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A New Common Fixed Point Theorem for Suzuki Type Contractions via Generalized Ψ-simulation Functions

Gholamreza Heidary Joonaghany[1], Ali Farajzadeh[2*], Mahdi Azhini[3], and Farshid Khojasteh[4]

Abstract. In this paper, a new stratification of mappings, which is called Ψ-simulation functions, is introduced to enhance the study of the Suzuki type weak-contractions. Some well-known results in weak-contractions fixed point theory are generalized by our researches. The methods have been appeared in proving the main results are new and different from the usual methods. Some suitable examples are furnished to demonstrate the validity of the hypothesis of our results and reality of our generalizations.

1. Introduction

The Banach contraction principle [4] was appeared in 1922. It is the wellspring of many future researchers and load of generalizations were presented in a short run. The concept of the weak contraction was defined by Alber and Guerre Dlabriere [1] for single valued maps on Hilbert spaces in 1997. In 2001 Rhoades generalized this concept to complete metric spaces [18]. Many authors made effort to find a way for characterizing the contractions which are the combination of two terms $d(x,y)$ and $d(Tx,Ty)$ in which X is a metric space, T is a self map on X and $x,y \in X$ (see [5–9, 11–14, 17–25] and references therein).

In 2008 Suzuki [23] established a worth-full generalization of Banach contraction theorem. Suzuki-type contractions as one of the remarkable concepts in nonlinear analysis were studied by many authors and worthwhile results in this direction obtained. See [7, 8, 11, 12, 17, 21, 22, 24].

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* Corresponding author.

Particularly, in 2015 Sing et al. [21] obtained a weakly contractive version of Suzuki type contractions and generalized some results of Đorić [6], Zhang et al. [25] and others.

In 2015, Khojasteh et al. [10] introduced a new class of mappings called simulation functions, from which many single valued contractions could be obtained. They proved the existence and uniqueness of fixed points for the class of Z-contraction mappings. The advantage of this notion is in providing a unique point of view for several fixed point problems (for more details, we refer the reader to [15, 19] and the references therein).

In 2016 Olgun et al. [16] introduced the concept of generalized Z-contraction on metric spaces and proved a fixed point theorem for this contractions.

In this paper, a new class of mappings so called Ψ-simulation functions which is greater than simulation functions' class, is introduced. Moreover, an armative answer to a conjecture proposed by Singh et al. [21] is provided and some well-known weak-contractions fixed point theorems are generalized.

Definition 1.1 ([10])**.** Let $\eta : [0,\infty) \times [0,\infty) \to \mathbb{R}$ be a mapping, then η is called a simulation function if it satisfies the following conditions:

(η_i) $\eta(0,0) = 0$,
(η_{ii}) $\eta(t,s) < s - t$ for all $t, s > 0$,
(η_{iii}) if $\{t_n\}, \{s_n\}$ are sequences in $(0,\infty)$ such that

$$\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n > 0,$$

then

$$\limsup_{n\to\infty} \eta(t_n, s_n) < 0.$$

Example 1.2. We recall some examples of the simulation functions given in [10].

(a_1) For each $s, t \geq 0$, let $\eta(t,s) = \alpha s - t$, in which $\alpha \in [0,1)$.
(a_2) For each $s, t \geq 0$, let $\eta(t,s) = \phi(s) - t$, in which $\phi : [0,+\infty) \to (0,+\infty)$ be a mapping such that for each $s > 0$, $\phi(s) < s$ and

$$\limsup_{t\to s} \phi(t) < s.$$

(a_3) For each $s, t \geq 0$ let $\eta(t,s) = s\phi(s) - t$, in which $\phi : [0,+\infty) \to [0,1)$ be a mapping such that for each $s > 0$,

$$\limsup_{t\to s} \phi(t) < 1.$$

(a_4) For each $s, t \geq 0$, let $\eta(t,s) = s - \phi(s) - t$, in which $\phi : [0, +\infty) \to [0, +\infty)$ be a mapping such that for each $s > 0$,

$$\liminf_{t \to s} \phi(t) > 0,$$

or $\phi : [0, +\infty) \to [0, 1)$ is a continuous function such that $\phi(t) = 0$ if and only if $t = 0$.

(a_5) For each $s, t \geq 0$, let $\eta(t,s) = \psi(s) - \varphi(s)$ where $\psi, \phi : [0, +\infty) \to [0, +\infty)$ are two continuous functions such that $\psi(t) = \phi(t) = 0$ if and only if $t = 0$ and $\psi(t) < t \leq \phi(t)$ for all $t > 0$.

As in [10] the set of all simulation functions is denoted by $\mathcal{Z}$. The following theorem is proved by Khojasteh et al. in [10]. (See also [2, 19]).

Theorem 1.3. *Let (X, d) be a complete metric space and $T : X \to X$ be a $\mathcal{Z}$-contraction with respect to a certain simulation function η, that is,*

$$\eta(d(Tx, Ty), d(x, y)) \geq 0, \quad \forall x, y \in X. \tag{1.1}$$

Then T has a unique fixed point.

Note that the condition (η_i) was not used for the proof of Theorem 1.3. However, one can easily see that if $\eta(0,0) < 0$, the set of operators $T : X \to X$ satisfying (1.1) will be empty. Therefore, we can modify the Definition 1.1 by removing the condition (η_i), as we do it in sequel.

Definition 1.4 ([16])**.** Let (X, d) be a complete metric space and $T : X \to X$ be a mapping and $\zeta \in \mathcal{Z}$. Then T is called generalized Z-contraction with respect to ζ if the following condition is satisfied:

$$\eta \left(d(Tx, Ty), m_T(x, y) \right) \geq 0,$$

where

$$m_T(x, y) = \max \left\{ d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Ty) + d(y, Tx)}{2} \right\}.$$

Theorem 1.5 ([16], Theorem 2)**.** *Every generalized Z -contraction on a complete metric space has a unique fixed point.*

The following theorem is proved by Sing et al. [21].

