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Diameter Approximate Best Proximity Pair in Fuzzy Normed Spaces

Seyed Ali Mohammad Mohsenialhosseini[1] and Morteza Saheli[2*]

Abstract. The main purpose of this paper is to study the approximate best proximity pair of cyclic maps and their diameter in fuzzy normed spaces defined by Bag and Samanta. First, approximate best point proximity points on fuzzy normed linear spaces are defined and four general lemmas are given regarding approximate fixed point and approximate best proximity pair of cyclic maps on fuzzy normed spaces. Using these results, we prove theorems for various types of well-known generalized contractions in fuzzy normed spaces. Also, we apply our results to get an application of approximate fixed point and approximate best proximity pair theorem of their diameter.

1. Introduction

Katsaras [14], who while studying fuzzy topological vector spaces, was the first to introduce in 1984 the idea of fuzzy norm on a linear space. In 1992, Felbin [11] defined a fuzzy norm on a linear space with an associated metric of the Kaleva and Seikkala type [12]. A further development along this line of inquiry took place when in 1994, Cheng and Mordeson [8] evolved the definition of a further type of fuzzy norm having a corresponding metric of the Kramosil and Michalek type [15].

Chitra and Mordeson [9] introduced the definition of fuzzy norm and thereafter the concept of fuzzy normed space has been introduced and generalized in different ways by Bag and Samanta in [1–3]. Best approximation has important applications in diverse disciplines of mathematics, engineering and economics in dealing with problems arising in:

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* Corresponding author.

Fixed point theory, Approximation theory, game theory, mathematical economics, best proximity pairs, Equilibrium pairs, etc. Furthermore, the fixed point theory in this kind of spaces has been intensively studied. For details, one can refer to, Bag and Samanta, Mohsenalhosseini et al. [3, 18, 19]. Many authors have studied best approximation and best proximity pair in the both metric and fuzzy metric spaces. In 2011, Mohsenalhosseini et al. [18], introduced the approximate best proximity pairs in metric spaces. In 2013, Mohsenalhosseini et al. [17], introduced the approximate best proximity pairs in metric space for contraction maps.
In this paper, starting from the article of Mohsenalhosseini [17, 18] and MăDălina Berinde [4], we study well-known types of operators on fuzzy normed spaces, and we give some qualitative and quantitative results regarding approximate best proximity pair and approximate fixed point for cyclic map $T : A \cup B \to A \cup B$ i.e. $T(A) \subseteq B$ and $T(B) \subseteq A$, and by using this result, we can prove some diameter approximate best proximity pair theorems in fuzzy normed spaces.

2. Some Preliminary Results

We begin by recalling some needed definitions and results.

Definition 2.1 ([2])**.** Let X be a linear space over R (real number) and N be a fuzzy subset of $X \times R$ such that for all $x, y \in X$ and $c \in R$:

- $(N1)$ $N(x,t) = 0$ for all $t \leq 0$,
- $(N2)$ $x = 0$ if and only if $N(x,t) = 1$ for all $t > 0$,
- $(N3)$ If $c \neq 0$ then $N(cx,t) = N(x, t/|c|)$ for all $t \in R$,
- $(N4($) $N(x+y, s+t) \geq \min\{N(x,s), N(y,t)\}$ for all $s, t \in R$,
- $)N5$ $N(x, .)$ is a nondecreasing function of R and $\lim_{t\to\infty} N(x,t) = 1$.

Then N is called a fuzzy norm on X. The pair (X, N) will be referred to as a fuzzy normed linear space.

We assume that

- $(N6)$ $N(x,t) > 0$, for all $t > 0$ implies $x = 0$,
- $(N7)$ For $x \neq 0$, $N(x, .)$ is a continuous function of R and strictly increasing on the subset $\{t : 0 < N(x,t) < 1\}$ of R.

Definition 2.2 ([2])**.** Let (X, N) be a fuzzy normed linear space.

- (i) A sequence $\{x_n\} \subseteq X$ is said to converge to $x \in X$ ($\lim_{n\to\infty} x_n = x$), if $\lim_{n\to\infty} N(x_n - x, t) = 1$, for all $t > 0$.
- (ii) A sequence $\{x_n\} \subseteq X$ is called Cauchy, if $\lim_{n,m\to\infty} N(x_n - x_m, t) = 1$, for all $t > 0$.

