordered metric spaces. Furthermore, we present an example which shows that our theorem is generalization of the main results in [3, 8, 19]. Next, we introduce the Ćirić type generalized ordered weakly F-contraction and give some fixed point results for these type mappings on complete matric spaces.

2. Basic Definitions

First we recall the concept of F-contraction, which was introduced by Wardowski [19]:

Definition 2.1 ([19])**.** Let $\mathcal{F}$ be the set of all functions $F : (0, \infty) \to \mathbb{R}$ satisfying the following conditions:

(F1) F is strictly increasing, i.e., for all $\alpha, \beta \in (0, \infty)$ such that $\alpha < \beta$, $F(\alpha) < F(\beta)$.
(F2) For each sequence $\{a_n\}$ of positive numbers, $\lim_{n\to\infty} a_n = 0$ if and only if $\lim_{n\to\infty} F(a_n) = -\infty$.
(F3) There exists $k \in (0, 1)$ such that $\lim_{\alpha\to 0^+} \alpha^k F(\alpha) = 0$.

Let (X,d) be a metric space and $T : X \to X$ be a mapping. Then T is said to be a F-contraction, if $F \in \mathcal{F}$ and there exists $\tau > 0$ such that

$$\forall x,y \in X \left[d\left(Tx,Ty\right) > 0 \ \Rightarrow \ \tau + F\left(d\left(Tx,Ty\right)\right) \leq F\left(d\left(x,y\right)\right)\right], \tag{2.1}$$

By considering the different types of the mapping F in (2.1) by Wardowski, we obtain the variety of contractions. Some of them are well known in the literature. We bring the following examples from [19]:

Example 2.2. Define $F_1 : (0,\infty) \to \mathbb{R}$ by $F\left(\alpha\right) = \ln \alpha$. It can be seen that F satisfies in (F1)-(F3). ((F3) for any $k \in (0,1)$). Each mapping $T : X \to X$ satisfying (2.1) is a F-contraction such that

$$d\left(Tx,Ty\right) \leq e^{-\tau} d\left(x,y\right), \quad \forall x,y \in X, Tx \neq Ty. \tag{2.2}$$

It is easy to show that if $Tx = Ty$ for $x,y \in X$, then the inequality $d\left(Tx,Ty\right) \leq e^{-\tau} d\left(x,y\right)$ also holds, i.e., T is a Banach contraction.

Example 2.3. If $F\left(\alpha\right) = \ln \alpha + \alpha$, $\alpha > 0$ then F satisfies (F1)-(F3) and the condition (2.1) is of the following form

$$\frac{d\left(Tx,Ty\right)}{d\left(x,y\right)} e^{d(Tx,Ty)-d(x,y)} \leq e^{-\tau}, \quad \forall x,y \in X, Tx \neq Ty. \tag{2.3}$$

The following corollary has been concluded by Wardowski.

Corollary 2.4. *Every F-contraction T is a contractive mapping, i.e.,*

$$d\left(Tx,Ty\right) < d\left(x,y\right), \quad \forall x,y \in X, Tx \neq Ty.$$

Therefore, every F-contraction is a continuous mapping. Furthermore, Wardowski concluded that if $F_1, F_2 \in \mathcal{F}$ with $F_1\left(\alpha\right) \leq F_2\left(\alpha\right)$ for all $\alpha > 0$ and $G = F_2 - F_1$ is nondecreasing, then every F_1-contraction T is an F_2-contraction.

He showed that for the mappings $F_1\left(\alpha\right) = \ln \alpha$, and $F_2\left(\alpha\right) = \ln \alpha + \alpha$, $F_1 < F_2$ and $F_2 - F_1$ is strictly increasing. Hence, every Banach contraction (2.2) satisfies the contractive condition (2.3). On the other side, Example 2.5 in [19] shows that the mapping T which is not F_1-contraction (Banach contraction), is a F_2-contraction. Thus, the following theorem, which was given by Wardowski, is a proper generalization of Banach Contraction Principle.