Theorem 1.6 ([21], Theorem 2.1)**.** *Let (X, d) be a complete metric space and $T : X \to X$ such that for all $x, y \in X$, $\frac{1}{2} d(x, Tx) \leq d(x, y)$ implies that*

$$\psi \left(d(Tx, Ty) \right) \leq \psi \left(m_T(x, y) \right) - \phi \left(m_T(x, y) \right),$$

where

(i) $\psi : [0, +\infty) \to [0, +\infty)$ *is a continuous nondecreasing function and* $\psi(t) = 0$ *if and only if* $t = 0$,
(ii) $\varphi : [0, +\infty) \to [0, +\infty)$ *is lower semi-continuous with* $\varphi(t) = 0$ *if and only if* $t = 0$,

and

$$m_T(x, y) = \max\left\{d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Ty) + d(y, Tx)}{2}\right\}.$$

Then T *has a unique fixed point.*

The following theorem is the conjecture which was proposed by Singh et al. in [21].

Theorem 1.7. *Let* (X, d) *be a complete metric space and* T *and* S *be two self-maps on* X *such that for every* $x, y \in X$, $\frac{1}{2}\min\{d(x, Tx), d(y, Sy)\} \leq d(x, y)$ *implies*

$$\psi(d(Tx, Sy)) \leq \psi(m(x, y)) - \phi(m(x, y)),$$

where ψ, φ *are defined as in Theorem 1.6 and*

$$m(x, y) = \max\left\{d(x, y), d(x, Tx), d(y, Sy), \frac{d(x, Sy) + d(y, Tx)}{2}\right\}.$$

Then T *and* S *have a unique common fixed point.*

We state the following lemma which is useful in proving our main result.

Lemma 1.8 ([6])**.** *Let* (X, d) *be a metric space, and* $\{x_n\}$ *be a sequence in* X *such that* $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$. *If* $\{x_{2n}\}$ *is not a Cauchy sequence then there exists* $\epsilon > 0$ *and two sequences of positive integers* $\{n_k\}$ *and* $\{m_k\}$ *such that* n_k *is smallest index for which* $n_k > m_k > k$ *and* $d(x_{2m_k}, x_{2n_k}) > \epsilon$ *and*

(1) $\lim_{k\to\infty} d(x_{2m_k}, x_{2n_k}) = \epsilon$,
(2) $\lim_{k\to\infty} d(x_{2m_k-1}, x_{2n_k}) = \epsilon$,
(3) $\lim_{k\to\infty} d(x_{2m_k}, x_{2n_k+1}) = \epsilon$,
(4) $\lim_{k\to\infty} d(x_{2m_k-1}, x_{2n_k+1}) = \epsilon$.

Proof. See the proof of Theorem 2.1 in [6]. □

2. Strong Convergence

In this section, the concept of Ψ-simulation functions is introduced and our main result is presented.

Denote $\Psi([0, +\infty))$ the set of all non-decreasing and continuous functions $\psi : [0, +\infty) \to [0, +\infty)$ such that $\psi(t) = 0$ if and only if $t = 0$.

Definition 2.1. A function $\eta : [0, \infty)\times[0, \infty) \to \mathbb{R}$ is called Ψ-simulation, if there exists $\psi \in \Psi([0, +\infty))$ such that:

(η1) $\eta(t,s) < \psi(s) - \psi(t)$ for all $s,t > 0$,

(η2) if $\{t_n\},\{s_n\}$ are sequences in $(0,\infty)$ such that

$$\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n > 0,$$

then

$$\limsup_{n\to\infty} \eta(t_n, s_n) < 0.$$

It is clear that, all of the previous results in fixed point theory which are obtained by simulation functions can be established by Ψ-simulation functions.

Example 2.2. Let $\psi \in \Psi([0,+\infty))$. The following are some examples of Ψ-simulation functions:

(e_1) For each $s,t \geq 0$ let $\eta(t,s) = \alpha\psi(s) - \psi(t)$, in which $\alpha \in [0,1)$.

(e_2) For each $s,t \geq 0$ let $\eta(t,s) = \phi(\psi(s)) - \psi(t)$, in which $\phi : [0,+\infty) \to [0,+\infty)$ is a function such that $\phi(0) = 0$ and for each $s > 0$, $0 < \phi(s) < s$ and $\limsup_{t\to s} \phi(t) < s$. (For example $\phi(s) = \alpha s$ in which $0 \leq \alpha < 1$).

(e_3) For each $s,t \geq 0$ let $\eta(t,s) = \phi(s)\psi(s) - \psi(t)$, in which $\phi : [0,+\infty) \to [0,1)$ is a function such that $\limsup_{t\to s} \phi(t) < 1$, for each $s > 0$.

(e_4) For each $s,t \geq 0$ let $\eta(t,s) = \psi(s) - \phi(s) - \psi(t)$, in which $\phi : [0,+\infty) \to [0,+\infty)$ is a function such that, for each $s > 0$, $\liminf_{t\to s} \phi(t) > 0$.

Remark 2.3. Every simulation function is obviously a Ψ-simulation function because, ψ can be considered as identity function on $[0,\infty)$. So all of the simulation functions presented in Example 1.2 are Ψ-simulation functions. However, the following example shows that every Ψ-simulation function is not necessarily a simulation function.

