Sahand Communications in Mathematical Analysis (SCMA) Vol. 16 No. 1 (2019), 35-46
http://scma.maragheh.ac.ir
DOI: 10.22130/scma.2018.72368.289

Fixed Point Theory in ε-connected Orthogonal Metric Space

Madjid Eshaghi[1] and Hasti Habibi[2*]

Abstract. The existence of fixed point in orthogonal metric spaces has been initiated by Eshaghi and et. al [7]. In this paper, we prove existence and uniqueness theorem of fixed point for mappings on ε-connected orthogonal metric space. As a consequence of this, we obtain the existence and uniqueness of fixed point for analytic function of one complex variable. The paper concludes with some illustrating examples.

1. Introduction

Concept of ε-connected (locally) contractive mappings and generalization of Banach contraction principle in ε-connected (chainable) metric space has been established in [3]. The ε-connected (locally) contractive type mappings has been studied by many authors and important results have been obtained by [2, 4–6, 8, 9, 12]. Recently, notions of orthogonal set and orthogonal metric space have been introduced in [7]. The existence of fixed point in orthogonal metric spaces has been initiated in [7] and generalizations of this theorem has been obtained in [1, 10, 11]. In this paper, we are interested to define a new concept of ε-connected orthogonal ($(\varepsilon, \perp)$-connected) metric space. We obtain existence and uniqueness theorem of fixed point for mappings on ε-connected orthogonal metric space. We state some examples to our obtained result. The following proposition will be proved, which guarantees the existence and uniqueness of fixed point for analytic functions of one complex variable [3].

2010 *Mathematics Subject Classification.* 47H10, 54H25.
Key words and phrases. Fixed point, ε-connected, Orthogonal set, Solution, Metric space, Analytic function.
Received: 23 September 2017, Accepted: 28 August 2018.
* Corresponding author.

Proposition 1.1. *Let $f(z)$ be an analytic function in a domain D of the complex z-plane; let $f(z)$ map a compact and connected subset C of D into itself. If in addition $|f'(z)| < 1$ for every $z \in C$, then the equation $f(z) = z$ has one and only one solution in C.*

The paper is organized as follows: In Section 2, we state some definitions and recall extension of Banach fixed point theorem in an orthogonal metric space. In Section 3, we present some new definitions which are needed to prove the main result and we show the existence and uniqueness of fixed point for mappings on an ε-connected orthogonal metric space. This section contains some examples illustrating our result. In Section 4, applying the result of Section 3, we prove the existence and uniqueness of fixed point for an analytic function of one complex variable.

2. Preliminaries

In this section, some preliminaries which are necessary for later are recalled.

Definition 2.1 ([7]). Let $X \neq \phi$ and $\perp \subseteq X \times X$ be a binary relation. If $\perp$ satisfies the following condition

$$\exists x_0 \in X; ((\forall y \in X; y \perp x_0) \quad \text{or} \quad (\forall y \in X; x_0 \perp y)),$$

it is called an orthogonal set (briefly O-set). We denote this O-set by $(X, \perp)$.

In the following, we give some examples of orthogonal sets.

Example 2.2. Let $X = [2, \infty)$, we define $x \perp y$ if $x \leq y$. Then by putting $x_0 = 2$, $(X, \perp)$ is an O-set.

By the following examples, we can see that x_0 is not necessarily unique.

Example 2.3. Let $X = \left\{(1, \varphi); 0 \leq \varphi \leq \frac{3}{2}\pi\right\}$ be the set of points of the plane $\mathbb{R}^2$ defined in polar coordinates. We define the relation $\perp$ on X as follows:

$$(1, \varphi_1) \perp (1, \varphi_2) \quad \Longleftrightarrow \quad \varphi_1 \leq \varphi_2.$$

It is easy to see that $(1, 0) \perp (1, \varphi)$ and $(1, \varphi) \perp \left(1, \frac{3}{2}\pi\right)$ for all $(1, \varphi) \in X$.

Example 2.4. Suppose that $\mathcal{M}(n)$ is the set of all $n \times n$ matrices and Q is a positive definite matrix. Define the relation $\perp$ on $\mathcal{M}(n)$ by

$$A \perp B \quad \Longleftrightarrow \quad \exists X \in \mathcal{M}(n);\ AX = B.$$

One can see that $I \perp B$, $B \perp 0$ and $Q^{\frac{1}{2}} \perp B$ for all $B \in \mathcal{M}(n)$.