Theorem 2.5 ([19])**.** *Let (X,d) be a complete metric space and let $T : X \to X$ be a F-contraction. Then T has a unique fixed point $x^* \in X$ and for every $x_0 \in X$ a sequence $\left\{T^n x_0\right\}_{n\in\mathbb{N}}$ is convergent to x^*.*

3. Main Results

In this section, we first bring the notion of regularity from [8] and then we define the ordered weakly F-contraction mapping. We use these concepts to prove a theorem which shows that every ordered weakly F-contraction has a fixed point. Furthermore, we give an example to show that our theorem is generalization of the main theorem of [3].

Let $(X,\leq)$ be an ordered set and d be a metric on X. The tripled $(X,\leq,d)$ is called an ordered metric space. If (X,d) is complete, then $(X,\leq,d)$ will be called ordered complete metric space. Also, X is said to be regular, if the ordered metric space $(X,\leq,d)$ provides the following condition:

- If x_n is non-decreasing sequence with $x_n \to x$ in X, then $x_n \leq x$ for all n.

Definition 3.1. Let $(X,\leq,d)$ be an ordered metric space and $T : X \to X$ be a non-constant mapping. Let

$$Y = \{(x,y) \in X \times X : x \leq y, d\left(Tx,Ty\right) > 0\}.$$

Also, suppose that there is a non-decreasing function $\psi : [0,\infty) \to [0,\infty)$ with $\lim_{n\to\infty} \psi^n\left(t\right) = 0$ for each $t > 0$. We say that T is an ordered weakly F-contraction if $F \in \mathcal{F}$ and there exists $\tau > 0$ such that

$$\forall\left(x,y\right) \in Y \quad \Rightarrow \quad \left[\tau + F\left(d\left(Tx,Ty\right)\right) \leq F\left(\psi\left(d\left(x,y\right)\right)\right)\right]. \tag{3.1}$$

In the following theorem we prove that an ordered weakly F-contraction mapping has a fixed point. This theorem is a generalization of [3, Theorem 2.1] and [19, Theorem 2.1].

Theorem 3.2. *Let $(X,\leq)$ be a partially ordered set and assume that there is a metric d on X such that (X,d) is a complete metric space. Assume there is a non-decreasing function $\psi : [0,\infty) \to [0,\infty)$ with $\lim_{n\to\infty} \psi^n\left(t\right) = 0$ for each $t > 0$ and also suppose $T : X \to X$ is an ordered weakly F-contraction. Let T is non-decreasing mapping and there exists $x_0 \in X$ such that $x_0 \leq Tx_0$. If T is continuous or X is regular, then T has a fixed point.*

Proof. We show that $\psi\left(t\right) < t$ for $t > 0$. To see this, suppose there exists $t_0 > 0$ with $t_0 \leq \psi\left(t_0\right)$. Since ψ is non-decreasing, by induction $t_0 \leq \psi^n\left(t_0\right)$ for $n \in \{1,2,\ldots\}$. This is a contradiction. In addition, $\psi\left(0\right) = 0$.

Let $x_0 \in X$ be as mentioned in the hypotheses. We define the sequence $\{x_n\}$ in X by $x_n = Tx_{n-1} = T^n x_0$ for all $n \in \mathbb{N}$. If there exists $n_0 \in \mathbb{N}$ such that $x_{n_0} = x_{n_0+1}$, then x_{n_0} is a fixed point of T and so the proof is completed. Thus, suppose that for every $n \in \mathbb{N}$, $x_{n+1} \neq x_n$. Since

$x_0 \leq Tx_0$ and T is non-decreasing, we obtain

$$x_0 \leq x_1 \leq x_2 \leq \cdots \leq x_n \leq \cdots .$$

Since $x_n \leq x_{n+1}$ and $d(Tx_n, Tx_{n-1}) > 0$ for every $n \in \mathbb{N}$, we have $(x_n, x_{n+1}) \in Y$. If we use the inequality 3.1 for the consecutive terms of $\{x_n\}$, then we get

$$\begin{aligned} F\left(d\left(x_{n+1}, x_n\right)\right) &= F\left(d\left(Tx_n, Tx_{n-1}\right)\right) \\ &\leq F\left(\psi\left(d\left(x_n, x_{n-1}\right)\right)\right) - \tau. \end{aligned} \tag{3.2}$$