Example 2.4. Define $\eta : [0,\infty) \times [0,\infty) \to \mathbb{R}$ by $\eta(t,s) = 2s\varphi(s) - 2t$ where $\phi : \mathbb{R} \to [0,1)$ is a mapping such that $\limsup_{t\to s} \phi(t) < 1$, for each $s > 0$ (for example $\phi(t) = \alpha$, where $0 \leq \alpha < 1$). Then, it is easily seen that η is a Ψ-simulation function, with $\psi(t) = 2t, t \geq 0$. However, $\eta \notin \mathcal{Z}$.

Denote by $\mathcal{Z}_\Psi$ the set of all Ψ-simulation functions. We proved:

Proposition 2.5. $\mathcal{Z} \subsetneqq \mathcal{Z}_\Psi$.

Here, we establish our main result:

Theorem 2.6. *Let (X,d) be a complete metric space and $T,S : X \to X$ be two mappings such that for all $x,y \in X$,*
$\frac{1}{2}\min\{d(x,Tx),d(y,Sy)\} \le d(x,y)$ implies that

$$\eta(d(Tx,Sy),m(x,y)) \ge 0, \tag{2.1}$$

where $\eta \in \mathcal{Z}_\Psi$ and

$$m(x,y) = \max\left\{d(x,y),d(x,Tx),d(y,Sy),\frac{d(x,Sy)+d(y,Tx)}{2}\right\}.$$

Then T and S have a unique common fixed point.

Proof. At first, let

$$\mathcal{A} = \left\{(x,y) \in X \times X : \frac{1}{2}\min\{d(x,Tx),d(y,Sy)\} \le d(x,y)\right\},$$

and note that for each $x \in X$, we have $(x,Tx) \in \mathcal{A}$. So $\mathcal{A} \neq \emptyset$.

We prove the theorem in several steps. At first, the existence of common fixed point will be proved. Let $x_0 \in X$ be an arbitrary element. We construct the sequence $\{x_n\}_{n\ge 0}$ recursively as

$$x_{2n+1} = Tx_{2n},$$

and

$$x_{2n+2} = Sx_{2n+1}.$$

If there exists $k_0 \in \mathbb{N}$ such that $x_{k_0} = x_{k_0+1}$, then we claim that $x_k = x_{k_0}$ for all $k \ge k_0$. To see this, suppose that $k_0 = 2n$ for some $n \in \mathbb{N}$. In this case, we have $x_{2n} = x_{2n+1}$. Now, if $m(x_{2n},x_{2n+1}) = 0$ then by the definition of $m(x,y)$, we have $x_{2n+1} = x_{2n+2}$. So, we can suppose that $m(x_{2n},x_{2n+1}) \neq 0$. Furthermore, we have

$$\begin{aligned}
&\frac{1}{2}\min\{d(x_{2n},Tx_{2n}),d(x_{2n+1},Sx_{2n+1})\}\\
&\qquad= \frac{1}{2}\min\{d(x_{2n},x_{2n+1}),d(x_{2n+1},x_{2n+2})\}\\
&\qquad\le d(x_{2n},x_{2n+1}),
\end{aligned}$$

thus from (2.1) and ($\eta 1$) we have

$$\psi(d(x_{2n+1},x_{2n+2})) < \psi(m(x_{2n},x_{2n+1})),$$

where, $\psi \in \Psi([0,+\infty))$. So, since ψ is a nondecreasing function, we have

$$d(x_{2n+1},x_{2n+2}) < m(x_{2n},x_{2n+1}).$$

But

$$\begin{aligned}
m(x_{2n},x_{2n+1}) = \max\Big\{&d(x_{2n},x_{2n+1}),d(x_{2n},Tx_{2n}),d(x_{2n+1},Sx_{2n+1})\\
&,\frac{d(x_{2n},Sx_{2n+1})+d(x_{2n+1},Tx_{2n})}{2}\Big\}
\end{aligned}$$

$$= \max\left\{d(x_{2n+1}, x_{2n+2}), \frac{d(x_{2n}, x_{2n+2})}{2}\right\}$$
$$= d(x_{2n+1}, x_{2n+2}),$$

which contradiction. So $d(x_{2n+1}, x_{2n+2}) = 0$ or $m(x_{2n}, x_{2n+1}) = 0$ i.e. $x_{2n+1} = x_{2n+2}$. Hence $x_{k_0} = x_{k_0+1} = x_{k_0+2}$. Similarly, if $k_0 = 2n + 1$ for some $n \geq 0$, we can prove that $x_{k_0} = x_{k_0+1} = x_{k_0+2}$. Therefore, x_{k_0} is a common fixed point of T and S. So, we can suppose that, for all $n \geq 0$, $d(x_n, x_{n+1}) > 0$ and $m(x_{2n}, x_{2n+1}) \neq 0$.

For convenience, we divide the rest of the proof into four steps.

Step (1): We prove that $\lim_{k\to\infty} d(x_k, x_{k+1}) = 0$.

To prove it, at first we claim that

$$\begin{aligned} d(x_{k+1}, x_{k+2}) &\leq m(x_k, x_{k+1}) \\ &= d(x_k, x_{k+1}), \quad \forall k \in \mathbb{N}. \end{aligned} \tag{2.2}$$

To see this, suppose that $k = 2n$ for some $n \in \mathbb{N}$. We have

$$\begin{aligned} &\frac{1}{2}\min\{d(x_{2n}, Tx_{2n}), d(x_{2n+1}, Sx_{2n+1})\} \\ &\qquad = \frac{1}{2}\min\{d(x_{2n}, x_{2n+1}), d(x_{2n+1}, x_{2n+2})\} \\ &\qquad \leq d(x_{2n}, x_{2n+1}). \end{aligned}$$

So from (2.1) and (η1) we have:

$$\begin{aligned} \psi(d(x_{2n+2}, x_{2n+1})) &= \psi(d(Sx_{2n+1}, Tx_{2n})) \\ &< \psi(m(x_{2n}, x_{2n+1})). \end{aligned}$$