Let $(X, \perp)$ be an O-set. We consider the notion of O-sequence.

Definition 2.5 ([7])**.** A sequence $\{x_n\}_{n\in\mathbb{N}}$ is called an orthogonal sequence (briefly O-sequence) if

$$((\forall n; x_n \perp x_{n+1}) \quad \text{or} \quad (\forall n; x_{n+1} \perp x_n)).$$

Let $(X, d, \perp)$ be an orthogonal metric space ($(X, \perp)$ is an O-set and (X, d) is a metric space). Now, we consider the following definitions.

Definition 2.6 ([7])**.** The orthogonal metric space $(X, d, \perp)$ is said to be orthogonally complete (briefly O-complete) if every Cauchy O-sequence is convergent.

Definition 2.7 ([7])**.** Let $(X, d, \perp)$ be an orthogonal metric space and $0 < \lambda < 1$. Let f be a mapping of $(X, d, \perp)$ into itself.

(i) f is said to be an orthogonal contractive ($\perp-$contractive) mapping with Lipschitz constant λ if

$$d(fx, fy) \leq \lambda d(x, y) \quad \text{if} \quad x \perp y. \tag{2.1}$$

(ii) f is called an orthogonal preserving ($\perp-$preserving) mapping if $x \perp y$ then $f(x) \perp f(y)$.

(iii) f is an orthogonal continuous ($\perp-$continuous) mapping in $a \in X$ if for each O-sequence $\{a_n\}_{n\in\mathbb{N}}$ in X such that $a_n \to a$ then $f(a_n) \to f(a)$. Also, f is $\perp-$continuous on X if f is $\perp-$continuous in each $a \in X$.

Example 2.8. Let $X = [0, 10)$ and the metric on X be the Euclidian metric. Define $x \perp y$ if $xy \leq \max\{x, y\}$. The space X is not complete but it is O-complete. Let $x \perp y$ and $xy \leq x$. If $\{x_k\}$ is an arbitrary Cauchy O-sequence in X, then there exists a subsequence $\{x_{k_n}\}$ of $\{x_k\}$ for which $x_{k_n} = 0$ for all n, or there exists a subsequence $\{x_{k_n}\}$ of $\{x_k\}$ such that $x_{k_n} \leq 1$ for all n. It follows that $\{x_{k_n}\}$ converges to some $x \in [0, 10)$. On the other hand, we know that every Cauchy sequence with a convergent subsequence is convergent. It follows that $\{x_k\}$ is convergent.

Let $f : X \to X$ be a mapping defined by

$$f(x) = \begin{cases} \frac{x}{2}\ , & x \leq 2, \\ 0\ , & x > 2. \end{cases}$$

Also, $x \perp y$ and $xy \leq x$. We have the following cases:

- Case 1) $x = 0$ and $y \leq 2$. Then $f(x) = 0$ and $f(y) = \frac{y}{2}$.
- Case 2) $x = 0$ and $y > 2$. Then $f(x) = f(y) = 0$.
- Case 3) $y \leq 1$ and $x \leq 2$. Then $f(y) = \frac{y}{2}$ and $f(x) = \frac{x}{2}$.
- Case 4) $y \leq 1$ and $x > 2$. Then $x - y > y$, $f(y) = \frac{y}{2}$ and $f(x) = 0$.

These imply that $f(x)f(y) \leq f(x)$. Hence f is $\perp$-preserving.

Also, one can see that $|f(x) - f(y)| \leq \frac{1}{2}|x - y|$. Hence f is $\perp-$contraction. But f is not a contraction. Otherwise, for two points 2 and 3 and for all $0 < c < 1$ we have $|f(3) - f(2)| \leq c|3 - 2|$ and one can conclude that, it is a contradiction.

If $\{x_n\}$ is an arbitrary O-sequence in X such that $\{x_n\}$ converges to $x \in X$. Since f is $\perp-$contraction, then for each $n \in \mathbb{N}$ we have

$$|f(x_n) - f(x)| \leq \frac{1}{2}|x_n - x|.$$

As n goes to infinity, f is $\perp$-continuous. But, as it can be seen easily, f is not continuous.

Now, we can state the following theorem which can be considered as a real extension of Banach fixed point theorem.

