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# Some Properties of $*$-frames in Hilbert Modules Over Pro-C*-algebras

Mona Naroei Irani[1] and Akbar Nazari[2*]

Abstract. In this paper, by using the sequence of adjointable operators from pro-C*-algebra $\mathcal{A}$ into a Hilbert $\mathcal{A}$-module $E$. We introduce frames with bounds in pro-C*-algebra $\mathcal{A}$. New frames in Hilbert modules over pro-C*-algebras are called standard $*$-frames of multipliers. Meanwhile, we study several useful properties of standard $*$-frames in Hilbert modules over pro-C*-algebras and investigate conditions that under which the sequence $\{h_i\}_{i\in I}$ is a standard $*$-frame of multipliers for Hilbert modules over pro-C*-algebras. Also the effect of operators on standard $*$-frames of multipliers for $E$ is examined. Finally, compositions of standard $*$-frames in Hilbert modules over pro-C*-algebras are studied.

## 1. Introduction

The concept of frames in Hilbert spaces was introduced by Duffin and Schaeffer [4], to deal with some problems in nonharmonic Fourier series. In 1986, the frames were reintroduced and developed by Daubechies et al. [3]. By using a sequence of bounded linear operators, the frames were generalized. Many generalizations of frames were suggested like frames of subspaces by Casazza and Kutyniok [2] and g-frames by Sun [12].

In the Frank and Larson's studyings of frames in Hilbert modules over C*-algebras [5], the properties of the frames were obtained. Finally, A. Khosravi and B. Khosravi [9] generalized the concept of fusion frames and g-frames in Hilbert C*-modules. In 2003, Raeburn and Thompson [11] considered a more general notion of countably generated

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* Corresponding author.

Hilbert C*-modules in which generators were multipliers of the module. Also they offered the notion of a standard frame of multipliers for a Hilbert-C*-module and proved some basic properties of standard frames of multipliers. In 2008, Joita reconsidered the ideas of Raeburn and Thompson in Hilbert modules over pro-C*-algebras [8]. Then fusion frames in Hilbert modules over pro-C*-algebras were stadied by Azhini and Haddadzadeh [1]. Also Haddadzadeh surveyed the generalization of frames as g-frames in Hilbert modules over pro-C*-algebras [6]. In this paper, the researchers introduced standard $*$-frames of multipliers in Hilbert modules over pro-C*-algebras.

The paper is organized as follows. In Section 2, we recall some facts about pro-C*-algebras and Hilbert modules over pro-C*-algebras. In Section 3, standard $*$-frames are introdused and some properties of standard $*$-frames are studied. Finally, in Section 4, the combination of operators on standard $*$-frames is investigated.

Throughout this manuscript, let $\mathcal{A}$ be a unital pro-C*-algebra with respect to the family of continuous C*-seminorms $\rho = \{\rho_\lambda\}_{\lambda\in\Lambda}$ and let $E, F$ be finitely or countably generated Hilbert $\mathcal{A}$-modules. We use $I, J$ to denote finite or countably infinite index sets.

## 2. PRELIMINARIES

In the following, we remind some basic definitions and properties of pro-C*-algebras and Hilbert modules over pro-C*-algebras. We refer the reader to [7] for more details.

**Definition 2.1** ([7])**.** A pro-C*-algebra is a Hausdorff complete complex topological $*$-algebra $\mathcal{A}$ whose topology is determined by its continuous C*-seminorms in the sense that a net $\{a_\lambda\}$ converges to 0 iff $\rho(a_\lambda) \longrightarrow 0$ for any continuous C*-seminorm $\rho$ on $\mathcal{A}$ and we have:

(1) $\rho(ab) \leq \rho(a)\rho(b)$,
(2) $\rho(a^*a) = \rho(a)^2$,

for all $a, b \in \mathcal{A}$.

**Example 2.2.** [7] Every C*-algebra is a pro-C*-algebra.

**Remark 2.3.** The notion $a \geq 0$ denotes $a \in \mathcal{A}^+$ and $a \geq b$ denotes $a - b \geq 0$.

We denote by $S(\mathcal{A})$, the set of all continuous C*-seminorms on $\mathcal{A}$. An element $a \in \mathcal{A}$ is bounded if

$$\sup\{\rho(a); \rho \in S(\mathcal{A})\} < \infty.$$

We denote by $b(\mathcal{A})$ the set of all bounded elements in $\mathcal{A}$.

**Definition 2.4** ([8])**.** A pre-Hilbert module over the pro-C*-algebra $\mathcal{A}$ is a complex vector space $E$ which is also a right $\mathcal{A}$-module, compatible with the complex algebra structure, equipped with an $\mathcal{A}$-valued inner product $\langle ., . \rangle : E \times E \to \mathcal{A}$ which is $\mathbb{C}$-and $\mathcal{A}$-linear in its second variable and satisfies the following relations:

(i) $\langle x, y \rangle^* = \langle y, x \rangle$, for every $x, y \in E$.
(ii) $\langle x, x \rangle \geq 0$, for every $x \in E$.
(iii) $\langle x, x \rangle = 0$ iff $x = 0$, for every $x \in E$.