Theorem 2.3 ([2]). *Let (X,N) be a fuzzy normed linear space satisfying (N6). Define $\|x\|_\alpha = \inf\{t : N(x,t) \geq \alpha\}$, for all $x \in X$ and all $\alpha \in (0,1)$. Then $\{\|.\|_\alpha : \alpha \in (0,1)\}$ is an ascending family of norms on X. We call these norms as $\alpha-$norm on X corresponding to the fuzzy norm N on X.*

Lemma 2.4 ([1]). *Let (X,N) be a fuzzy normed space satisfying (N6) and (N7). Let $\|x\|_\alpha = \inf\{t : N(x,t) \geq \alpha\}$, for all $\alpha \in (0,1)$ and $N' : X \times \mathbb{R} \longrightarrow [0,1]$ be a function defined by*

$$N'(x,t) = \begin{cases} \sup\{\alpha \in (0,1) : \|x\|_\alpha \leq t\}, & (x,t) \neq (0,0), \\ 0, & (x,t) = (0,0). \end{cases}$$

Then

(i) *N' is a fuzzy norm on X,*
(ii) *$N = N'$.*

Lemma 2.5 ([1]). *Let (X,N) be a fuzzy normed space satisfying (N6) and (N7) and $\{x_n\}$ be a sequence in X. Then $\lim_{n\to\infty} N(x_n - x, t) = 1$, for all $t > 0$ if and only if $\lim_{n\to\infty} \|x_n - x\|_\alpha = 0$, for all $\alpha \in (0,1)$.*

Definition 2.6 ([16]). Let (X,N) be a fuzzy normed space, $\epsilon > 0$ and $x_0 \in X$. Let $T : X \to X$ be a function. Then x_0 is called an F^z_ϵ-approximate fixed point (fuzzy approximate fixed point) of T if there exists $\alpha \in (0,1)$ such that $\inf\{t > 0 : N(x_0 - Tx_0, t) \geq \alpha\} \leq \epsilon$. We denote the set of all F^z_ϵ-approximate fixed points of T by $F^z_\epsilon(T)$.

Definition 2.7 ([16]). Let (X,N) be a fuzzy normed space. A function $T : X \to X$ is said to have the fuzzy approximate fixed point property (f.a.f.p.p.) if $F^z_\epsilon(T) \neq \emptyset$, for all $\epsilon > 0$.

3. Fuzzy Approximate Best Proximity Pairs

In this section, we begin with two lemmas which will be used in order to prove all the results given in the same section.

Definition 3.1. Let (X,N) be a fuzzy normed space. Furthermore, let A, B be nonempty subsets of X and $t > 0$. We define

$$N(A-B,t) = \sup\{N(x-y,t) : \ x \in A, \ y \in B\}.$$

Definition 3.2. Let (X,N) be a fuzzy normed space, $\epsilon > 0$, $\alpha \in (0,1)$ and $(\emptyset \neq) A, B \subseteq X$. Moreover, let $T : A \cup B \to A \cup B$ be a function such that $T(A) \subseteq B$ and $T(B) \subseteq A$. A point $x_0 \in A \cup B$ is called an $F^z_{\alpha,\epsilon}$-approximate best proximity point of the pair (A,B), if

$$N(A-B,t) \geq \alpha \text{ implies that } N(x_0 - Tx_0, t+\epsilon) \geq \alpha, \text{ for all } t > 0.$$

We denote the set of all $F^z_{\alpha,\epsilon}$-approximate best proximity point of the pair (A,B) by $F^z_{\alpha,\epsilon}(A,B)$.

The following result guarantees the existence of approximate best proximity pairs for a function on a fuzzy normed linear space X.

Lemma 3.3. *Let (X,N) be a fuzzy normed space, $\epsilon>0$, $\alpha\in(0,1)$ and $(\emptyset\neq)\,A,B\subseteq X$. Moreover, let $T:A\cup B\to A\cup B$ be a function such that $T(A)\subseteq B$, $T(B)\subseteq A$ and*

$$\lim_{n\to\infty}\left[\inf\left\{t>0:N\left(T^nx_0-T^{n+1}x_0,t\right)\geq\alpha\right\}\right]$$
$$=\inf\left\{t>0:N\left(A-B,t\right)\geq\alpha\right\},$$

for some $x_0\in X$. Then $F^z_{\alpha,\epsilon}(A,B)\neq\emptyset$.

Proof. Let $\epsilon>0$ and $\alpha\in(0,1)$.

Assume that $x_0\in A\cup B$, $C_n=\left\{t>0:N\left(T^nx_0-T^{n+1}x_0,t\right)\geq\alpha\right\}$ and

$$\lim_{n\to\infty}\left[\inf C_n\right]=\inf\left\{t>0:N\left(A-B,t\right)\geq\alpha\right\}.$$

Then there exists $N_0>0$ such that

$$\inf C_n<\inf\left\{t>0:N\left(A-B,t\right)\geq\alpha\right\}+\epsilon,\ \text{for all } n\geq N_0.$$

Hence

$$\inf C_{N_0}<\inf\left\{t>0:N\left(A-B,t\right)\geq\alpha\right\}+\epsilon.$$

Let $s>0$ and $N(A-B,s)\geq\alpha$. Then

$$\inf\left\{t>0:N\left(A-B,t\right)\geq\alpha\right\}+\epsilon\leq s+\epsilon.$$

Thus

$$\inf\left\{t>0:N\left(T^{N_0}x_0-T\left(T^{N_0}x_0\right),t\right)\geq\alpha\right\}<s+\epsilon.$$