So, we have

$$d(x_{2n+1}, x_{2n+2}) < m(x_{2n}, x_{2n+1}). \tag{2.3}$$

On the other hand,

$$\begin{aligned} m(x_{2n}, x_{2n+1}) &= \max\Big\{d(x_{2n}, x_{2n+1}), d(x_{2n}, Tx_{2n}) \\ &\qquad , d(x_{2n+1}, Sx_{2n+1}), \frac{d(x_{2n}, Sx_{2n+1}) + d(x_{2n+1}, Tx_{2n})}{2}\Big\} \\ &= \max\left\{d(x_{2n}, x_{2n+1}), d(x_{2n+1}, x_{2n+2}), \frac{d(x_{2n}, x_{2n+2})}{2}\right\} \\ &\leq \max\Big\{d(x_{2n}, x_{2n+1}), d(x_{2n+1}, x_{2n+2}) \\ &\qquad , \frac{d(x_{2n}, x_{2n+1}) + d(x_{2n+1}, x_{2n+2})}{2}\Big\} \\ &\leq \max\{d(x_{2n}, x_{2n+1}), d(x_{2n+1}, x_{2n+2})\}. \end{aligned}$$

So, if $d(x_{2n_0+1}, x_{2n_0+2}) \geq d(x_{2n_0}, x_{2n_0+1})$ for some $n_0 \in \mathbb{N}$, then

$$m(x_{2n_0}, x_{2n+1}) \leq d(x_{2n_0+1}, x_{2n_0+2}),$$

which contradicts (2.3). Hence, for each $n \in \mathbb{N}$,

$$d(x_{2n+1}, x_{2n+2}) < d(x_{2n}, x_{2n+1}),$$

and so

$$m(x_{2n}, x_{2n+1}) \leq d(x_{2n}, x_{2n+1}).$$

Consequently, (2.2) is proved when $k \geq 0$ is an even number. By the same argument, one can verify that (2.2) holds when k is an odd number. Thus, the sequence $\{d(x_n, x_{n+1})\}_{n\geq 1}$ is non increasing and bounded below, so it converges to a real number $\gamma \geq 0$. Hence

$$\begin{aligned} \lim_{n\to\infty} d(x_n, x_{n+1}) &= \lim_{n\to\infty} m(x_n, x_{n+1}) \\ &= \gamma. \end{aligned} \tag{2.4}$$

We claim that $\gamma = 0$. To prove the claim, at first suppose that

$$\Omega = \{(d(Tx, Sy), m(x, y)) : (x, y) \in \mathcal{A}\}.$$

By (2.1) and definition of $\mathcal{A}$, one can easily see that

$$\eta(t, s) \geq 0, \quad \forall (t, s) \in \Omega \setminus \{(0, 0)\}. \tag{2.5}$$

For each $n \geq 0$ we have

$$\begin{aligned} &\frac{1}{2} \min \{d(x_{2n}, Tx_{2n}), d(x_{2n+1}, Sx_{2n+1})\} \\ &\qquad = \frac{1}{2} \min \{d(x_{2n}, x_{2n+1}), d(x_{2n+1}, x_{2n+2})\} \\ &\qquad \leq d(x_{2n}, x_{2n+1}). \end{aligned}$$

Thus $(x_{2n}, x_{2n+1}) \in \mathcal{A}$ for each $n \geq 0$. Consequently, (2.5) implies that

$$\eta\left(d(Tx_{2n}, Sx_{2n+1}), m(x_{2n}, x_{2n+1})\right) \geq 0.$$

So

$$\limsup_{n\to\infty} \eta\left(d(x_{2n+1}, x_{2n+2}), m(x_{2n}, x_{2n+1})\right) \geq 0. \tag{2.6}$$

Now, in contrary, suppose that $\gamma > 0$. From (2.4) we have

$$\begin{aligned} \lim_{n\to\infty} d(x_{2n+1}, x_{2n+2}) &= \lim_{n\to\infty} m(x_{2n}, x_{2n+1}) \\ &= \gamma \\ &> 0. \end{aligned}$$

Therefore, from $(\eta 2)$

$$\limsup_{n\to\infty} \eta\left(d(x_{2n+1}, x_{2n+2}), m(x_{2n}, x_{2n+1})\right) < 0,$$

which contradicts (2.6). So the claim is proved and we obtain that

$$\lim_{n\to\infty} d(x_n, x_{n+1}) \le \lim_{n\to\infty} m(x_n, x_{n+1}) = \lim_{n\to\infty} m(x_n, x_{n+1}). \tag{2.7}$$

Step (2): $\{x_n\}$ is a Cauchy sequence.
To show that $\{x_n\}$ is a Cauchy sequence, because of (2.7), it is enough to show that the subsequence $\{x_{2n}\}$ is a Cauchy sequence.

On contrary, suppose that $\{x_{2n}\}$ is not a Cauchy sequence. Then by Lemma (1.8) there exist $\epsilon_0 > 0$ and subsequences $\{x_{2m_k}\}$ and $\{x_{2n_k}\}$ of $\{x_n\}$ such that n_k is the smallest index for which $n_k > m_k > k$ and $d(x_{2m_k}, x_{2n_k)}) \ge \epsilon_0$ and

(l_1) $\lim_{k\to\infty} d(x_{2m_k}, x_{2n_k}) = \epsilon_0$,
(l_2) $\lim_{k\to\infty} d(x_{2m_k-1}, x_{2n_k}) = \epsilon_0$,
(l_3) $\lim_{k\to\infty} d(x_{2m_k}, x_{2n_k+1}) = \epsilon_0$,
(l_4) $\lim_{k\to\infty} d(x_{2m_k-1}, x_{2n_k+1}) = \epsilon_0$.