We say that $E$ is a Hilbert $\mathcal{A}$-module (or Hilbert pro-C*-module over $\mathcal{A}$) if $E$ is complete with respect to the topology determined by the family of seminorms

$$\bar{\rho}_E(x) = \sqrt{\rho(\langle x, x \rangle)}, \quad x \in E, \rho \in S(\mathcal{A}).$$

Let $E$ be a pre-Hilbert $\mathcal{A}$-module. For every $\rho \in S(\mathcal{A})$ and for all $x, y \in E$, the following Cauchy-Schwarz inequality holds [7]

$$\rho(\langle x, y \rangle)^2 \leq \rho(\langle x, x \rangle)\rho(\langle y, y \rangle).$$

Consequently, for each $\rho \in S(\mathcal{A})$, we have:

$$\bar{\rho}_E(ax) \leq \rho(a)\bar{\rho}_E(x), \quad a \in \mathcal{A}, x \in E.$$

Let $\mathcal{A}$ be a pro-C*-algebra and $E, F$ be two Hilbert $\mathcal{A}$-modules. A module morphism $T : E \to F$ is continuous if for each $\rho \in S(\mathcal{A})$ there exists $C_\rho > 0$ such that $\bar{\rho}_F(Tx) \leq C_\rho \bar{\rho}_E(x)$ for all $x \in E$, and it is adjointable if there exists a module morphism $T^* : F \to E$ such that $\langle Tx, y \rangle = \langle x, T^* y \rangle$ for every $x \in E$ and $y \in F$. Any adjointable module morphism is continuous. The set of all adjointable module morphisms from $E$ to $F$ is denoted by $L(E, F)$. If $E = F$, $L(E, F)$ is denoted by $L(E)$ and it is a pro-C*-algebra [8].

**Definition 2.5.** An element $T \in L(E, F)$ is bounded if

$$\sup\{\bar{\rho}_{L(E,F)}(T); \rho \in S(\mathcal{A})\} < \infty.$$

The set of all bounded elements in $L(E, F)$, is denoted by $b(L(E, F))$.

**Definition 2.6.** [6] Let $E$ and $F$ be two Hilbert modules over the pro-C*-algebra $\mathcal{A}$. Then the operator $T : E \to F$ is called uniformly bounded (below), if there exists $C > 0$ such that for each $\rho \in S(\mathcal{A})$,

$$\bar{\rho}_F(Tx) \leq C\bar{\rho}_E(x), \ \textit{for all } x \in E, \tag{2.1}$$

$$(\bar{\rho}_F(Tx) \geq C\bar{\rho}_E(x), \ \textit{for all } x \in E). \tag{2.2}$$

The number $C$ in (2.1) is called an upper bound for $T$ and we set:

$$\|T\|_\infty = \inf\{C : C \text{ is an upper bound for } T\}.$$

Clearly, in this case we have:

$$\hat{\rho}(T) \leq \|T\|_{\infty}, \quad \forall \rho \in S(\mathcal{A}),$$

where

$$\hat{\rho}(T) = \sup\{\bar{\rho}_F(Tx) : x \in E, \bar{\rho}_E(x) \leq 1\}.$$

In [8] a multiplier of the Hilbert $\mathcal{A}$-module $E$ is an adjointable module morphism from $\mathcal{A}$ to $E$. The Hilbert $M(\mathcal{A})$-module $L(\mathcal{A}, E)$ is called the multiplier module of $E$ and it is denoted by $M(E)$. For all $h \in M(E)$ and $x \in E$, we set

$$\langle h, x \rangle_{M(E)} = h^*(x).$$

Moreover, if $a \in \mathcal{A}$ and $h \in M(E)$, then $h.a$ can be identified with $h(a)$.

Let $H_{\mathcal{A}}$ be the set of all sequences $(a_n)_n$ with $a_n \in \mathcal{A}$ such that $\sum a_n^* a_n$ convergens in $\mathcal{A}$ be a Hilbert $\mathcal{A}$-module with the inner product

$$\langle (a_n)_n, (b_n)_n \rangle_{H_{\mathcal{A}}} = \sum_n a_n^* b_n.$$

Joita in [8] introduced standard frames of multipliers for Hilbert pro-C*-modules as follows. Let $E$ be a Hilbert pro-C*-module. The sequence $\{h_n\}_n$ in $M(E)$ is called a standard frame of multipliers in $E$ if for each $x \in E$, the series $\sum_n \langle x, h_n \rangle_{M(E)} \langle h_n, x \rangle_{M(E)}$ converges in $\mathcal{A}$, and there exist two positive constants $C$ and $D$ such that:

$$C\langle x, x \rangle_E \leq \sum_n \langle x, h_n \rangle_{M(E)} \langle h_n, x \rangle_{M(E)} \leq D\langle x, x \rangle_E,$$

for all $x \in E$ .

**Proposition 2.7** ([6])**.** *Let $T$ be a uniformly bounded below operator in $L(E, F)$. Then $T$ is closed and injective.*

**Definition 2.8** ([7])**.** An adjointable operator $U$ from $E$ to $F$ is said to be unitary if $U^*U = id_E$ and $UU^* = id_F$.