From (F1) we have

$$\begin{aligned} F\left(d\left(T^{n+1}\left(x_0\right), T^n(x_0)\right)\right) &\leq F\left(\psi\left(d\left(T^n\left(x_0\right), T^{n-1}\left(x_0\right)\right)\right)\right) - \tau \\ &\;\;\vdots \\ &\leq F\left(\psi\left(d\left(T\left(x_0\right), x_0\right)\right)\right) - n\tau. \end{aligned}$$

By using (F1) and inequality 3.1, we deduce

$$\begin{aligned} F\left(d\left(x_{n+1}, x_n\right)\right) &= F\left(d\left(Tx_n, Tx_{n-1}\right)\right) \\ &\leq F\left(\psi\left(d\left(x_n, x_{n-1}\right)\right)\right) - \tau. \end{aligned}$$

Denote $\gamma_n = d(x_n, x_{n+1})$ for $n \in \mathbb{N}$. Then, from 3.2 we obtain

$$F\left(\gamma_n\right) \leq F\left(\psi\left(\gamma_{n-1}\right)\right) - \tau \leq \cdots \leq F\left(\psi\left(\gamma_0\right)\right) - n\tau. \tag{3.3}$$

From 3.3, we get $\lim_{n\to\infty} F(\gamma_n) = -\infty$. Thus, (F2) implies $\lim_{n\to\infty} \gamma_n = 0$. From (F3), there exists $k \in (0,1)$ such that $\lim_{n\to\infty} \gamma_n^k F(\gamma_n) = 0$. By 3.3, the following inequality holds for all $n \in \mathbb{N}$;

$$\gamma_n^k F\left(\gamma_n\right) - \gamma_n^k F\left(\psi\left(\gamma_0\right)\right) \leq -\gamma_n^k n\tau \leq 0. \tag{3.4}$$

As $n \to \infty$ in 3.4, we obtain that

$$\lim_{n\to\infty} n\gamma_n^k = 0. \tag{3.5}$$

From 3.5, there exists $n_1 \in \mathbb{N}$ such that $n\gamma_n^k \leq 1$ for all $n \geq n_1$. So, we have

$$\gamma_n \leq \frac{1}{n^{\frac{1}{k}}}, \tag{3.6}$$

for all $n \geq n_1$.

In order to show that $\{x_n\}$ is a Cauchy sequence consider $m, n \in \mathbb{N}$ such that $m > n \geq n_1$. By 3.6 and triangular inequality for the metric we have

$$\begin{aligned} d\left(x_n, x_m\right) &\leq d\left(x_n, x_{n+1}\right) + d\left(x_{n+1}, x_{n+2}\right) + \cdots + d\left(x_{m-1}, x_m\right) \\ &= \gamma_n + \gamma_{n+1} + \cdots + \gamma_{m-1} \end{aligned}$$

$$= \sum_{i=n}^{m-1} \gamma_i$$
$$\leq \sum_{i=n}^{\infty} \gamma_i$$
$$\leq \sum_{i=n}^{\infty} \frac{1}{i^{\frac{1}{k}}}.$$

Taking limit as $n \to \infty$, we deduce that $d(x_n, x_m) \to 0$, which yields that $\{x_n\}$ is a Cauchy sequence in X.

Since X is complete, the sequence $\{x_n\}$ converges to some point $z \in X$. Now, if T is continuous, then we obtain

$$z = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} Tx_n = T \lim_{n\to\infty} x_n = Tz,$$

which shows that z is a fixed point of T.

Now, suppose X is regular. Then $x_n \leq z$ for all $n \in \mathbb{N}$. There are only two cases to consider:

1) If there exists $n_0 \in \mathbb{N}$ for which $x_{n_0} = z$, then clearly

$$Tz = Tx_{n_0} = x_{n_0+1} \leq z.$$

Also, since $x_{n_0} \leq x_{n_0+1}$, we get $z \leq Tz$ and thus, $z = Tz$.