So $N\left(T^{N_0}x_0-T\left(T^{N_0}x_0\right),s+\epsilon\right)\geq\alpha$. Therefore $T^{N_0}x_0\in F^z_{\alpha,\epsilon}(A,B)$. Then $F^z_{\alpha,\epsilon}(A,B)\neq\emptyset$. □

Definition 3.4. Let (X,N) be a fuzzy normed space and A, B be nonempty subsets of X. We define fuzzy diameter $F^z_{\alpha,\epsilon}(A,B)\,(\neq\emptyset)$ as follows

$$\delta\left(F^z_{\alpha,\epsilon}(A,B)\right)=\sup\left\{\inf\left\{t>0:N\left(x-y,t\right)\geq\alpha\right\}:x,y\in F^z_{\alpha,\epsilon}(A,B)\right\}.$$

Lemma 3.5. *Let (X,N) be a fuzzy normed space, $\epsilon>0$, $\alpha\in(0,1)$, $(\emptyset\neq)\,A,B\subseteq X$ and $F^z_{\alpha,\epsilon}(A,B)\neq\emptyset$. Moreover, let $T:A\cup B\to A\cup B$ be a function such that $T(A)\subseteq B$, $T(B)\subseteq A$. Furthermore, assume for all $\mu>0$ there exists $\varphi(\mu)>0$ such that*

$$\inf\left\{t>0:N\left(x-y,t\right)\geq\alpha\right\}-\inf\left\{t>0:N\left(Tx-Ty,t\right)\geq\alpha\right\}\leq\mu$$

implies that

$$\inf\left\{t>0:N\left(x-y,t\right)\geq\alpha\right\}\leq\varphi(\mu),\ \textit{for all } x,y\in F^z_{\alpha,\epsilon}(A,B).$$

Then

$$\delta\left(F^z_{\alpha,\epsilon}(A,B)\right) \leq \varphi\left(2\left(\inf\{t>0 : N(A-B,t)\geq\alpha\}+\epsilon\right)\right).$$

Proof. Let $\epsilon>0$, $\alpha\in(0,1)$ and $x,y\in F^z_{\alpha,\epsilon}(A,B)$. Then

$$\inf\{t>0 : N(x-Tx,t)\geq\alpha\} \leq \inf\{t>0 : N(A-B,t)\geq\alpha\}+\epsilon,$$

and

$$\inf\{t>0 : N(y-Ty,t)\geq\alpha\} \leq \inf\{t>0 : N(A-B,t)\geq\alpha\}+\epsilon.$$

Hence

$$\begin{aligned}\inf\{t>0 : N(x-y,t)\geq\alpha\} &\leq \inf\{t>0 : N(x-Tx,t)\geq\alpha\}\\ &\quad+\inf\{t>0 : N(Tx-Ty,t)\geq\alpha\}\\ &\quad+\inf\{t>0 : N(y-Ty,t)\geq\alpha\}\\ &\leq \inf\{t>0 : N(Tx-Ty,t)\geq\alpha\}\\ &\quad+2\left(\inf\{t>0 : N(A-B,t)\geq\alpha\}\right)+2\epsilon.\end{aligned}$$

Therefore

$$\begin{aligned}&\inf\{t>0 : N(x-y,t)\geq\alpha\}-\inf\{t>0 : N(Tx-Ty,t)\geq\alpha\}\\ &\qquad\leq 2\left(\inf\{t>0 : N(A-B,t)\geq\alpha\}+\epsilon\right).\end{aligned}$$

Thus

$$\inf\{t>0 : N(x-y,t)\geq\alpha\} \leq \varphi\left(2\left(\inf\{t>0 : N(A-B,t)\geq\alpha\}\right)+\epsilon\right),$$

so

$$\delta\left(F^z_{\alpha,\epsilon}(A,B)\right) \leq \varphi\left(2\left(\inf\{t>0 : N(A-B,t)\geq\alpha\}\right)+\epsilon\right).$$

□

Lemma 3.6. *Let (X,N) be a fuzzy normed space, $\epsilon>0$, $\alpha\in(0,1)$ and $(\emptyset\neq)\,A,B\subseteq X$. Let $T : A\cup B\to A\cup B$ be a function such that $T(A)\subseteq B$, $T(B)\subseteq A$. Moreover, assume that there exists $x_0\in A\cup B$ such that*

$$\begin{aligned}&\lim_{n\to\infty}\left[\inf\left\{t>0 : N\left(T^n x_0-T^{n+1}x_0,t\right)\geq\alpha\right\}\right]\\ &\qquad=\inf\{t>0 : N(A-B,t)\geq\alpha\},\end{aligned}$$

and for all $\mu>0$ there exists $\varphi(\mu)>0$ such that

$$\inf\{t>0 : N(x-y,t)\geq\alpha\}-\inf\{t>0 : N(Tx-Ty,t)\geq\alpha\}\leq\mu$$

implies that

$$\inf\{t>0 : N(x-y,t)\geq\alpha\}\leq\varphi(\mu),\ \textit{for all } x,y\in F^z_{\alpha,\epsilon}(A,B).$$

Then

$$\delta\left(F^z_{\alpha,\epsilon}(A,B)\right) \leq \varphi\left(2\left(\inf\{t>0 : N(A-B,t)\geq\alpha\}\right)+\epsilon\right).$$

4. Approximate Fixed Point for Cyclic Maps in Fuzzy Normed Space

We begin with two lemmas which will be used in order to prove all the results given in the same section.