**Proposition 2.9** ([7])**.** *Let $U$ be a linear map from $E$ to $F$. Then the following statements are equivalent:*

1. *$U$ is a unitary operator from $E$ to $F$;*
2. *$\langle Ux, Ux \rangle = \langle x, x \rangle$ for all $x \in E$ and $U$ is surjective;*
3. *$\overline{\rho}_F(Ux) = \overline{\rho}_E(x)$ for all $x \in E$ and $\rho \in S(\mathcal{A})$ and $U$ is a surjective module homomorphism from $E$ to $F$.*

**Definition 2.10** ([7])**.** Two Hilbert $\mathcal{A}$-modules $E$ and $F$ are isomorphic if there exists a surjective module homomorphism $\varphi$ from $E$ onto $F$ such that

$$\langle \varphi x, \varphi y \rangle_F = \langle x, y \rangle_E,$$

for all $x, y \in E$.

## 3. SOME PROPERTIES

In this section, we introduce standard $*$-frames of multipliers in $E$, and investigate an example of standard $*$-frames of multipliers.

**Definition 3.1.** Let $E$ be a Hilbert pro-$C^*$-module. The sequence $\{h_n\}_n$ in $M(E)$ is called a standard $*$-frame of multipliers for $E$ if for each $x \in E$, the series $\sum_n \langle x, h_n\rangle_{M(E)} \langle h_n, x\rangle_{M(E)}$ converges in $\mathcal{A}$ and there exist two strictly nonzero elements $C$ and $D$ in $\mathcal{A}$ such that

$$C\langle x, x\rangle_E C^* \leq \sum_n \langle x, h_n\rangle_{M(E)} \langle h_n, x\rangle_{M(E)} \leq D\langle x, x\rangle_E D^*,$$

for all $x \in E$.

The sequence $\{h_n\}_n$ is a standard $\lambda$-tight $*$-frame of multipliers if $\lambda = C = D$, and the sequence $\{h_n\}_n$ is a standard normalized $*$-frame of multipliers if $C = D = 1_{\mathcal{A}}$.

If $\{h_n\}_n$ possess an upper $*$-frame bound but not necessarily a lower $*$-frame bound, we call it a standard $*$-Bessel sequence of multipliers for $E$.

If $\{h_n\}_n$ is a standard $*$-frame of multipliers for $E$ with $*$-frame bounds $C, D$, then the pre-$*$-frame operator $T : E \to H_{\mathcal{A}}$ defined by $T(x) = \{\langle h_n, x\rangle_{M(E)}\}_n$ has a unique $*$-frame operator $S : E \to E$ defined by

$$Sx = \sum_{n\in\mathbb{N}} h_n \langle h_n, x\rangle_{M(E)}.$$

Moreover, $S$ is positive, self-adjoint and invertible.

**Example 3.2.** Let $H_{\mathcal{A}}$ be a Hilbert $\mathcal{A}$-module. Then $L(\mathcal{A}, H_{\mathcal{A}})$ is a $L(\mathcal{A})$ module with the following operations:

$$uv := \{u_i v_i\}_{i\in\mathbb{N}}, \qquad u^* := \{\bar{u}_i\}_{i\in\mathbb{N}}, \qquad \langle\{u_i\}, \{v_i\}\rangle := \sum_{i\in\mathbb{N}} u_i^* v_i,$$

$$\bar{\rho}_{H_{\mathcal{A}}}(u) = (\rho(\langle u, u\rangle_{H_{\mathcal{A}}}))^{\frac{1}{2}}, \quad \forall\ u = \{u_i\}_{i\in\mathbb{N}}, v = \{v_i\}_{i\in\mathbb{N}}.$$

Let $J = \mathbb{N}$ and define $h_j \in L(\mathcal{A}, H_{\mathcal{A}})$ by $h_j = \{h_i^j\}_{i\in\mathbb{N}}$. For any constant $C$ we define:

$$h_i^j(a) = \begin{cases} \langle a, C1_{\mathcal{A}}\rangle, & i = j, \\ 0, & i \neq j. \end{cases}$$

Then we have:

$$\begin{aligned} \rho\left(\sum_j h_i^j(a)\overline{h_i^j(a)}\right) &= \rho(h_i^i(a)\overline{h_i^i(a)}) \\ &= \rho(\langle a, C1_{\mathcal{A}}\rangle\, \overline{\langle a, C1_{\mathcal{A}}\rangle}) \end{aligned}$$

$$= \rho(\langle a, C1_{\mathcal{A}}\rangle)^2 < \infty.$$

This shows that $h_j$ is well-defined and adjointable. Now we define $h_j^* \in L(H_{\mathcal{A}}, \mathcal{A})$ by $h_j^* = \{h_i^{j^*}\}_{i\in\mathbb{N}}$, $h_i^{j^*}(\{x_i\}) = C1_{\mathcal{A}}x_j$. Then we have:

$$\begin{aligned}\langle\{x_i\}_{i\in\mathbb{N}}, h_j\rangle_{M(H_{\mathcal{A}})}\langle h_j, \{x_i\}_{i\in\mathbb{N}}\rangle_{M(H_{\mathcal{A}})} &= \overline{h_j^*(\{x_i\}_{i\in\mathbb{N}})}h_j^*(\{x_i\}_{i\in\mathbb{N}})\\ &= \overline{C1_{\mathcal{A}}}\overline{x}_j C1_{\mathcal{A}}x_j.\end{aligned}$$