Therefore, from the definition of $m(x, y)$ we have:

$$\begin{aligned}\lim_{k\to\infty} m(x_{2n_k}, x_{2m_k-1}) = \lim_{k\to\infty} \max\Big\{ & d(x_{2n_k}, x_{2m_k-1}), d(x_{2n_k}, x_{2n_k+1}) \\ & , d(x_{2m_k-1}, x_{2m_k}) \\ & , \frac{d(x_{2n_k}, x_{2m_k}) + d(x_{2m_k-1}, x_{2n_k+1})}{2} \Big\} \\ = \max & \left\{ \epsilon_0, 0, 0, \frac{\epsilon_0 + \epsilon_0}{2} \right\} \\ = \epsilon_0 & .\end{aligned}$$

So

$$\begin{aligned}\lim_{k\to\infty} d(x_{2m_k}, x_{2n_k+1}) &= \lim_{k\to\infty} m(x_{2m_k-1, x_{2n_k}}) \\ &= \epsilon_0 \\ &> 0.\end{aligned}$$

Hence, ($\eta 2$) implies that

$$\limsup_{n\to\infty} \eta(d(x_{2m_k}, x_{2n_k+1}), m(x_{2m_k-1, x_{2n_k}})) < 0. \tag{2.8}$$

On the other hand, we claim that for sufficiently large $k \in \mathbb{N}$,
if $n_k > m_k > k$, then

$$\frac{1}{2} \min\{d(x_{2n_k}, Tx_{2n_k}), d(x_{2m_k-1}, Sx_{2m_k-1})\} \le d(x_{2n_k}, x_{2m_k-1}). \tag{2.9}$$

Indeed, since $n_k > m_k$ and $\{d(x_n, x_{n+1})\}$ is non-increasing we have

$$d(x_{2n_k+1}, Tx_{2n_k}) = d(x_{2n_k}, x_{2n_k+1})$$

$$\begin{aligned}&\leq d(x_{2m_k+1}, x_{2m_k})\\&\leq d(x_{2m_k}, x_{2m_k-1})\\&= d(x_{2m_k-1}, Sx_{2m_k-1}).\end{aligned}$$

And so, the left hand side of inequality (2.9) is equal to

$$\frac{1}{2}d(x_{2n_k}, Tx_{2n_k}) = \frac{1}{2}d(x_{2n_k}, x_{2n_k+1}).$$

Therefore, we must show that, for sufficiently large $k \in \mathbb{N}$, if $n_k > m_k > k$, then

$$d(x_{2n_k}, x_{2n_k+1}) \leq d(x_{2n_k}, x_{2m_k-1}).$$

According to (2.7), there exists $k_1 \in \mathbb{N}$ such that for any $k > k_1$,

$$d(x_{2n_k}, x_{2n_k+1}) < \frac{1}{2}\epsilon_0.$$

Also, there exists $k_2 \in \mathbb{N}$ such that for any $k > k_2$,

$$d(x_{2m_k-1}, x_{2m_k}) < \frac{1}{2}\epsilon_0.$$

Hence, for any $k > \max\{k_1, k_2\}$ and $n_k > m_k > k$, we have

$$\begin{aligned}\epsilon_0 &\leq d(x_{2n_k}, x_{2m_k})\\&\leq d(x_{2n_k}, x_{2m_k-1}) + d(x_{2m_k-1}, x_{2m_k})\\&\leq d(x_{2n_k}, x_{2m_k-1}) + \frac{\epsilon_0}{2}.\end{aligned}$$

So, one concludes that

$$\frac{\epsilon_0}{2} \leq d(x_{2n_k}, x_{2m_k-1}).$$

Thus we obtain that for any $k > \max\{k_1, k_2\}$ and $n_k > m_k > k$,

$$\begin{aligned}d(x_{2n_k}, x_{2n_k+1}) &\leq \frac{\epsilon_0}{2}\\&\leq d(x_{2n_k}, x_{2m_k-1}).\end{aligned}$$

So (2.9) is proved. Therefore, by (2.1) and definition of $\mathcal{A}$, for sufficiently large $k \in \mathbb{N}$, if $n_k > m_k > k$, then $(x_{2n_k}, x_{2m_k-1}) \in \mathcal{A}$. Consequently, for sufficiently large $k \in \mathbb{N}$, if $n_k > m_k > k$ then

$$\eta\left(d(Tx_{2n_k}, Sx_{2m_k-1}), m(x_{2n_k}, x_{2m_k-1})\right) \geq 0.$$

So

$$\limsup_{k\to\infty} \eta(d(x_{2n_k+1}, x_{2m_k}), m(x_{2n_k}, x_{2m_k-1})) \geq 0, \tag{2.10}$$

which contradicts (2.8). So $\{x_n\}$ is a Cauchy sequence and since X is complete, there exists $u \in X$ such that $x_n \to u$ as $n \to \infty$.

Step (3): u is a common fixed point of T and S.
Without loosing of generality, one can suppose that $d(x_n, u) \neq 0$ for each $n \geq 0$. In fact, if $x_{n_k} = u$ for some subsequence $\{x_{n_k}\}_{k\geq 0}$ of $\{x_n\}_{n\geq 0}$, then $x_n = u$ for any $n \geq n_0$. So u will be a common fixed point of T and S.