Definition 4.1. Let (X, N) be a fuzzy normed space, $\epsilon > 0$, $\alpha \in (0,1)$ and $(\emptyset \neq) A, B \subseteq X$. Moreover, let $T : A \cup B \to A \cup B$ be a function such that $T(A) \subseteq B$ and $T(B) \subseteq A$. A point $x_0 \in A \cup B$ is called an $F^z_{\alpha,\epsilon}$-approximate fixed point, if $N(x_0 - Tu_0, \epsilon) \geq \alpha$. We denote the set of all $F^z_{\alpha,\epsilon}$-approximate fixed point of T by $F^z_{\alpha,\epsilon}(T)$.

Definition 4.2. Let (X, N) be a fuzzy normed space, $\alpha \in (0,1)$ and $(\emptyset \neq) A, B \subseteq X$. Moreover, let $T : A \cup B \to A \cup B$ be a function such that $T(A) \subseteq B$ and $T(B) \subseteq A$. Then T has the fuzzy approximate fixed point property (f.a.f.p.p.) if $F^z_{\alpha,\epsilon}(T) \neq \emptyset$, for all $\epsilon > 0$.

Definition 4.3. Let (X, N) be a fuzzy normed space, $\alpha \in (0,1)$ and $(\emptyset \neq) A, B \subseteq X$. Moreover, let $T : A \cup B \to A \cup B$ be a function such that $T(A) \subseteq B$ and $T(B) \subseteq A$. The function $T : A \cup B \to A \cup B$ is called an α-asymptotically regular at $x_0 \in A \cup B$, if

$$\lim_{n\to\infty} \left[\inf\left\{t > 0 : N\left(T^n x_0 - T^{n+1}x_0, t\right) \geq \alpha\right\}\right] = 0.$$

Lemma 4.4. *Let (X, N) be a fuzzy normed space, $\alpha \in (0,1)$ and $(\emptyset \neq) A, B \subseteq X$. Moreover, let $T : A\cup B \to A\cup B$ be an α-asymptotically regular at $x_0 \in A \cup B$. Then T has the fuzzy approximate fixed point property.*

Proof. Let $\alpha \in (0,1)$ and $T : A \cup B \to A \cup B$ be an α-asymptotically regular at $x_0 \in A \cup B$. Assume that $\epsilon > 0$ and

$$\lim_{n\to\infty} \left[\inf\left\{t > 0 : N\left(T^n x_0 - T^{n+1}x_0, t\right) \geq \alpha\right\}\right] = 0.$$

Then there exists $N_0 > 0$ such that

$$\inf\left\{t > 0 : N\left(T^n x_0 - T^{n+1}x_0, t\right) \geq \alpha\right\} < \epsilon, \text{ for all } n \geq N_0.$$

Hence

$$\inf\left\{t > 0 : N\left(T^{N_0} x_0 - T\left(T^{N_0}x_0\right), t\right) \geq \alpha\right\} < \epsilon.$$

Thus $T^{N_0}x_0 \in F^z_{\alpha,\epsilon}(T)$. So $F^z_{\alpha,\epsilon}(T) \neq \emptyset$. Therefore T has the fuzzy approximate fixed point property. □

Definition 4.5. Let (X, N) be a fuzzy normed space, $\alpha \in (0,1)$ and $(\emptyset \neq) A, B \subseteq X$. Moreover, let $T : A \cup B \to A \cup B$ be a function such that $T(A) \subseteq B$ and $T(B) \subseteq A$. The function $T : A \cup B \to A \cup B$ is called an F-s-contraction, if there exists $s \in (0,1)$ such that

$$N(x - y, t) \geq \alpha \text{ implies that } N(Tx - Ty, t \inf s) \geq \alpha,$$

for all $x, y \in A \cup B$ and all $t > 0$.

Theorem 4.6. *Let (X, N) be a fuzzy normed space, $\alpha \in (0,1)$ and $(\emptyset \neq) A, B \subseteq X$. Moreover, let $T : A \cup B \to A \cup B$ be an F-s-contraction. Then T has the fuzzy approximate fixed point property.*

Proof. Let $\epsilon > 0$, $\alpha \in (0,1)$, $x \in A \cup B$ and

$$C_n = \left\{t > 0 : N\left(T^n x - T^{n+1}x, t\right) \geq \alpha\right\}.$$

We have

$$\begin{aligned}\inf C_n &= \inf\left\{t > 0 : N\left(T\left(T^{n-1}x\right) - T\left(T^n x\right), t\right) \geq \alpha\right\} \\ &\leq s \wedge \left[\inf\left\{t > 0 : N\left(T^{n-1}x - T^n x, t\right) \geq \alpha\right\}\right] \\ &\vdots \\ &\leq s^n \wedge \left[\inf\left\{t > 0 : N\left(x - Tx, t\right) \geq \alpha\right\}\right].\end{aligned}$$

Since $s \in (0,1)$ it follows that

$$\lim_{n\to\infty}\left[\inf\left\{t > 0 : N\left(T^n x - T^{n+1}x, t\right) \geq \alpha\right\}\right] = 0.$$

By Lemma 4.4, T has the fuzzy approximate fixed point property. □

In 1968, Kannan (see [6] [13]) proved a fixed point theorem for operators which need not be continuous. We apply it to a fuzzy normed space for fuzzy approximate fixed point.