So

$$\begin{aligned}\sum_{j\in J}\langle x, h_j\rangle_{M(H_{\mathcal{A}})}\langle h_j, x\rangle_{M(H_{\mathcal{A}})} &= \sum_{j\in J}\left\langle\{x_i\}_{i\in\mathbb{N}}, h_j\right\rangle_{M(H_{\mathcal{A}})}\left\langle h_j, \{x_i\}_{i\in\mathbb{N}}\right\rangle_{M(H_{\mathcal{A}})}\\ &= \sum_{j\in J}\overline{C1_{\mathcal{A}}}\overline{x}_j x_j C1_{\mathcal{A}}\\ &= C1_{\mathcal{A}}\sum_{j\in J}\overline{x}_j x_j C1_{\mathcal{A}}\\ &= C1_{\mathcal{A}}\langle x, x\rangle_{H_{\mathcal{A}}} C1_{\mathcal{A}}.\end{aligned}$$

Consequently $\{h_j\}_{j\in I}$ is a standard $C1_{\mathcal{A}}$-tight $*$- frame of multipliers for $H_{\mathcal{A}}$.

**Theorem 3.3.** *Let the sequence $\{h_i\}_{i\in I}$ be a standard $*$-frame of multipliers for $E$ with some $*$-frame bounds in $b(\mathcal{A})$. Then the pre-$*$-frame operator $T$ is in $b(L(E, H_{\mathcal{A}}))$ and the $*$-frame operator $S$ is in $b(L(E))$.*

*Proof.* Let $x$ be an arbitrary element in $E$ and $C, D \in b(\mathcal{A})$ be lower and upper $*$-frame bounds for $\{h_i\}_{i\in I}$, respectively. So

$$C\langle x, x\rangle_E C^* \leq \sum_{i\in I}\langle x, h_i\rangle_{M(E)}\langle h_i, x\rangle_{M(E)} \leq D\langle x, x\rangle_E D^*.$$

Since $\{h_i\}_{i\in I}$ is a standard $*$-frame of multipliers for $E$ with the $*$-pre-frame operator $T : E \to H_{\mathcal{A}}$ defined by $T(x) = \{\langle h_i, x\rangle_{M(E)}\}_{i\in I}$ and

$$\langle Tx, Tx\rangle_E = \sum_{i\in I}\langle x, h_i\rangle_{M(E)}\langle h_i, x\rangle_{M(E)}.$$

So $(\overline{\rho}_{H_{\mathcal{A}}}(Tx))^2 \leq (\rho(D))^2(\overline{\rho}_E(x))^2$ and $\widehat{\rho}(T) \leq \rho(D)$ since $D \in b(\mathcal{A})$, then $\sup_{\rho\in S(\mathcal{A})}\widehat{\rho}(T) \leq \sup_{\rho\in S(\mathcal{A})}\rho(D)$. This shows that $T \in b(L(E, H_{\mathcal{A}}))$.

Since $\widehat{\rho}(T^*) = \widehat{\rho}(T)$, therefore $T^* \in b(L(H_{\mathcal{A}}, E))$. Moreover $\widehat{\rho}(S) = \widehat{\rho}(T^* o T) = (\widehat{\rho}(T))^2$, hence $S \in b(L(E))$. □

**Remark 3.4.** If $\{h_i\}_{i\in I}$ is a standard $*$-frame of multipliers for $E$ with some $*$-frame bounds in $b(\mathcal{A})$, then $h_i \in b(M(E))$ for all $i \in I$.

**Remark 3.5.** For every standard frame of multipliers for $E$, the frame operator $S$ belongs to $b(L(E))$, but for a standard $*$-frame of multipliers for $E$ with some $*$-frames bounds in $b(\mathcal{A})$, the $*$-frame operator $S$ belongs to $b(L(E))$.

**Theorem 3.6.** *Let the sequence $\{h_i\}_{i\in I}$ in $M(E)$ be a standard $*$-frame of multipliers for $E$. Then, there exist strictly nonzero elements $C, D \in \mathcal{A}$ such that for every $x \in E$*

$$\begin{aligned}(\rho(C^{-1}))^{-2}(\overline{\rho}_E(x))^2 &\leq \rho\left(\sum_{i\in I}\langle x, h_i\rangle_{M(E)}\langle h_i, x\rangle_{M(E)}\right)\\ &\leq (\overline{\rho}_E(x))^2(\rho(D))^2. \end{aligned} \tag{3.1}$$

*Conversely, if (3.1) holds for some $C, D \in b(\mathcal{A})$, then $\{h_i\}_{i\in I}$ is a standard $*$-frame of multipliers for $E$.*