2) $x_n \neq z$ for every $n \in \mathbb{N}$ and $d(z, Tz) > 0$. Since $\lim_{n\to\infty} x_n = z$, there exists $n_1 \in \mathbb{N}$ such that $d(x_{n+1}, Tz) > 0$ and $d(x_n, z) < \frac{d(z,Tz)}{2}$ for all $n \geq n_1$. Moreover, in this case $(x_n, z) \in Y$. Hence, by (F1), we get for $n \geq n_1$

$$\tau + F(d(Tx_n, Tz)) \leq F(\psi(d(x_n, z))) \leq F\left(\psi\left(\frac{d(z, Tz)}{2}\right)\right),$$

which implies

$$d(x_{n+1}, Tz) \leq \frac{d(z, Tz)}{2}. \tag{3.7}$$

So letting $n \to \infty$ yields

$$d(z, Tz) \leq \frac{d(z, Tz)}{2},$$

a contradiction. Therefore, we conclude that $d(z, Tz) = 0$, i.e. $z = Tz$.

□

Now we give an example to illustrate the usability of Theorem 3.2.

Example 3.3. Let $A = \left\{ \frac{1}{n^2} : n \in \mathbb{N} \right\} \cup \{0\}$, $B = \{2, 3\}$ and $X = A \cup B$. Define an order relation $\preceq$ on X by

$$x \preceq y \quad \Leftrightarrow \quad \{x = y \text{ or } x, y \in A \text{ with } x \leq y\}.$$

where $\leq$ is usual order. One can see that $(X, \preceq, d)$ is an ordered complete metric space with the usual metric d. Let $T : X \to X$ be a mapping defined by

$$T(x) = \begin{cases} \dfrac{1}{(n+1)^2}, & \text{if } x = \frac{1}{n^2}, \\ \\ x, & \text{if } x \in \{0, 2, 3\}. \end{cases}$$

Clearly T is nondecreasing. Also, for $x_0 = 0$ we have $x_0 \preceq Tx_0$. Define F by

$$F(\alpha) = \begin{cases} \dfrac{\ln \alpha}{\sqrt{\alpha}}, & \text{if } 0 < \alpha < e^2, \\ \\ \alpha - e^2 + \frac{2}{e}, & \text{if } \alpha > e^2. \end{cases}$$

Then conditions (F1), (F2) and (F3) (for $k = \frac{2}{3}$) are satisfied.

The mapping $\psi : [0, \infty) \to [0, \infty)$ defined by $\psi(t) = \frac{t}{2}$ is a nondecreasing mapping with $\lim_{n\to\infty} \psi^n(t) = 0$ for each $t > 0$. We claim that T is an ordered weakly F-contraction with $\tau = \ln 2$. To see this, let us consider the following calculations:

We have

$$\begin{aligned} Y &= \{(x, y) \in X \times X : x \preceq y, d(Tx, Ty) > 0\} \\ &= \{(x, y) \in X \times X : x, y \in A \text{ and } x < y\}. \end{aligned}$$

Therefore, to see (3.1), it is sufficient to show that

$$\forall (x, y) \in Y \quad \Rightarrow \quad \ln 2 + F(d(Tx, Ty)) \leq F(\psi(d(x, y))). \tag{3.8}$$

It is equivalent to show that

$$\forall x, y \in A \quad \text{and} \quad x < y \quad \Rightarrow \quad d(Tx, Ty)^{\frac{1}{\sqrt{d(Tx,Tx)}}} \, d(x, y)^{\frac{-1}{\sqrt{d(x,y)}}} \leq 1,$$

or equivalently

$$\forall x, y \in A \quad \text{and} \quad x < y \quad \Rightarrow \quad |Tx - Ty|^{\frac{1}{\sqrt{|Tx-Ty|}}} \, |x - y|^{\frac{-1}{\sqrt{|x-y|}}} \leq 1.$$

By using [11, Example 3], it can be shown that (3.8) is true. Also, T is continuous (and X is regular). Thus, all conditions of Theorem 3.2 are satisfied, and so, T has a fixed point in X.

On the other hand, since $d(T2, T3) = d(2, 3) = 1$ and

$$d(T2, T3) = 1 \nleq \psi(d(2, 3)) = \frac{1}{2},$$

Theorem 2.1 of [3] is not applicable to this example.

Again, for all $F \in \mathcal{F}$ and $\tau > 0$ we have

$$\tau + F\left(d\left(T2, T3\right)\right) > F\left(d\left(2, 3\right)\right).$$

Therefore, Theorem 2.1, which is the main result of [19], is not applicable to this example.

Now, by combining the ideas of Minak [11], Agarwal [3] and Wardowski [19], we define the concept of Ćirić type generalized ordered weakly F-contraction, which is the extension of the above ideas.