Theorem 2.9 ([7])**.** *Let $(X, d, \perp)$ be an O-complete metric space (not necessarily a complete metric space) and $0 < \lambda < 1$. Let $f : X \to X$ be $\perp-$continuous, $\perp-$contraction (with Lipschitz constant λ) and $\perp-$preserving. Then f has a unique fixed point x^* in X. Also, f is a Picard operator, that is,* $\lim f^n(x) = x^*$ *for all $x \in X$.*

3. Main Results

In this section, we state and prove our existence and uniqueness results. Let $\varepsilon > 0$. At first, we consider the following definitions.

Definition 3.1. A sequence $\{x_n\}$ is called an ε-connected orthogonal sequence (briefly $(\varepsilon, \perp)$-connected sequence) if $d(x_n, x_{n+1}) < \varepsilon \ \ (n \in \mathbb{N})$ and

$$((\forall n \in \mathbb{N}; x_n \perp x_{n+1}) \quad \text{or} \quad (\forall n \in \mathbb{N}; x_{n+1} \perp x_n)).$$

Definition 3.2. An orthogonal metric space X is called $(\varepsilon, \perp)$-connected if for any points $x \in, y \in X$, one can find an $(\varepsilon, \perp)$-connected sequence $x = x_0, x_1, \dots, x_n = y$.

Let $(X, d, \perp)$ be an $(\varepsilon, \perp)$-connected metric space. We turn our consideration to the following definition.

Definition 3.3. A mapping $f : X \to X$ is called ε-connected orthogonal preserving ($(\varepsilon, \perp)$-connected preserving) if $d(x, y) < \varepsilon$ and $x \perp y$ imply $d(f(x), f(y)) < \varepsilon$ and $f(x) \perp f(y)$.

Now, we give an example of $(\varepsilon, \perp)$-connected preserving maps.

Example 3.4. Let $X = [0, 1)$. Define $x \perp y$ if $xy \leq \min\left\{\frac{x}{2}, \frac{y}{2}\right\}$. Let the metric on X be the Euclidian metric and $f : X \to X$ be a mapping defined by

$$f(x)=\begin{cases}\frac{x}{8}\ , & 0\leq x<\frac{1}{2},\\ 0\ , & \frac{1}{2}\leq x<1.\end{cases}$$

Suppose $x\perp y$, $xy \leq \frac{x}{2}$, $\varepsilon = \frac{1}{10}$ and $d(x,y) < \varepsilon$. Then the following cases are hold.

- Case 1) $x = 0$ and $y < \frac{1}{2}$. Then $f(x) = 0$ and $f(y) = \frac{y}{8}$.
- Case 2) $y \leq \frac{1}{2}$ and $x < \frac{1}{2}$. Then $f(y) = \frac{y}{8}$ and $f(x) = \frac{x}{8}$.
- Case 3) $y \leq \frac{1}{2}$ and $\frac{1}{2} \leq x < 1$. Then $f(y) = \frac{y}{8}$ and $f(x) = 0$.

This implies that $d(f(x), f(y)) < \frac{1}{16} < \varepsilon$ and $f(x)f(y) \leq \frac{f(x)}{2}$. Hence f is $(\varepsilon,\perp)$-connected preserving.

Let $(X, d, \perp)$ be an $(\varepsilon,\perp)$-connected metric space and $0 < \lambda < 1$. We turn our attention to the concept of $(\varepsilon,\perp)$-connected contractive maps.

Definition 3.5. A mapping $f : X \to X$ is said to be ε-connected orthogonal contractive ($(\varepsilon,\perp)$-connected contractive) with Lipschitz constant $\lambda\,(0 \leq \lambda < 1)$ if

$$d(fx, fy) \leq \lambda d(x,y), \tag{3.1}$$

for any $x \in X$ and $y \in X$ such that $d(x,y) < \varepsilon$ and $x\perp y$.

In the following, we present some examples of $(\varepsilon,\perp)$-connected contractive maps.

Example 3.6. Using Example 3.4, one can see that

$$|f(x) - f(y)| \leq \frac{1}{8}\,|x-y|\,.$$

Hence f is an $(\varepsilon,\perp)$-connected contractive map with $\lambda = \frac{1}{8}$.

The following example shows that there exist $(\varepsilon,\perp)$-connected contractive maps which are not $\perp$-contractive.