*Proof.* If $\{h_i\}_{i\in I}$ is a standard $*$-frame of multipliers for $E$, then for every $x \in E$ we have

$$\langle x, x\rangle_E \leq C^{-1}\sum_{i\in I}\langle x, h_i\rangle_{M(E)}\langle h_i, x\rangle_{M(E)}C^{*^{-1}},$$

since for $a \in \mathcal{A}$ and $x \in E$ we have $\bar{\rho}_E(ax) \leq \rho(a)\bar{\rho}_E(x)$. Thus

$$\left(\rho\left(C^{-1}\right)\right)^{-1}\rho\left(\langle x, x\rangle_E\right)\left(\rho\left(C^{*^{-1}}\right)\right)^{-1} \leq \rho\left(\sum_{i\in I}\langle x, h_i\rangle_{M(E)}\langle h_i, x\rangle_{M(E)}\right).$$

Similarly

$$\rho\left(\sum_{i\in I}\langle x, h_i\rangle_{M(E)}\langle h_i, x\rangle_{M(E)}\right) \leq \rho(D)\rho(\langle x, x\rangle_E)\rho(D^*).$$

Conversely, we define the operator $U : E \to H_{\mathcal{A}}$ by $U(x) = \{\langle h_i, x\rangle_{M(E)}\}_{i\in I}$. Then

$$\begin{aligned}\langle Ux, Ux\rangle_E &= \left\langle\left\{\langle h_i, x\rangle_{M(E)}\right\}_{i\in I}, \left\{\langle h_i, x\rangle_{M(E)}\right\}_{i\in I}\right\rangle\\ &= \sum_{i\in I}\langle x, h_i\rangle_{M(E)}\langle h_i, x\rangle_{M(E)}.\end{aligned}$$

So $\overline{\rho}_{H_{\mathcal{A}}}(Ux) \leq \rho(D)\overline{\rho}_E(x)$. This shows that $U$ is well-defined. Hence $U^* : H_{\mathcal{A}} \to E$ given by $U^*(\{a_i\}_{i\in I}) = \sum_{i\in I} h_i a_i$ is the adjoint of $U$.

We denote $U^*U = K$ since $K$ is positive and self-adjoint, so $K^{\frac{1}{2}}$ is positive and self-adjoint. Also

$$\begin{aligned}\left\langle K^{\frac{1}{2}}x, K^{\frac{1}{2}}x\right\rangle_E &= \langle Kx, x\rangle_E\\ &= \sum_{i\in I}\langle x, h_i\rangle_{M(E)}\langle h_i, x\rangle_{M(E)},\end{aligned}$$

and

$$\left(\rho\left(C^{-1}\right)\right)^{-2}(\overline{\rho}_E(x))^2 \leq \rho\left(\left\langle K^{\frac{1}{2}}x, K^{\frac{1}{2}}x\right\rangle_E\right)$$

$$\leq (\overline{\rho}_E(x))^2(\rho(D))^2.$$

Suppose that $K^{\frac{1}{2}}(x) = 0$, then we have $\overline{\rho}_E(x) = 0$ for all $\rho \in S(\mathcal{A})$. Therefore $x = 0$. It follows that $K^{\frac{1}{2}}$ is injective and thus invertible. Also since $C, D \in b(\mathcal{A})$, then $K^{\frac{1}{2}} \in b(L(E))$ [8], and

$$\left\|K^{-\frac{1}{2}}\right\|_\infty^{-2} \langle x,x\rangle_E \leq \left\langle K^{\frac{1}{2}}x, K^{\frac{1}{2}}x\right\rangle_E \leq \left\|K^{\frac{1}{2}}\right\|_\infty^2 \langle x,x\rangle_E .$$

Then

$$\begin{aligned}\left\|K^{-\frac{1}{2}}\right\|_\infty^{-1} 1_{\mathcal{A}} \langle x,x\rangle_E \left\|K^{-\frac{1}{2}}\right\|_\infty^{-1} 1_{\mathcal{A}} &\leq \sum_{i\in I} \langle x, h_i\rangle_{M(E)} \langle h_i, x\rangle_{M(E)} \\ &\leq \left\|K^{\frac{1}{2}}\right\|_\infty 1_{\mathcal{A}} \langle x,x\rangle_E \left\|K^{\frac{1}{2}}\right\|_\infty 1_{\mathcal{A}}.\end{aligned}$$

This shows that $\{h_i\}_{i\in I}$ is a standard $*$-frame of multipliers for $E$ with lower and upper $*$-frame bounds $\left\|K^{-\frac{1}{2}}\right\|_\infty^{-1} 1_{\mathcal{A}}, \left\|K^{\frac{1}{2}}\right\|_\infty 1_{\mathcal{A}}$. □

## 4. OPERATORS ON $*$-FRAMES

In this section, we study the result of combining of operators with standard $*$-frames of multipliers for $E$.