Definition 4.7. Let (X, N) be a fuzzy normed space, $\alpha \in (0,1)$ and $(\emptyset \neq) A, B \subseteq X$. Moreover, let $T : A \cup B \to A \cup B$ be a function such that $T(A) \subseteq B$ and $T(B) \subseteq A$. The function $T : A \cup B \to A \cup B$ is called an F-Kannan contraction, if there exists $s \in (0, 1/2)$ such that $N(x - Tx, t_1) \geq \alpha$ and $N(y - Ty, t_2) \geq \alpha$ imply that

$$N(Tx - Ty, s(t_1 + t_2)) \geq \alpha,$$

for all $x, y \in A \cup B$ and all $t_1, t_2 > 0$.

Theorem 4.8. *Let (X, N) be a fuzzy normed space, $\alpha \in (0,1)$ and $(\emptyset \neq) A, B \subseteq X$. Moreover, let $T : A \cup B \to A \cup B$ be a F-Kannan contraction. Then T has the fuzzy approximate fixed point property.*

Proof. Let $\epsilon > 0$, $\alpha \in (0,1)$, $x \in A \cup B$ and

$$C_n = \left\{t > 0 : N\left(T^n x - T^{n+1}x, t\right) \geq \alpha\right\}.$$

Since T is a F-Kannan contraction, there exists $s \in (0, 1/2)$ such that

$$\begin{aligned}\inf C_n &= \inf\left\{t > 0 : N\left(T\left(T^{n-1}x\right) - T\left(T^n x\right), t\right) \geq \alpha\right\} \\ &\leq s\left(\inf\left\{t > 0 : N\left(T^{n-1}x - T\left(T^{n-1}x\right), t\right) \geq \alpha\right\}\right) \\ &\quad + s\left(\inf\left\{t > 0 : N\left(T^n x - T\left(T^n x\right), t\right) \geq \alpha\right\}\right).\end{aligned}$$

Therefore

$$(1-s)\left[\inf C_n\right] \leq s\left[\inf\left\{t>0 : N\left(T^{n-1}x - T^n x, t\right) \geq \alpha\right\}\right].$$

Then

$$\begin{aligned}\inf C_n &\leq (s/1-s)\left[\inf\left\{t>0 : N\left(T^{n-1}x - T^n x, t\right) \geq \alpha\right\}\right]\\ &\ \vdots\\ &\leq (s/1-s)^n\left[\inf\left\{t>0 : N\left(x - Tx, t\right) \geq \alpha\right\}\right].\end{aligned}$$

Since $s \in (0, 1/2)$ it follows that $(s/1-s) \in (0,1)$. Therefore

$$\lim_{n\to\infty}\left[\inf\left\{t>0 : N\left(T^n x - T^{n+1}x, t\right) \geq \alpha\right\}\right] = 0.$$

By Lemma 4.4, T has the fuzzy approximate fixed point property. □

In 1972, Chatterjea (see [7]) considered another contraction which does not imply the continuity. We apply it to a fuzzy normed space for fuzzy approximate fixed point.

Definition 4.9. Let (X, N) be a fuzzy normed space, $\alpha \in (0,1)$ and $(\emptyset \neq) A, B \subseteq X$. Moreover, let $T : A \cup B \to A \cup B$ be a function such that $T(A) \subseteq B$ and $T(B) \subseteq A$. The function $T : A \cup B \to A \cup B$ is called an F-Chatterjea contraction, if there exists $s \in (0, 1/2)$ such that $N(x - Ty, t_1) \geq \alpha$ and $N(y - Tx, t_2) \geq \alpha$ imply that

$$N\left(Tx - Ty, s\left(t_1 + t_2\right)\right) \geq \alpha,$$

for all $x, y \in A \cup B$ and all $t_1, t_2 > 0$.