Now we prove that

$$\lim_{n\to\infty} m(u, x_{2n}) = d(Su, u). \tag{2.11}$$

For notice that

$$\begin{aligned} d(u, Su) &\leq m(x_{2n}, u) \\ &= \max\Big\{ d(x_{2n}, u), d(x_{2n}, x_{2n+1}), d(u, Su) \\ &\quad , \frac{d(x_{2n}, Su) + d(u, x_{2n+1})}{2} \Big\}. \end{aligned}$$

Letting $n \to \infty$ we obtain that

$$\begin{aligned} d(u, Su) &\leq \lim_{n\to\infty} m(u, x_{2n}) \\ &\leq \max\left\{ 0, 0, d(u, Su), \frac{d(u, Su) + 0}{2} \right\} \\ &= d(u, Su). \end{aligned}$$

Hence

$$\lim_{n\to\infty} m(u, x_{2n}) = d(Su, u).$$

This completes the proof of (2.11). In the same manner, one can show that

$$\lim_{n\to\infty} m(u, x_{2n+1}) = d(Tu, u). \tag{2.12}$$

Now, we claim that for each $n \geq 0$, at least one of the following inequalities is true:

$$\frac{1}{2} d(x_{2n}, x_{2n+1}) \leq d(x_{2n}, u), \tag{2.13}$$

or

$$\frac{1}{2} d(x_{2n+1}, x_{2n+2}) \leq d(x_{2n}, u). \tag{2.14}$$

In opposite, if for some $n_0 \geq 0$, both of them are false then we get

$$\begin{aligned} d(x_{2n_0}, x_{2n_0+1}) &\leq d(x_{2n_0}, u) + d(u, x_{2n_0+1}) \\ &< \frac{1}{2} d(x_{2n_0}, x_{2n_0+1}) + \frac{1}{2} d(x_{2n_0+1}, x_{2n_0+2}) \\ &\leq \frac{1}{2} d(x_{2n_0}, x_{2n_0+1}) + \frac{1}{2} d(x_{2n_0}, x_{2n_0+1}) \end{aligned}$$

$$= d(x_{2n_0}, x_{2n_0+1}),$$

which is a contradiction and the claim is proved. So, one can consider the following two cases:

Case (1): The relation (2.13) is established for infinitely many $n \geq 0$. In this case, for infinitely many $n \geq 0$ we have

$$\begin{aligned}\frac{1}{2}\min\{d(x_{2n}, Tx_{2n}), d(u, Su)\} &= \frac{1}{2}\min\{d(x_{2n}, x_{2n+1}), d(u, Su)\}\\ &\leq \frac{1}{2}d(x_{2n}, x_{2n+1})\\ &\leq d(x_{2n}, u).\end{aligned}$$

Therefore, $(x_{2n}, u) \in \mathcal{A}$. Thus

$$(d(Tx_{2n}, Su), m(x_{2n}, u)) \in \Omega \diagdown \{(0,0)\}.$$

Consequently, from (2.5), it is seen that for infinitely many $n \geq 0$,

$$\eta(d(Tx_{2n}, Su), m(x_{2n}, u)) \geq 0.$$

Therefore,

$$\limsup_{k\to\infty} \eta(d(x_{2n_k+1}, x_{2m_k}), m(x_{2n_k}, x_{2m_k-1})) \geq 0. \tag{2.15}$$

Now, we assert that $d(Su, u) = 0$. Otherwise, suppose that $d(Su, u) \neq 0$. Then, since

$$\begin{aligned}\lim_{n\to\infty} d(Tx_{2n}, Su) &= \lim_{n\to\infty} m(u, x_{2n})\\ &= d(u, Su)\\ &> 0,\end{aligned}$$

from $(\eta 2)$ we have

$$\limsup_{k\to\infty} \eta(d(x_{2n_k+1}, x_{2m_k}), m(x_{2n_k}, x_{2m_k-1})) < 0,$$

which contradicts (2.15). So $d(u, Su) = 0$, i.e. $Su = u$.

On the other hand, we have

$$\begin{aligned}m(u,u) &= \max\left\{d(u,u), d(u,Tu), d(u,Su), \frac{d(u,Su)+d(u,Tu)}{2}\right\}\\ &= \max\left\{0, d(u,Tu), 0, \frac{d(u,Tu)}{2}\right\}\\ &= d(u,Tu).\end{aligned}$$

So,

$$m(u,u) = d(u,Tu). \tag{2.16}$$

Furthermore,

$$\begin{aligned}\frac{1}{2}\min\{d(u,Tu),d(u,Su)\} &= \frac{1}{2}\min\{d(u,Tu),0\}\\ &= 0\\ &\leq d(u,u).\end{aligned}$$

Thus, if $d(Tu,u) > 0$, then (2.1) implies that $(u,u) \in \mathcal{A}$. Thus

$$(d(Tu,Su),m(u,u)) \in \Omega \diagdown \{(0,0)\}.$$

Consequently, it follows from (2.5), that

$$\eta(d(Tu,Su),m(u,u)) \geq 0.$$

So from (η1) one can observe that

$$d(Tu,Su) < m(u,u),$$

which contradicts (2.16). Hence $d(Tu,u) = 0$. i.e. $Tu = u$. Hence we obtain that $Tu = Su = u$.

Case (2): The relation (2.13) is established only for finitely many $n \geq 0$.

In this case, there exists $n_0 \geq 0$ such that (2.14) is true for any $n \geq n_0$. Similar to Case (1), one can prove that, (2.14) leads to a contradiction unless $Su = Tu = u$. So in any case, u is a common fixed point of T and S, and the proof is completed.