**Theorem 4.1.** *Let the sequence $\{h_i\}_{i\in I}$ in $M(E)$ be a standard $*$-frame of multipliers for $E$ and $Q \in b(L(E))$. Then $Q^*$ is uniformly bounded below if and only if $\{Qh_i\}_{i\in I}$ is a standard $*$-frame of multipliers for $E$. In this case $S_Q = QSQ^*$ is a $*$-frame operator.*

*Proof.* Since $Q^*$ is a uniformly bounded below operator, then by Proposition 2.7, $Q^*$ is invertible. Since $Q \in b(L(E))$, so $Q^* \in b(L(E))$ by [8], and we have

$$\left\|{Q^*}^{-1}\right\|_\infty^{-2} \langle x,x\rangle_E \leq \langle Q^*x, Q^*x\rangle_E \leq \|Q^*\|_\infty^2 \langle x,x\rangle_E. \tag{4.1}$$

Let $\{h_i\}_{i\in I}$ be a standard $*$-frame of multipliers for $E$ with $*$-frame bounds $C$ and $D$. Then

$$\begin{aligned}C\langle Q^*x, Q^*x\rangle_E C^* &\leq \sum_{i\in I} \langle Q^*x, h_i\rangle_{M(E)} \langle h_i, Q^*x\rangle_{M(E)} \\ &= \sum_{i\in I} \langle x, Qh_i\rangle_{M(E)} \langle Qh_i, x\rangle_{M(E)}.\end{aligned}$$

Similarly, we have

$$\begin{aligned}C\langle Q^*x, Q^*x\rangle_E C^* &\leq \sum_{i\in I} \langle x, Qh_i\rangle_{M(E)} \langle Qh_i, x\rangle_{M(E)} \\ &\leq D\langle Q^*x, Q^*x\rangle_E D^*,\end{aligned}$$

and using (4.1) we have

$$\left(\left\|Q^{*^{-1}}\right\|_{\infty}^{-1} C\right)\langle x,x\rangle_E\left(\left\|Q^{*^{-1}}\right\|_{\infty}^{-1} C\right)^* \leq \sum_{i\in I}\langle x,Qh_i\rangle_{M(E)}\langle Qh_i,x\rangle_{M(E)} \leq (\|Q^*\|_{\infty}D)\langle x,x\rangle_E(\|Q^*\|_{\infty}D)^*.$$

Consequently $\{Qh_i\}_{i\in I}$ is a standard $*$-frame of multipliers for $E$. Let $S$ be a $*$-frame operator for $\{h_i\}_{i\in I}$ and suppose $S_Q$ be a $*$-frame operator for $\{Qh_i\}_{i\in I}$. Since $Q$ is a linear map, then we have

$$\begin{aligned} QSQ^*(x) &= Q\left(\sum_{i\in I} h_i\langle h_i, Q^*x\rangle_{M(E)}\right)\\ &= \sum_{i\in I} Qh_i\langle Qh_i, x\rangle_{M(E)}, \end{aligned}$$

hence $S_Q = QSQ^*$.

Conversely, let $\{Qh_i\}_{i\in I}$ be a standard $*$-frame of multipliers for $E$ by $*$-frame bounds $M, N$, and $\{h_i\}_{i\in I}$ be a standard $*$-frame of multipliers for $E$ by $*$-frame bounds $C, D$.

Suppose that $Q^*x = 0$, then we have

$$\begin{aligned} 0 \leq M\langle x,x\rangle_E M^* &\leq \sum_{i\in I}\langle x,Qh_i\rangle_{M(E)}\langle Qh_i,x\rangle_{M(E)}\\ &= \sum_{i\in I}\langle Q^*x,h_i\rangle_{M(E)}\langle h_i,Q^*x\rangle_{M(E)}\\ &\leq D\langle Q^*x,Q^*x\rangle_E D^* = 0. \end{aligned}$$

Therefore $x = 0$, and it follows that $Q^*$ is injective. Since $Q^*$ is an invertible element in $b(L(E))$, then

$$\left\|Q^{*^{-1}}\right\|_{\infty}^{-2}\langle x,x\rangle_E \leq \langle Q^*x,Q^*x\rangle_E \leq \|Q^*\|_{\infty}^2\langle x,x\rangle_E,$$

hence

$$\left\|Q^{*^{-1}}\right\|_{\infty}^{-1}\overline{\rho}_E(x) \leq \overline{\rho}_E(Q^*x),$$

for each $\rho \in S(\mathcal{A})$. This shows that $Q^*$ is a uniformly bounded below operator. □

**Proposition 4.2.** *Let $\{h_i\}_{i\in I}$ be a sequence in $M(E)$. If there exists an invertible map $V \in b(L(E))$ such that $\{Voh_i\}_{i\in I}$ is a standard $*$-frame of multipliers for $E$, then $\{h_i\}_{i\in I}$ is a standard $*$-frame of multipliers for $E$.*

*Proof.* Since $V$ is an invertible element in $b(L(E))$, then from [8], for any $x \in E$, we have

$$\|V^*\|_{\infty}^{-2}\langle x,x\rangle_E \leq \left\langle V^{*^{-1}}x, V^{*^{-1}}x\right\rangle_E \leq \left\|V^{*^{-1}}\right\|_{\infty}^2\langle x,x\rangle_E. \tag{4.2}$$