Example 3.7. Let $X = \left\{(1,\varphi)\,;0 \leq \varphi \leq \frac{3}{2}\pi\right\}$ be the set of points of the plane $\mathbb{R}^2$ defined in polar coordinates. For $x = (1,\varphi_1)$ and $y = (1,\varphi_2)$ in X we define the relation $\perp$ as follows:

$$x\perp y \iff \varphi_1 \leq \varphi_2.$$

Let $f : X \to X$ be a mapping defined by $f(1,\varphi) = \left(1, \frac{4}{9}\varphi\right)$. In other words, X is an arc of the unit circle. Let d be the usual metric on the plane and $\varepsilon = \sqrt{2}$. If $x\perp y$ and $d(x,y) < \sqrt{2}$ then we have:

$$\begin{array}{ll} x = (1,\varphi_1)\,, & y = (1,\varphi_2)\,,\\ fx = \left(1,\frac{4}{9}\varphi_1\right), & fy = \left(1,\frac{4}{9}\varphi_2\right). \end{array}$$

In the plane $\mathbb{R}^2$, we have:

$$\begin{aligned} x &= (\cos\varphi_1, \sin\varphi_1), & y &= (\cos\varphi_2, \sin\varphi_2), \\ fx &= \left(\cos\tfrac{4}{9}\varphi_1, \sin\tfrac{4}{9}\varphi_1\right), & fy &= \left(\cos\tfrac{4}{9}\varphi_2, \sin\tfrac{4}{9}\varphi_2\right). \end{aligned}$$

So,

$$\begin{aligned} d(fx,fy) &= \sqrt{\left(\cos\frac{4}{9}\varphi_1 - \cos\frac{4}{9}\varphi_2\right)^2 + \left(\sin\frac{4}{9}\varphi_1 - \sin\frac{4}{9}\varphi_2\right)^2} \\ &= \sqrt{2 - 2\left(\sin\frac{4}{9}\varphi_1 \sin\frac{4}{9}\varphi_2 + \cos\frac{4}{9}\varphi_1 \cos\frac{4}{9}\varphi_2\right)} \\ &= \sqrt{2 - 2\cos\frac{4}{9}(\varphi_2 - \varphi_1)} \\ &= \sqrt{2 - 2\left(1 - 2\sin^2\left(\frac{2}{9}(\varphi_2 - \varphi_1)\right)\right)} \\ &= 2\sin\left(\frac{2}{9}(\varphi_2 - \varphi_1)\right). \end{aligned}$$

In this way, one obtains $d(x,y) = 2\sin\left(\frac{1}{2}(\varphi_2 - \varphi_1)\right)$, and

$$\begin{aligned} \frac{d(fx,fy)}{d(x,y)} &= \frac{2\sin\left(\frac{2}{9}(\varphi_2 - \varphi_1)\right)}{2\sin\left(\frac{1}{2}(\varphi_2 - \varphi_1)\right)} \\ &\leq \frac{\sin\left(\frac{2}{9}\frac{\pi}{2}\right)}{\sin\left(\frac{1}{2}\frac{\pi}{2}\right)} \\ &= \frac{\sin\frac{\pi}{9}}{\sin\frac{\pi}{4}} \\ &< 1. \end{aligned}$$

Therefore, if $d(x,y) < \varepsilon = \sqrt{2}$ and $x \perp y$ then

$$d(fx,fy) \leq \lambda d(x,y),$$

where $\lambda = \frac{\sin\frac{\pi}{9}}{\sin\frac{\pi}{4}} < 1$. This means that f is $(\sqrt{2}, \perp)$-connected contractive. Now, we show that f is not $\perp$-contraction. Otherwise, for $a = (1,0)$ and $b = \left(1, \frac{3}{2}\pi\right)$ such that $a \perp b$, $d(a,b) = \sqrt{2}$, $fa = a = (1,0)$ and $fb = \left(1, \frac{2}{3}\pi\right)$, we have $d(fa,fb) = \sqrt{3}$, $fa \perp fb$ and $d(fa,fb) > d(a,b)$. One can conclude that it is a contradiction.

Let X be an $(\varepsilon, \perp)$-connected metric space. In the following definition, we consider the notion of an $(\varepsilon, \perp)$-connected continuous map.