Step (4): The common fixed point of T and S is unique.
Suppose that u and v are two common fixed points of T and S. We have

$$\begin{aligned}\frac{1}{2}\min\{d(u,Tu),d(u,Su)\} &= \frac{1}{2}\min\{d(u,Tu),0\}\\ &= 0\\ &= d(u,u).\end{aligned}$$

In opposite, if $d(u,v) \neq 0$ then $m(u,v) \neq 0$ and so $(u,v) \in \mathcal{A}$. Thus

$$(d(Tu,Sv),m(u,v)) \in \Omega \diagdown \{(0,0)\}.$$

Consequently, from (2.5), it is seen that

$$\eta(d(Tu,Sv),m(u,v)) \geq 0.$$

So from (η1), one can conclude that

$$d(Tu,Sv) < m(u,v).$$

But

$$\begin{aligned}m(u,v) &= \max\left\{d(u,v),d(u,Tu),d(v,Sv),\frac{d(u,Sv)+d(v,Tu)}{2}\right\}\\ &= d(u,v),\end{aligned}$$

and it is a contradiction. So $d(u,v)=0$ and the proof of theorem is completed. □

The following corollary is an immediate consequence of Theorem 2.6:

Corollary 2.7. *Let (X,d) be a complete metric space and T and S be two self-maps on X such that for all $x,y\in X$,*

$$\eta(d(Tx,Sy),m(x,y))\geq 0, \tag{2.17}$$

where $\eta\in\mathcal{Z}_\Psi$ and $m(x,y)$ is defined as in Theorem 1.7. Then T and S have a unique common fixed point.

Putting $S=T$ in Theorem 2.6 we obtain:

Corollary 2.8. *Let (X,d) be a complete metric space and T be a self-map on X. If there exists $\eta\in\mathcal{Z}_\Psi$ such that for all $x,y\in X$,*
$\frac{1}{2}d(x,Tx)\leq d(x,y)$ implies that

$$\eta(d(Tx,Ty),m_T(x,y))\geq 0,$$

where

$$m_T(x,y)=\max\left\{d(x,y),d(x,Tx),d(y,Ty),\frac{d(x,Ty)+d(y,Tx)}{2}\right\},$$

then T has a unique fixed point.

Theorem 1.5 is an obvious result of Corollary 2.8.

Corollary 2.9 (Theorem 1.5)**.** *Let (X,d) be a complete metric space and T be a self-map on X. If there exists $\eta\in\mathcal{Z}_\Psi$ such that for all $x,y\in X$,*

$$\eta(d(Tx,Ty),m_T(x,y))\geq 0,$$

then T has a unique fixed point.

Corollary 2.10. *Let (X,d) be a complete metric space and T and S be two self-maps on X such that for every $x,y\in X$,*
$\frac{1}{2}\min\{d(x,Tx),d(y,Sy)\}\leq d(x,y)$ implies that

$$\psi(d(Tx,Sy))\leq\psi(m(x,y))-\phi(m(x,y)), \tag{2.18}$$

where ψ and $m(x,y)$ are defined as in Theorem 1.6, and $\varphi:[0,+\infty)\to[0,+\infty)$ is a function such that $\liminf_{s\to t}\varphi(s)>0$ for each $t>0$, and $\varphi(t)=0$ if and only if $t=0$.

Then T and S have a unique common fixed point.

Proof. Define $\eta(t,s) = \psi(s) - \phi(s) - \psi(t)$. Then η is a Ψ-simulation function. Indeed, $\psi \in \Psi([0,+\infty))$ and $(\eta 1)$ is clearly hold. On the other hand, if $\{t_n\}, \{s_n\}$ are two sequences in $(0,\infty)$ such that

$$\begin{aligned}\lim_{n\to\infty} t_n &= \lim_{n\to\infty} s_n\\ &= \ell\\ &> 0,\end{aligned}$$

then, we have

$$\begin{aligned}\limsup_{n\to\infty} \eta(t_n,s_n) &= \limsup_{n\to\infty}(\psi(s_n) - \phi(s_n) - \psi(t_n))\\ &\le 0 - \liminf_{n\to\infty} \phi(s_n)\\ &< 0.\end{aligned}$$

So $(\eta 2)$ holds and one can apply Theorem 2.6 to complete the proof. □

Here, the conjecture which is proposed by Singh et al. in [21], will be proved.

Corollary 2.11 (Theorem 1.7)**.** *Let (X,d) be a complete metric space and T and S be two self-maps on X such that for every $x, y \in X$, $\frac{1}{2}\min\{d(x,Tx), d(y,Sy)\} \le d(x,y)$ implies that*

$$\psi(d(Tx,Sy)) \le \psi(m(x,y)) - \phi(m(x,y)),$$

where ψ, φ are defined as in Theorem 1.6 and

$$m(x,y) = \max\left\{d(x,y), d(x,Tx), d(y,Sy), \frac{d(x,Sy)+d(y,Tx)}{2}\right\}.$$

Then T and S have a unique common fixed point.

Proof. Since ϕ is lower semi-continuous, if

$$\begin{aligned}\lim_{n\to\infty} s_n &= \ell\\ &> 0,\end{aligned}$$

then

$$\begin{aligned}\liminf_{n\to\infty} \varphi(s_n) &\ge \varphi(\ell)\\ &> 0.\end{aligned}$$

So, one can apply Corollary 2.10. □

Theorem 1.6 is an obvious result of Corollary 2.11.

Corollary 2.12. *Let (X,d) be a complete metric space and T and S be two self-maps on X such that for every $x,y \in X$,*
$\frac{1}{2}\min\{d(x,Tx),d(y,Sy)\} \leq d(x,y)$ *implies that*

$$\psi(d(Tx,Sy)) \leq \rho(m(x,y))\psi(m(x,y)), \tag{2.19}$$

where $\psi \in \Psi$ and $m(x,y)$ is defined as in Corollary 2.11 and $\rho : [0,+\infty) \to [0,1)$ is a function such that $\rho(t) = 0$ if and only if $t = 0$ and $\limsup_{t\to s}\rho(t) < 1$ for each $s > 0$. Then T and S have a unique common fixed point.