Also

$$\begin{aligned} C\Big\langle V^{*^{-1}}x, V^{*^{-1}}x\Big\rangle_E C^* &\leq \sum_{i\in I}\Big\langle V^{*^{-1}}x, Vh_i\Big\rangle_{M(E)}\Big\langle Vh_i, V^{*^{-1}}x\Big\rangle_{M(E)} \\ &= \sum_{i\in I}\langle x, V^{-1}Vh_i\rangle_{M(E)}\langle V^{-1}Vh_i, x\rangle_{M(E)} \\ &= \sum_{i\in I}\langle x, h_i\rangle_{M(E)}\langle h_i, x\rangle_{M(E)}. \end{aligned}$$

Now (4.2) implies that

$$\Big(C\left\|V^*\right\|_\infty^{-1}\Big)\langle x,x\rangle_E\Big(C\left\|V^*\right\|_\infty^{-1}\Big)^* \leq \sum_{i\in I}\langle x,h_i\rangle_{M(E)}\langle h_i,x\rangle_{M(E)}, \tag{4.3}$$

and similarly

$$\sum_{i\in I}\langle x,h_i\rangle_{M(E)}\langle h_i,x\rangle_{M(E)} \leq \Big(D\Big\|V^{*^{-1}}\Big\|_\infty\Big)\langle x,x\rangle_E\Big(D\Big\|V^{*^{-1}}\Big\|_\infty\Big)^*. \tag{4.4}$$

Hence (4.3) and (4.4) imply that $\{h_i\}_{i\in I}$ is a standard $*$-frame of multipliers in $E$. $\square$

**Theorem 4.3.** *Let $U$ be a unitary operator from $E$ to $F$. If the sequence $\{h_i\}_{i\in I}$ in $M(E)$ is a standard $*$-frame of multipliers for $E$, then $\{Uh_i\}_{i\in I}$ is a standard $*$-frame of multipliers for $F$ with the $*$-frame operator $USU^*$.*

*Proof.* Let $J$ be a finite subset of $I$. Since $\{h_i\}_{i\in I}$ is a standard $*$-frame of multipliers for $E$ with $*$-frame bounds $C, D$, then for any $y\in F$ we have

$$\sum_{i\in J}\langle y, Uh_i\rangle_{M(F)}\langle Uh_i, y\rangle_{M(F)} = \sum_{i\in J}\langle U^*y, h_i\rangle_{M(E)}\langle h_i, U^*y\rangle_{M(E)},$$

so

$$\sum_{i\in I}\langle y, Uh_i\rangle_{M(F)}\langle Uh_i, y\rangle_{M(F)},$$

is convergent in $\mathcal{A}$. Also we have:

$$\begin{aligned} \langle y,y\rangle_F &= \langle U^*y, U^*y\rangle_E \\ &\leq C^{-1}\sum_{i\in I}\langle U^*y, h_i\rangle_{M(E)}\langle h_i, U^*y\rangle_{M(E)}C^{*^{-1}} \\ &= C^{-1}\sum_{i\in I}\langle y, Uh_i\rangle_{M(F)}\langle Uh_i, y\rangle_{M(F)}C^{*^{-1}}. \end{aligned}$$

So

$$C\langle y,y\rangle_F C^* \leq \sum_{i\in I}\langle y, Uh_i\rangle_{M(F)}\langle Uh_i, y\rangle_{M(F)},$$

and similarly

$$\sum_{i\in I} \langle y, Uh_i\rangle_{M(F)} \langle Uh_i, y\rangle_{M(F)} \leq D\langle y, y\rangle_F D^*.$$

This shows that $\{Uh_i\}_{i\in I}$ is a standard $*$-frame of multipliers for $F$.

Let $S$ be a $*$-frame operator for $\{h_i\}_{i\in I}$ and $y \in F$. Then

$$\begin{aligned} USU^*(y) &= U\left(\sum_{i\in J} h_i \langle h_i, U^* y\rangle_{M(E)}\right) \\ &= \sum_{i\in J} Uh_i \langle Uh_i, y\rangle_{M(F)}. \end{aligned}$$

□

**Corollary 4.4.** *Let $U$ be a linear and surjective map from $E$ to $F$ such that $\langle Ux, Ux\rangle_F = \langle x, x\rangle_E$ for all $x \in E$. If the sequence $\{h_i\}_{i\in I}$ in $M(E)$ is a standard $*$-frame of multipliers for $E$, then $\{Uh_i\}_{i\in I}$ is a standard $*$-frame of multipliers for $F$.*

**Corollary 4.5.** *Let $E_0$ be an orthogonally complemented submodule of $E$. Suppose that $\pi$ denotes the orthogonal projection of $E$ onto $E_0$. Then the following holds:*

i. *If the sequence $\{h_i\}_{i\in I}$ is a standard $*$-frame of multipliers for $E$ with $*$-frame bounds $C, D$, then $\{\pi h_i\}_{i\in I}$ is a standard $*$-frame of multipliers for $E_0$ with $*$-frame bounds $C, D$.*

ii. *If the sequence $\{h_i\}_{i\in I}$ is a standard $*$-frame of multipliers for $E$ with the $*$-frame operator $S$, then*

$$\pi(x) = \sum_{i\in I} h_i \langle S^{-1} h_i, x\rangle_{M(E)}.$$

**Corollary 4.6.** *Let $E$ and $F$ be isomorphic. If the sequence $\{h_i\}_{i\in I}$ in $M(E)$ is a standard $*$-frame of multipliers for $E$, then there exists $\varphi$ from $E$ onto $F$ such that $\{\varphi h_i\}_{i\in I}$ is a standard $*$-frame of multipliers for $F$.*

In the following, we show that the combination of $*$-frames with some special conditions is also a $*$-frame of multipliers.