Definition 3.8. A mapping $f : X \to X$ is ε-connected orthogonal continuous (briefly$(\varepsilon, \perp)$-connected continuous) in $a \in X$ if for each $(\varepsilon, \perp)$-connected sequence $\{a_n\}_{n\in\mathbb{N}}$ in X if $a_n \to a$ then $f(a_n) \to f(a)$. Also f is $(\varepsilon, \perp)$-connected continuous on X if f is $(\varepsilon, \perp)$-connected continuous in each $a \in X$.

Let us consider some examples of $(\varepsilon, \perp)$-connected continuous map.

Example 3.9. In Example 3.4, f is an $(\varepsilon, \perp)$-connected continuous map as can be proved in the following way:

Let $\{x_n\}_{n\in\mathbb{N}}$ be an $(\varepsilon, \perp)$-connected sequence in X converging to $x \in X$. Since f is an $(\varepsilon, \perp)$-connected preserving map so $\{fx_n\}$ is an $(\varepsilon, \perp)$-connected sequence. In Example 3.6, we have shown that f is $(\varepsilon, \perp)$-connected contractive with $\lambda = \frac{1}{8}$. For each $n \in \mathbb{N}$ we have

$$|f(x_n) - f(x)| < \frac{1}{8} |x_n - x| .$$

As n goes to infinity, f is an $(\varepsilon, \perp)$-connected continuous map.

Example 3.10. Let $(X, d, \perp)$, ε and f be as defined in Example 3.7. Suppose that $\{x_n\}_{n\in\mathbb{N}} = \{(1, \varphi_n)\}_{n\in\mathbb{N}}$ is an $(\varepsilon, \perp)$-connected sequence in X converging to $x \in X$. It is obvious that f is an $(\varepsilon, \perp)$-connected preserving map. So, $\{fx_n\}$ is an $(\varepsilon, \perp)$-connected sequence. Since, f is $(\varepsilon, \perp)$-connected contractive, for each $n \in \mathbb{N}$ we have

$$|f(x_n) - f(x)| < \lambda |x_n - x| .$$

As n goes to infinity, it follows that f is an $(\varepsilon, \perp)$-connected continuous map.

Let X be an $(\varepsilon, \perp)$-connected metric space. We turn our consideration to the following definition.

Definition 3.11. The space X is called ε-connected orthogonal complete (briefly $(\varepsilon, \perp)$-connected complete) if every Cauchy $(\varepsilon, \perp)$-connected sequence is convergent.

At this stage, we can state the main theoretical result of this paper which proves existence and uniqueness of fixed point for mappings on $(\varepsilon, \perp)$-connected metric space.

Theorem 3.12. *Let $(X, d, \perp)$ be an $(\varepsilon, \perp)$-connected complete metric space and f be some $(\varepsilon, \perp)$-connected preserving, $(\varepsilon, \perp)$-connected contractive and $(\varepsilon, \perp)$-connected continuous map. Then there exists a unique fixed point x^* of f and $\lim_{n\to\infty} f^n x = x^*$ for any point $x \in X$.*

Proof. We set

$$d^*(x, y) = \inf \sum_{i=0}^{n-1} d(x_i, x_{i+1}),$$

where the sum is taken over all $(\varepsilon, \perp)$-connected sequences $x = x_0, x_1, \ldots, x_n = y$. Since $(\varepsilon, \perp)$ is an $(\varepsilon, \perp)$-connected metric space, then $d^*(x, y)$ is defined for all $x \in X$, $y \in X$.

First, we note that

$$d(x, y) \leq \sum_{i=0}^{n-1} d(x_i, x_{i+1}),$$

if $x = x_0$, $y = x_n$ and hence

$$\begin{aligned} d(x, y) &\leq d^*(x, y) \\ &= \inf \sum_{i=0}^{n-1} d(x_i, x_{i+1}). \end{aligned}$$

Now, we will prove that d^* is a metric on $(X, \perp)$. The relations $d^*(x, y) = d^*(y, x)$ and $d^*(x, x) = 0$ follow directly from definition of d^*.

If $x \neq y$ and $x \perp y$ then $d(x, y) > 0$ and since $d(x, y) \leq d^*(x, y)$, one has $d^*(x, y) > 0$.