Theorem 4.10. *Let (X, N) be a fuzzy normed space, $\alpha \in (0,1)$ and $(\emptyset \neq) A, B \subseteq X$. Moreover, let $T : A \cup B \to A \cup B$ be an F-Chatterjea contraction. Then T has the fuzzy approximate fixed point property.*

Proof. Let $\epsilon > 0$, $\alpha \in (0,1)$, $x \in A \cup B$ and

$$C_n = \left\{t>0 : N\left(T^n x - T^{n+1}x, t\right) \geq \alpha\right\}.$$

Since T is an F-Chatterjea contraction, there exists $s \in (0, 1/2)$ such that

$$\begin{aligned}\inf C_n &= \inf\left\{t>0 : N\left(T\left(T^{n-1}x\right) - T\left(T^n x\right), t\right) \geq \alpha\right\}\\ &\leq s\left(\inf\left\{t>0 : N\left(T^{n-1}x - T\left(T^n x\right), t\right) \geq \alpha\right\}\right)\\ &\quad + s\left(\inf\left\{t>0 : N\left(T^n x - T\left(T^{n-1}x\right), t\right) \geq \alpha\right\}\right)\\ &= s\left[\inf\left\{t>0 : N\left(T^{n-1}x - T^{n+1}x, t\right) \geq \alpha\right\}\right].\end{aligned}$$

On the other hand, we have

$$\inf\left\{t>0 : N\left(T^{n-1}x - T^{n+1}x, t\right) \geq \alpha\right\} \leq \inf C_n + \inf C_{n-1}.$$

Then

$$(1-s)\left[\inf C_n\right] \leq s\left[\inf C_{n-1}\right].$$

Hence

$$\begin{aligned}\inf C_n &\leq (s/1-s)\left[\inf C_{n-1}\right]\\ &\vdots\\ &\leq (s/1-s)^n\left[\inf C_0\right].\end{aligned}$$

Since $s \in (0,1/2)$ it follows that $(s/1-s) \in (0,1)$. Therefore

$$\lim_{n\to\infty}\left[\inf\left\{t>0 : N\left(T^n x - T^{n+1}x, t\right) \geq \alpha\right\}\right] = 0.$$

By Lemma 4.4, T has the fuzzy approximate fixed point property. □

Combining the three independent contraction conditions above, we obtain another fuzzy approximate fixed point result for operators which satisfy the following.

Definition 4.11. Let (X,N) be a fuzzy normed space, $\alpha \in (0,1)$ and $(\emptyset \neq) A, B \subseteq X$. Moreover, let $T : A\cup B \to A\cup B$ be a function such that $T(A) \subseteq B$ and $T(B) \subseteq A$. The function $T : A\cup B \to A\cup B$ is called an F-Zamfirescu contraction, if there exist $s_1 \in (0,1)$ and $s_2, s_3 \in (0,1/2)$ such that for all $x, y \in A\cup B$ at least one of the following is true:

- (F_{Z1}) $N(x-y,t) \geq \alpha$ implies that $N(Tx-Ty, t\inf s_1) \geq \alpha$, for all $x, y \in A\cup B$, and all $t>0$,
- (F_{Z2}) $N(x-Tx,t_1) \geq \alpha$ and $N(y-Ty,t_2) \geq \alpha$ imply that $N(Tx-Ty, s_2(t_1+t_2)) \geq \alpha$, for all $x, y \in A\cup B$, and all $t_1, t_2 > 0$,
- (F_{Z3}) $N(x-Ty,t_1) \geq \alpha$ and $N(y-Tx,t_2) \geq \alpha$ imply that $N(Tx-Ty, s_3(t_1+t_2)) \geq \alpha$, for all $x, y \in A\cup B$, and all $t_1, t_2 > 0$.

Corollary 4.12. *Let (X,N) be a fuzzy normed space, $\alpha \in (0,1)$ and $(\emptyset \neq) A, B \subseteq X$. Moreover, let $T : A\cup B \to A\cup B$ be an F-Zamfirescu contraction. Then T has the fuzzy approximate fixed point property.*

Proof. By Theorems 4.6, 4.8 and 4.10, proof is clear. □

Now, we consider the contraction condition given in 2004 by Y. Berinde, who also formulated a corresponding fixed point theorem, see [6], for example.

Definition 4.13. Let (X,N) be a fuzzy normed space, $\alpha \in (0,1)$ and $(\emptyset \neq) A, B \subseteq X$. Moreover, let $T : A\cup B \to A\cup B$ be a function such that $T(A) \subseteq B$ and $T(B) \subseteq A$. The function $T : A\cup B \to A\cup B$ is called an F-weak contraction, if there exist $s_1 \in (0,1)$ and $s_2 \geq 0$ such that $N(x-y,t_1) \geq \alpha$ and $N(y-Tu,t_2) \geq \alpha$ imply that

$$N(Tx-Ty,(s_1 \inf t_1)+(s_2 \inf t_2)) \geq \alpha,$$

for all $x,y \in A \cup B$ and all $t_1,\ t_2 > 0$.