Proof. Take $\eta(t,s) = \rho(s)\psi(s) - \psi(t)$. One can easily show that η is a Ψ -simulation function. Now the corollary follows from Theorem 2.6. □

Corollary 2.13. *Let (X,d) be a complete metric space and T and S be two maps on X such that for every $x,y \in X$, $\frac{1}{2}\min\{d(x,Tx),d(y,Sy)\} \leq d(x,y)$ implies that*

$$\psi(d(Tx,Sy)) + \phi(d(Tx,Sy)) \leq \psi(m(x,y)),$$

where $\psi \in \Psi$ and $m(x,y)$ is defined as in Corollary 2.11 and $\varphi : [0,+\infty) \to [0,+\infty)$ is a function such that $\liminf_{s\to t}\varphi(s) > 0$ for each $t > 0$, and $\varphi(t) = 0$ if and only if $t = 0$.

Then T and S have a unique common fixed point.

Proof. Take $\eta(t,s) = \psi(s) - \phi(s) - \psi(t)$. Then similar to the proof of Corollary 2.10, one can see that η is a Ψ-simulation function. So by applying Theorem 2.6 the proof will be completed. □

Corollary 2.14. *Let (X,d) be a complete metric space and T and S be two self-maps on X such that for every $x,y \in X$,*
$\frac{1}{2}\min\{d(x,Tx),d(y,Sy)\} \leq d(x,y)$ *implies that*

$$\psi(d(Tx,Sy)) \leq \phi(\psi(m(x,y))),$$

where $\psi \in \Psi$ and $m(x,y)$ is defined as in Corollary 2.11 and
$\varphi : [0,+\infty) \to [0,+\infty)$ is a function such that for each $t > 0$, $\varphi(t) < t$ and $\limsup_{s\to t}\varphi(s) < t$ and $\varphi(t) = 0$ if and only if $t = 0$.

Then T and S have a unique common fixed point.

Proof. Define $\eta(t,s) = \phi(\psi(s)) - \psi(t)$. Then $(\eta 1)$ is holds. On the other hand, if $\{t_n\},\{s_n\}$ are two sequences in $(0,\infty)$ such that

$$\begin{aligned}\lim_{n\to\infty} t_n &= \lim_{n\to\infty} s_n \\ &= \ell \\ &> 0,\end{aligned}$$

then, the continuity of ψ and the properties of ϕ show that

$$\begin{aligned}\limsup_{n\to\infty}\eta(t_n,s_n) &= \limsup_{n\to\infty}\phi(\psi(s_n)) - \lim_{n\to\infty}\psi(t_n)\\ &< \psi(\ell)-\psi(\ell)\\ &= 0.\end{aligned}$$

So $(\eta 2)$ holds and one can apply Theorem 2.6 to complete the proof. □

Remark 2.15. Suppose that $\psi \in \Psi([0,+\infty))$ and $\varphi : [0,+\infty) \to [0,+\infty)$ is an upper semi-continuous function such that $\varphi(t) < t$ for each $t > 0$ and $\varphi(t) = 0$ if and only if $t = 0$. Then for any sequence $\{s_n\}$ in $(0,\infty)$ with $\lim\limits_{n\to\infty} s_n = \ell > 0$, one can obtain that

$$\limsup_{n\to\infty}\phi(\psi(s_n)) < \psi(\ell).$$

So, by applying the same argument as in Corollary 2.14, one can prove the following corollary:

Corollary 2.16. *Let (X,d) be a complete metric space and T and S be two self-maps on X such that for every $x,y \in X$, $\frac{1}{2}\min\{d(x,Tx),d(y,Sy)\} \le d(x,y)$ implies that*

$$\psi(d(Tx,Sy)) \le \phi(\psi(m(x,y))),$$

where $\psi \in \Psi$ and $m(x,y)$ is defined as in Corollary 2.11 and $\varphi : [0,+\infty) \to [0,+\infty)$ is an upper semi-continuous function such that $\varphi(t) < t$ for each $t > 0$, and $\varphi(t) = 0$ if and only if $t = 0$.

Then T and S have a unique common fixed point.

The following example shows that Theorem 2.6 is a genuine generalization of the Corollary 2.7.

Example 2.17. Let $X = \{(0,0),(0,5),(5,0),(5,6)\}$ be endowed with the metric d defined by

$$d((x_1,x_2),(y_1,y_2)) = |x_1-y_1| + |x_2-y_2|.$$

Let T and S be two self-mappings on X as follows:

$$T(x_1,x_2) = (\min\{x_1,x_2\},0),$$

and

$$S(x_1,x_2) = (0,\min\{x_1,x_2\}).$$

For any $\eta \in \mathcal{Z}_\Psi$, the mappings T and S do not satisfy the condition (2.17) of Corollary 2.7 at $x = y = (5,6)$. However, by Choosing $\eta(t,s) = \frac{11s}{12} - t$, it is readily verified that η is a Ψ-simulation function where ψ is the identity function on $[0,\infty)$ and all the hypothesis of Theorem 2.6 are verified.

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[1] Department of Mathematics, Faculty of Science, Science and Research Branch, Islamic Azad University, Tehran, Iran.
E-mail address: ghr.hj1350@gmail.com

[2] Department of Mathematics, Faculty of Science, Razi University, Kermanshah 67149, Iran.
E-mail address: a.farajzadeh@razi.ac.ir

[3] Department of Mathematics, Faculty of Science, Science and Research Branch, Islamic Azad University, Tehran, Iran.
E-mail address: mahdi.azhini@gmail.com

[4] Department of Mathematics, Faculty of Science, Arak Branch, Islamic Azad University, Arak, Iran.
E-mail address: f-khojaste@iau-arak.ac.ir