Moreover, the triangle axiom is satisfied, as can be proved in the following way:

For $\forall x, y, z \in X$ such that $x \perp y$, $y \perp z$ and $x \perp z$, we have

$$\begin{aligned} d^*(x, y) + d^*(y, z) &= \inf_{x = x_0, y = x_n} \sum_{i=0}^{n-1} d(x_i, x_{i+1}) + \inf_{y = y_0, z = y_m} \sum_{j=0}^{m-1} d(y_j, y_{j+1}) \\ &= \inf_{x = x_0, y = x_n, z = x_{m+n}} \sum_{i=0}^{n+m-1} d(x_i, x_{i+1}) \\ &\geq \inf_{x = x_0, z = x_{n+m}} \sum_{i=0}^{n+m-1} d(x_i, x_{i+1}) \\ &= d^*(x, z). \end{aligned}$$

If $x \perp y$ and $d(x, y) < \varepsilon$ then among the sums

$$\sum_{i=0}^{n-1} d(x_i, x_{i+1}),$$

$x = x_0$, $y = x_n$, $d(x_i, x_{i+1}) < \varepsilon$ figures the sum

$$\sum_{i=0}^{0} d(x_i, x_{i+1}),$$

consisting of one summand $d(x, y)$ i.e. in this case $d^*(x, y) \leq d(x, y)$ and taking account of the inequality $d(x, y) \leq d^*(x, y)$ which was proved earlier we get the relation

$$d(x,y) < \varepsilon \quad \Rightarrow \quad d(x,y) = d^*(x,y).$$

We note that the $\perp$-preserving map f is $\perp$-contractive with respect to the metric d^*, since, for $x \perp y$ we have

$$\begin{aligned} d^*(fx,fy) &= \inf_{fx=x_0, fy=x_n} \sum_{i=0}^{n-1} d(z_i, z_{i+1}) \\ &\leq \inf_{x=x_0, y=x_n} \sum_{i=0}^{n-1} d(fx_i, fx_{i+1}) \\ &\leq \inf_{x=x_0, y=x_n} \sum_{i=0}^{n-1} \lambda d(x_i, x_{i+1}) \\ &= \lambda d^*(x,y). \end{aligned}$$

Finally, the $\perp$-preserving map f is $\perp$-continuous with respect to the metric d^*, since for O-sequence $\{a_n\}$ such that $a_n \longrightarrow a$ we have

$$\begin{aligned} d^*(fa_n, fa) &\leq \lambda d^*(a_n, a) \\ &< \varepsilon\lambda. \end{aligned}$$

Thus, a metric d^* on $(X,\perp)$ has been found in which the $\perp$-preserving map f is $\perp$-continuous and $\perp$-contractive for any λ $(0 < \lambda < 1)$.

For $x \perp y$, we have $d(x,y) = d^*(x,y)$ if $d(x,y) < \varepsilon$ and $d^*(x,y) < \varepsilon$. Hence, Cauchy systems in $(X,d,\perp)$ coincide with Cauchy systems in $(X,d^*,\perp)$ and in this way, convergence in $(X,d,\perp)$ coincides with convergence in $(X,d^*,\perp)$. Thus, if $(X,d,\perp)$ is $(\varepsilon,\perp)$-connected complete metric space then $(X,d^*,\perp)$ is a O-complete metric space, too. Therefore, f is an $\perp$-preserving, $\perp$-contractive and $\perp$-continuous map of $(X,d^*,\perp)$ into itself. So, by Theorem 2.9, there exists a unique fixed point x^* of f and $\lim_{n\to\infty} f^n x = x^*$ in the metric d^* and consequently in the metric d for any point $x \in X$. $\square$

In the following, we show how the classical fixed point theorem on ε-connected metric spaces of [3] is a consequence of the previous theorem.

Theorem 3.13. *Let (X,d) be a complete ε-connected metric space and f be some ε-contractive map. Then there exists a unique fixed point x^* of f and $\lim_{n\to\infty} f^n x = x^*$ for any point $x \in X$.*

Proof. Let $x,y \in X$ and $d(x,y) < \varepsilon$. Define $x \perp y$ if $d(fx,fy) \leq d(x,y)$. Fix $x_0 \in X$. Since f is an ε-contractive then for each $y \in X$, $x_0 \perp y$. Hence X is $(\varepsilon,\perp)$-connected metric space.

Let $x, y \in X$, $d(x,y) < \varepsilon$ and $x \perp y$. Then by definition of $\perp$, we have $d(f(x), f(y)) \leq d(x,y) < \varepsilon$. Since f is ε-contractive, $f(x) \perp f(y)$ and hence, f is an $(\varepsilon, \perp)$-connected preserving map.