**Theorem 4.7.** *Let the sequence $\{h_j\}_{j\in J}$ in $M(E)$ be a standard $*$-frame of multipliers for $E$, and the sequence $\{t_i\}_{i\in I}$ in $M(\mathcal{A})$ be a standard $*$-frame of multipliers for $\mathcal{A}$. If for any $j \in J$, $h_j^* \in b(L(E, \mathcal{A}))$ is a uniformly bounded below operator, then $\{h_j t_i\}_{i\in I}$ is a standard $*$-frame of multipliers in $E$, for all $j \in J$.*

*Proof.* Suppose that $C, D$ be lower and upper $*$-frame bounds for $\{h_j\}_{j\in J}$ respectively. Also, assume that $A, B$ be lower and upper $*$-frame bounds for $\{t_i\}_{i\in I}$ respectively. Since for each $j \in J$, $h_j$ is an adjointable operator and $\{t_i\}_{i\in I}$ is a standard $*$-frame of multipliers for $\mathcal{A}$, then we have

$$\sum_{i\in I} \langle x, h_j t_i\rangle_{M(E)} \langle h_j t_i, x\rangle_{M(E)} = \sum_{i\in I} \left\langle h_j^* x, t_i\right\rangle_{M(\mathcal{A})} \left\langle t_i, h_j^* x\right\rangle_{M(\mathcal{A})},$$

so

$$\sum_{i\in I} \langle x, h_j t_i\rangle_{M(E)} \langle h_j t_i, x\rangle_{M(E)},$$

is convergent in $\mathcal{A}$.

For each $j \in J$ and $x \in E$, we have

$$\begin{aligned}\sum_{i\in I} \langle x, h_j t_i\rangle_{M(E)} \langle h_j t_i, x\rangle_{M(E)} &= \sum_{i\in I} \left\langle h_j^* x, t_i\right\rangle_{M(\mathcal{A})} \left\langle t_i, h_j^* x\right\rangle_{M(\mathcal{A})} \\ &\leq B\left\langle h_j^* x, h_j^* x\right\rangle_{\mathcal{A}} B^* \\ &= B\langle x, h_j\rangle_{M(E)} \langle h_j, x\rangle_{M(E)} B^* \\ &\leq B \sum_{j\in J} \langle x, h_j\rangle_{M(E)} \langle h_j, x\rangle_{M(E)} B^* \\ &\leq BD\langle x, x\rangle_E B^* D^*.\end{aligned}$$

This shows that for each $j \in J$, $\{h_j t_i\}_{i\in I}$ is a $*$-Bessel sequence. By Proposition 2.7, for each $j \in J$, $h_j^*$ is an invertible element in $b(L(E, \mathcal{A}))$. By [10], we have

$$\begin{aligned}\left\|h_j^{*^{-1}}\right\|_\infty^{-2} \langle x, x\rangle_E &\leq \left\langle h_j^* x, h_j^* x\right\rangle_{\mathcal{A}} \\ &\leq A^{-1} \sum_{i\in I} \left\langle h_j^* x, t_i\right\rangle_{M(\mathcal{A})} \left\langle t_i, h_j^* x\right\rangle_{M(\mathcal{A})} A^{*^{-1}} \\ &= A^{-1} \sum_{i\in I} \langle x, h_j t_i\rangle_{M(E)} \langle h_j t_i, x\rangle_{M(E)} A^{*^{-1}},\end{aligned}$$

hence

$$\begin{aligned}\left(\left\|h_j^{*^{-1}}\right\|_\infty^{-1} A\right) \langle x, x\rangle_E \left(\left\|h_j^{*^{-1}}\right\|_\infty^{-1} A\right)^* &\leq \sum_{i\in I} \langle x, h_j t_i\rangle_{M(E)} \langle h_j t_i, x\rangle_{M(E)} \\ &\leq BD\langle x, x\rangle_E B^* D^*.\end{aligned}$$

This implies that for all $j \in J$, $\{h_j t_i\}_{i\in I}$ is a standard $*$-frame of multipliers for $E$. □

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[1] Department of Mathematics, Kerman Branch, Islamic Azad University, Kerman, Iran.

*E-mail address*: M.naroei.math@gmail.com

[2] Department of Pure Mathematics, Faculty of Mathematics and Computer, Shahid Bahonar University of Kerman, Kerman, Iran.

*E-mail address*: nazari@uk.ac.ir