Theorem 4.14. *Let (X,N) be a fuzzy normed space, $\alpha \in (0,1)$ and $(\emptyset \neq) A,B \subseteq X$. Moreover, let $T : A \cup B \to A \cup B$ be an F-weak contraction. Then T has the fuzzy approximate fixed point property.*

Proof. Let $x \in A \cup B$ and $C_n = \left\{t > 0 : N\left(T^n x - T^{n+1}x, t\right) \geq \alpha\right\}$. Since T is an F-weak contraction, there exist $s_1 \in (0,1)$ and $s_2 \geq 0$ such that

$$\begin{aligned}
\inf C_n &= \inf\left\{t > 0 : N\left(T\left(T^{n-1}x\right) - T\left(T^n x\right), t\right) \geq \alpha\right\} \\
&\leq s_1 \wedge \left[\inf\left\{t > 0 : N\left(T^{n-1}x - T^n x, t\right) \geq \alpha\right\}\right] \\
&\quad + s_2 \wedge \left[\inf\left\{t > 0 : N\left(T^n x - T^n x, t\right) \geq \alpha\right\}\right] \\
&\leq s_1 \wedge \left[\inf\left\{t > 0 : N\left(T^{n-1}x - T^n x, t\right) \geq \alpha\right\}\right] \\
&\ \ \vdots \\
&\leq s_1^n \inf\left\{t > 0 : N(x - Tx, t) \geq \alpha\right\}.
\end{aligned}$$

Since $s_1 \in (0,1)$ it follows that

$$\lim_{n\to\infty}\left[\inf\left\{t > 0 : N\left(T^n x - T^{n+1}x, t\right) \geq \alpha\right\}\right] = 0, \text{ for all } x \in A \cup B.$$

Now by Lemma 4.4, T has the fuzzy approximate fixed point property. □

In 1974, Ciric [10] obtained a contraction condition for which the map satisfying it, is still a Picard operator. We apply it to fuzzy normed space for fuzzy approximate fixed point.

Definition 4.15. Let (X,N) be a fuzzy normed space, $\alpha \in (0,1)$ and $(\emptyset \neq) A,B \subseteq X$. Moreover, let $T : A \cup B \to A \cup B$ be a function such that $T(A) \subseteq B$ and $T(B) \subseteq A$. The function $T : A \cup B \to A \cup B$ is called an F-quasi contraction, if there exists $s \in (0,1)$ such that

$$\begin{aligned}
&N(x-y,t_1) \geq \alpha, \qquad &&N(y-Ty,t_3) \geq \alpha, \\
&N(x-Tx,t_2) \geq \alpha, \qquad &&N(x-Ty,t_4) \geq \alpha,
\end{aligned}$$

and $N(y-Tx,t_5) \geq \alpha$ imply that

$$N(Tu-Tv,s(\max\{t_1,t_2,t_3,t_4,t_5\})) \geq \alpha,$$

for all $x,y \in A \cup B$, and all $t_1,\ t_2,\ t_3,\ t_4,\ t_5 > 0$.

Corollary 4.16. *Let (X, N) be a fuzzy normed space, $\alpha \in (0,1)$ and $(\emptyset \neq) A, B \subseteq X$. Moreover, let $T : A \cup B \to A \cup B$ be an F-quasi contraction with $s \in (0, 1/2)$. Then T has the fuzzy approximate fixed point property.*

Proof. By Proposition 3 in [5] any F-quasi contraction with $s \in (0, 1/2)$ is an F-weak contraction. By Theorem 4.14, T has the fuzzy approximate fixed point property. □

5. Fuzzy Diameter Approximate Best Proximity Pair for Various Types of Operators

In this section, we study the fuzzy diameter approximate best proximity pair.

Theorem 5.1. *Let (X, N) be a fuzzy normed space, $\alpha \in (0,1)$ and $(\emptyset \neq) A, B \subseteq X$. Moreover, let $T : A \cup B \to A \cup B$ be an F-s-contraction. Then*

$\delta\left(F^z_{\alpha,\epsilon}(A,B)\right) \leq s + 2\left(\inf\{t > 0 : N(A - B, t) \geq \alpha\}\right) + 2\epsilon,$ *for all* $\epsilon > 0$.

Proof. Let $\epsilon > 0$, $\alpha \in (0,1)$ and $x, y \in F^z_{\alpha,\epsilon}(A,B)$. Then

$$\inf\{t > 0 : N(x - Tx, t) \geq \alpha\} \leq \inf\{t > 0 : N(A - B, t) \geq \alpha\} + \epsilon,$$

and

$$\inf\{t > 0 : N(y - Ty, t) \geq \alpha\} \leq \inf\{t > 0 : N(A - B, t) \geq \alpha\} + \epsilon.$$

Since $T : A \cup B \to A \cup B$ is an F-s-contraction,

$$\begin{aligned}
\inf\{t > 0 : N(x - y, t) \geq \alpha\} &\leq \inf\{t > 0 : N(x - Tx, t) \geq \alpha\} \\
&\quad + \inf\{t > 0 : N(Tx - Ty, t) \geq \alpha\} \\
&\quad + \inf\{t > 0 : N(y - Ty, t) \geq \alpha\} \\
&\leq s \wedge [\inf\{t > 0 : N(x - y, t) \geq \alpha\}] \\
&\quad + 2\left(\inf\{t > 0 : N(A - B, t) \geq \alpha\}\right) + 2\epsilon \\
&\leq 2\left(\inf\{t > 0 : N(A - B, t) \geq \alpha\}\right) \\
&\quad + s + 2\epsilon.
\end{aligned}$$