Let $x, y \in X$, $d(x,y) < \varepsilon$ and $x \perp y$. Since f is ε-contractive, we have $d(f(x), f(y)) \leq \lambda d(x,y)$. Hence f is an $(\varepsilon, \perp)$-connected contractive map.

Let $\{x_n\}_{n\in\mathbb{N}}$ be an $(\varepsilon, \perp)$-connected sequence in X converging to $x \in X$. Since f is an $(\varepsilon, \perp)$-connected preserving map, so $\{fx_n\}$ is an $(\varepsilon, \perp)$-connected sequence. Also, f is $(\varepsilon, \perp)$-connected contractive with λ $(0 \leq \lambda < 1)$. For each $n \in \mathbb{N}$ we have $d(f(x_n), f(x)) \leq d(x_n, x)$. As n goes to infinity, f is an $(\varepsilon, \perp)$-connected continuous map.

It is obvious that X is an $(\varepsilon, \perp)$-connected complete metric space and f is an $(\varepsilon, \perp)$-connected continuous map. Applying previous theorem, f has a unique fixed point x^* and $\lim_{n\to\infty} f^n x = x^*$ for any point $x \in X$. □

Next examples illustrate some of the assumptions involved in Theorem 3.12.

Example 3.14. For $(X, d, \perp)$ and f as in Example 3.4, it is obvious that X is $(\varepsilon, \perp)$-connected complete. We have shown in Example 3.4 that f is $(\varepsilon, \perp)$-connected preserving. Example 3.6 shows that f is $(\varepsilon, \perp)$-connected contractive and also Example 3.9 shows that f is an $(\varepsilon, \perp)$-connected continuous map. Applying Theorem 3.12, f has a unique fixed point in X.

Example 3.15. Let $(X, d, \perp)$ and f be as in Example 3.7. It is obvious that X is $(\varepsilon, \perp)$-connected complete and f is $(\varepsilon, \perp)$-connected preserving. In Example 3.7, we have shown that f is $(\varepsilon, \perp)$-connected contractive and Example 3.10 shows that f is an $(\varepsilon, \perp)$-connected continuous map. Applying Theorem 3.12, f has a unique fixed point in X.

4. An Application to Analytic Function of one Complex Variable

In this section, we apply the obtained results in the previous section to the particular case of analytic functions in a domain of the complex z-plane. This approach provides a new proof of the following fixed point theorem for analytic functions of one complex variable.

Proposition 4.1. *Let $f(z)$ be an analytic function in a domain D of the complex z-plane; let $f(z)$ map a compact and connected subset C of D into itself. If in addition $|f'(z)| < 1$ for every $z \in C$ then the equation $f(z) = z$ has one and only one solution in C.*

Proof. Let $X = C$ and d be the usual metric on the complex z-plane. As $|f'(z)|$ is continuous on X, it follows from the compactness of X that there exists λ such that $|f'(z)| \leq \lambda < 1$ on X.

Define $z_1 \perp z_2$ if there exists $\delta > 0$ such that $d(z_1, z_2) < \delta$ for $z_1, z_2 \in X$. Fix $z_0 \in X$. Consider a cover of X by a family of open discs $S(z, \rho)$ centered at points $z \in X$ and a radius ρ such that $f(z)$ is analytic and $|f'(z)| < \lambda$ in $S(z, 2\rho)$. This cover contains, again by compactness of X, a finite subcover $\{S(z_i, \rho_i)\}$, $(i = 0, 1, 2, \ldots, n)$. Put $\delta = \Sigma_{i=0}^{n} 2\rho_i$. We have $d(z, z_0) < \delta$ for any $z \in X$. So $z \perp z_0$ for any $z \in X$. Hence, $(X, d, \perp)$ is an orthogonal metric space.

Now, put $\varepsilon = \min_i \rho_i$. Since X is connected, we can deduce that X is an $(\varepsilon, \perp)$-connected complete metric space.

We break the end of the proof into the following steps:

Step 1) f is $(\varepsilon, \perp)$-connected preserving.