Definition 3.4. Let $(X, \leq, d)$ be an ordered metric space and $T : X \to X$ be a mapping. Let

$$Y = \left\{(x, y) \in X \times X : x \leq y, d\left(Tx, Ty\right) > 0\right\}.$$

Also, suppose that there is a non-decreasing function $\psi : [0, \infty) \to [0, \infty)$ with $\lim_{n\to\infty} \psi^n\left(t\right) = 0$ for each $t > 0$. We say that T is Ćirić type generalized ordered weakly F-contraction if $F \in \mathcal{F}$ and there exists $\tau > 0$ such that

$$\forall\left(x, y\right) \in Y \quad \Rightarrow \quad \left[\tau + F\left(d\left(Tx, Ty\right)\right) \leq F\left(\psi\left(M\left(x, y\right)\right)\right)\right], \tag{3.9}$$

where

$$M\left(x, y\right) = \max\left\{d\left(x, y\right), d\left(x, Tx\right), d\left(y, Ty\right), \frac{1}{2}\left[d\left(x, Ty\right) + d\left(y, Tx\right)\right]\right\}.$$

In the following theorem, we prove that a Ćirić type generalized ordered weakly F-contraction mapping has a fixed point. It is obvious that this theorem is a generalization of the main theorems in [3, 11, 19].

Theorem 3.5. *Let $(X, \leq)$ be a partially ordered set and let there be a metric d on X such that (X, d) is a complete metric space. Assume $T : X \to X$ is a Ćirić type generalized ordered weakly F-contraction. Also suppose T is a non-decreasing mapping and there exists $x_0 \in X$ such that $x_0 \leq Tx_0$. If T is continuous or X is regular, then T has a fixed point.*

Proof. Let $x_0 \in X$ be as mentioned in the hypotheses. We define the sequence $\{x_n\}$ in X such that $x_n = Tx_{n-1} = T^n x_0$ for all $n \in \mathbb{N}$. If there exists $n_0 \in \mathbb{N}$ for which $x_{n_0} = x_{n_0+1}$, then x_{n_0} is a fixed point of T and so the proof is completed. Thus, suppose that for every $n \in \mathbb{N}$, $x_{n+1} \neq x_n$. Since $x_0 \leq Tx_0$ and T is non-decreasing, we obtain

$$x_0 \leq x_1 \leq x_2 \leq \cdots \leq x_n \leq \cdots.$$

Since $x_n \leq x_{n+1}$ and $d\left(Tx_n, Tx_{n-1}\right) > 0$ for every $n \in \mathbb{N}$, $(x_n, x_{n+1}) \in Y$, and so we can use the inequality 3.9 for the consecutive terms of

$\{x_n\}$. We have

$$F\left(d\left(x_{n+1},x_n\right)\right) = F\left(d\left(Tx_n,Tx_{n-1}\right)\right) \leq F\left(\psi\left(M\left(x_n,x_{n-1}\right)\right)\right)-\tau \leq F\left(\psi\left(\max\left\{d\left(x_n,x_{n-1}\right),d\left(x_n,x_{n+1}\right)\right\}\right)\right)-\tau. \tag{3.10}$$

Denote $\gamma_n = d\left(x_{n+1},x_n\right)$ for $n\in\mathbb{N}$. From 3.10 we have

$$F\left(\gamma_n\right) \leq F\left(\psi\left(\max\left\{\gamma_{n-1},\gamma_n\right\}\right)\right)-\tau. \tag{3.11}$$

If $\gamma_n \geq \gamma_{n-1}$ for some $n\in\mathbb{N}$, then from 3.11, we get

$$F\left(\gamma_n\right) \leq F\left(\psi\left(\gamma_n\right)\right)-\tau \leq F\left(\gamma_n\right)-\tau,$$

which is a contradiction, because $\tau>0$. Therefore $\gamma_n<\gamma_{n-1}$ for all $n\in\mathbb{N}$ and so from 3.11 we obtain

$$F\left(\gamma_n\right) \leq F\left(\psi\left(\gamma_{n-1}\right)\right)-\tau.$$