Let $z_1, z_2 \in X$, $d(z_1, z_2) < \varepsilon$ and $z_1 \perp z_2$. Since X is $(\varepsilon, \perp)$-connected complete metric space, so there exists an $(\varepsilon, \perp)$-connected sequence $f(z_1) = x_0, x_1, \ldots, x_n = f(z_2)$ such that $d(x_i, x_{i+1}) < \varepsilon$. Put $\delta = n\varepsilon$. So $d(f(z_1), f(z_2)) < \delta$. This means that $f(z_1) \perp f(z_2)$. Thus, f is $(\varepsilon, \perp)$-connected preserving.

Step 2) f is $(\varepsilon, \perp)$-connected contractive.

Any two point z_1, z_2 of X with distant less than ε will, evidently, fall into some $S(z_j, 2\rho_j)$. Hence,

$$|f(z_2) - f(z_1)| = \left| \int_{z_1}^{z_2} f'(z) dz \right| < \lambda \, |z_2 - z_1| \, .$$

This means that f is $(\varepsilon, \perp)$-connected contractive.

Step 3) f is $(\varepsilon, \perp)$-connected continuous.

Suppose $\{z_n\}_{n \in \mathbb{N}}$ is an $(\varepsilon, \perp)$-sequence in X converging to $z \in X$. Because f is $(\varepsilon, \perp)$-connected preserving, $\{f(z_n)\}_{n \in \mathbb{N}}$ is an $(\varepsilon, \perp)$-connected sequence. For each $n \in \mathbb{N}$, since f is $(\varepsilon, \perp)$-connected contractive we have

$$|f(z_n) - f(z)| < \lambda \, |z_n - z| \, .$$

As n goes to infinity, it follows that f is $(\varepsilon, \perp)$-connected continuous map on X.

Applying Theorem 3.12, then $f(z) = z$ has one and only one solution in X. □

References

1. H. Baghani, M. Eshaghi Gordji, and M. Ramezani, *Orthogonal sets: their relation to the axiom of choice and a generalized fixed point theorem*, J. Fixed Point Theory Appl., 18 (2016), pp. 465-477.
2. I. Beg and A.R. Butt, *Fixed point of set-valued graph contractive mappings*, J. Inequa. Appl., (2013), 2013:252.

3. M. Edelstein, *An extension of Banach's contraction principle*, Proc. Amer. Math. Soc., 12 (1961), pp. 7-10.
4. M. Eshaghi Gordji, H. Baghani, H. Khodaei, and M. Ramezani, *A generalization of Nadler's fixed point theorem*, J. Nonlinear Sci. Appl., 3 (2010), pp. 148-151.
5. M. Eshaghi Gordji, H. Baghani, H. Khodaei, and M. Ramezani, *Generalized multi valued contraction mappings*, J. Comput. Anal. Appl., 13 (2011), pp. 730-733.
6. M. Eshaghi Gordji, H. Baghani, H. Khodaei, and M. Ramezani, *Geraghty's fixed point theorem for special multi-valued mappings*, Thai. J. Math., 10 (2012), pp. 225-231.
7. M. Eshaghi Gordji, M. Ramezani, M. De La Sen, and Y.J. Cho, *On orthogonal sets and Banach fixed point theorem*, Fixed Point Theory, 18 (2017), pp. 569-578.
8. R. Espinola, E.S. Kim, and W.A. Kirk, *Fixed point properties of mappings satisfying local contractive conditions*, Nonlinear Anal. Forum, 6 (2001), pp. 103-111.
9. N. Mehmood, A. Azam, and S. Aleksic, *Topological vector-space valued cone Banach spaces*, Int. J. Anal. Appl., 6 (2014), pp. 205-219.
10. M. Ramezani, *Orthogonal metric space and convex contractions*, Int. J. Nonlinear Anal. Appl., 6 (2015), pp. 127-132.
11. M. Ramezani, H. Baghani, *Contractive gauge functions in strongly orthogonal metric spaces*, Int. J. Nonlinear Anal. Appl., 8(2) (2017) pp. 23-28.
12. P. Shahi, J. Kaur, and S.S. Bhatia, *On fixed points of generalized α-ϕ contractive type mappings in partial metric spaces*, Int. J. Anal. Appl., 12 (2016), pp. 38-48.

[1] Department of Mathematics, Semnan University, P.O. Box 35195-363, Semnan, Iran.

E-mail address: meshaghi@semnan.ac.ir

[2] Department of Mathematics, Semnan University, Semnan, Iran.

E-mail address: hastihabibi1363@gmail.com