Hence, we deduce that

$$F\left(\gamma_n\right) \leq F\left(\psi\left(\gamma_{n-1}\right)\right)-\tau \leq F\left(\psi\left(\gamma_{n-2}\right)\right)-2\tau \leq \cdots \leq F\left(\psi\left(\gamma_0\right)\right)-n\tau. \tag{3.12}$$

From 3.12, we get $\lim_{n\to\infty} F\left(\gamma_n\right)=-\infty$. Thus (F2) implies $\lim_{n\to\infty}\gamma_n = 0$. From (F3) there exists $k\in(0,1)$ such that $\lim_{n\to\infty}\gamma_n^k F\left(\gamma_n\right)=0$. By 3.12, the following holds for all $n\in\mathbb{N}$,

$$\gamma_n^k F\left(\gamma_n\right)-\gamma_n^k F\left(\psi\left(\gamma_0\right)\right) \leq -\gamma_n^k n\tau \leq 0. \tag{3.13}$$

As $n\to\infty$ in 3.13, we obtain that

$$\lim_{n\to\infty} n\gamma_n^k = 0. \tag{3.14}$$

From 3.14, there exists $n_1\in\mathbb{N}$ such that $n\gamma_n^k\leq 1$ for all $n\geq n_1$. So, we have, for all $n\geq n_1$

$$\gamma_n \leq \frac{1}{n^{\frac{1}{k}}}. \tag{3.15}$$

In order to show that $\{x_n\}$ is a Cauchy sequence, consider $m,n\in\mathbb{N}$ such that $m>n\geq n_1$. By using the triangular inequality for the metric and from 3.15, we have

$$\begin{aligned} d\left(x_n,x_m\right) &\leq d\left(x_n,x_{n+1}\right)+d\left(x_{n+1},x_{n+2}\right)+\cdots+d\left(x_{m-1},x_m\right) \\ &= \sum_{i=n}^{m-1}\gamma_i \\ &\leq \sum_{i=n}^{\infty}\gamma_i \end{aligned}$$

$$\leq \sum_{i=n}^{\infty} \frac{1}{i^{\frac{1}{k}}}.$$

By the convergence of the series $\sum_{i=1}^{\infty} \frac{1}{i^{\frac{1}{k}}}$ passing to limit $n \to \infty$, we deduce that $d(x_n, x_m) \to 0$, which yields $\{x_n\}$ is a Cauchy sequence in X. Since (X, d) is a complete metric space, the sequence $\{x_n\}$ converges to some point $z \in X$, that is, $\lim_{n\to\infty} x_n = z$.

Now, if T is continuous, then we obtain

$$z = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} Tx_n = T \lim_{n\to\infty} x_n = Tz,$$

which yields that z is a fixed point of T.

Now, suppose X is regular. Then $x_n \leq z$ for all $n \in \mathbb{N}$. We will consider the following two cases:

1) If there exists $n_0 \in \mathbb{N}$ for which $x_{n_0} = z$, then

$$Tz = Tx_{n_0} = x_{n_0+1} \leq z.$$

Also, since $x_{n_0} \leq x_{n_0+1}$, $z \leq Tz$ and hence, $z = Tz$.

2) $x_n \neq z$ for every $n \in \mathbb{N}$ and $d(z, Tz) > 0$. Since $\lim_{n\to\infty} x_n = z$, there exists $n_1 \in \mathbb{N}$ such that $d(x_{n+1}, Tz) > 0$ for all $n \geq n_1$. We note that in this case $(x_n, z) \in Y$. Hence, by considering 3.9 and (F1), we get for $n \geq n_1$,

$$\begin{aligned}\tau + F(d(x_{n+1}, Tz)) &= \tau + F(d(Tx_n, Tz)) \\ &\leq F(\psi(M(x_n, z))) \\ &\leq F(\psi(\max\{d(x_n, z), d(x_n, Tx_n), d(z, Tz) \\ &\quad , \frac{1}{2}[d(x_n, Tz) + d(z, Tx_n)]\})).\end{aligned}$$

So taking limit as $n \to \infty$ yields

$$\tau + F(d(z, Tz)) \leq F(\psi(d(z, Tz))) \leq F(d(z, Tz)),$$

a contradiction. Therefore, we conclude that $d(z, Tz) = 0$, i.e. $z = Tz$.

□

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