Hence

$$\delta\left(F^z_{\alpha,\epsilon}(A,B)\right) \leq s + 2\left(\inf\{t > 0 : N(A - B, t) \geq \alpha\}\right) + 2\epsilon.$$

□

Theorem 5.2. *Let (X, N) be a fuzzy normed space, $\alpha \in (0,1)$ and $(\emptyset \neq) A, B \subseteq X$. Moreover, let $T : A \cup B \to A \cup B$ be an F-Kannan contraction. Then*

$$\delta\left(F^z_{\alpha,\epsilon}(A,B)\right) \leq 2\epsilon(1 + s) + 2s \inf\{t > 0 : N(A - B, t) \geq \alpha\},$$

for all $\epsilon > 0$.

Proof. Let $\epsilon > 0$, $\alpha \in (0,1)$ and $x, y \in F^z_{\alpha,\epsilon}(A,B)$. Then, assume that $\theta > 0$ and

$$\inf\{t > 0 : N(x-y,t) \geq \alpha\} - \inf\{t > 0 : N(Tu - Tv, t) \geq \alpha\} \leq \theta.$$

Since $T : A \cup B \to A \cup B$ is an F-Kannan contraction,

$$\begin{aligned}\inf\{t > 0 : N(x-y,t) \geq \alpha\} &\leq s(\inf\{t > 0 : N(x - Tu, t) \geq \alpha\}) \\ &\quad + s(\inf\{t > 0 : N(y - Tv, t) \geq \alpha\}) + \theta.\end{aligned}$$

Since $x, y \in F^z_{\alpha,\epsilon}(A,B)$, it follows that

$$\inf\{t > 0 : N(x - Tx, t) \geq \alpha\} \leq \inf\{t > 0 : N(A - B, t) \geq \alpha\} + \epsilon,$$

and

$$\inf\{t > 0 : N(y - Ty, t) \geq \alpha\} \leq \inf\{t > 0 : N(A - B, t) \geq \alpha\} + \epsilon.$$

Hence

$$\inf\{t > 0 : N(x-y,t) \geq \alpha\} \leq 2s[\inf\{t > 0 : N(A - B, t) \geq \alpha\}] + 2s\epsilon + \theta.$$

So for every $\theta > 0$ there exists

$$\phi(\theta) = \theta + 2s(\inf\{t > 0 : N(A - B, t) \geq \alpha\} + \epsilon) > 0,$$

such that

$$\inf\{t > 0 : N(x-y,t) \geq \alpha\} - \inf\{t > 0 : N(Tx - Ty, t) \geq \alpha\} \leq \theta,$$

which implies that

$$\inf\{t > 0 : N(x-y,t) \geq \alpha\} \leq \phi(\theta).$$

Now by Lemma 3.5, we obtain that

$$\delta\left(F^z_{\alpha,\epsilon}(A,B)\right) \leq \phi(2\epsilon).$$

Therefore

$$\delta\left(F^z_{\alpha,\epsilon}(A,B)\right) \leq 2\epsilon(1+s) + 2s[\inf\{t > 0 : N(A - B, t) \geq \alpha\}].$$

□

Theorem 5.3. *Let* (X,N) *be a fuzzy normed space,* $\alpha \in (0,1)$ *and* $(\emptyset \neq) A, B \subseteq X$*. Moreover, let* $T : A \cup B \to A \cup B$ *be an* F*-Chatterjea contraction. Then*

$$\delta\left(F^z_{\alpha,\epsilon}(A,B)\right) \leq (2\epsilon(1+s) + 2s\inf\{t > 0 : N(A - B, t) \geq \alpha\}/1 - 2s),$$

for all $\epsilon > 0$.

Proof. Let $\epsilon > 0$, $\alpha \in (0,1)$ and $x, y \in F^z_{\alpha,\epsilon}(A,B)$.

Assume that $\theta > 0$ and

$$\inf\{t>0 : N(x-y,t) \geq \alpha\} - \inf\{t>0 : N(Tx-Ty,t) \geq \alpha\} \leq \theta.$$

Since $T : A \cup B \to A \cup B$ is an F-Chatterjea contraction,

$$\begin{aligned}\inf\{t>0 : N(x-y,t) \geq \alpha\} &\leq s(\inf\{t>0 : N(x-Ty,t) \geq \alpha\}) \\ &\quad + s(\inf\{t>0 : N(y-Tu,t) \geq \alpha\}) + \theta \\ &\leq s(\inf\{t>0 : N(x-Ty,t) \geq \alpha\}) \\ &\quad + s(\inf\{t>0 : N(y-Tx,t) \geq \alpha\}) + \theta \\ &\leq s(\inf\{t>0 : N(x-y,t) \geq \alpha\}) \\ &\quad + s(\inf\{t>0 : N(y-Ty,t) \geq \alpha\}) \\ &\quad + s(\inf\{t>0 : N(y-x,t) \geq \alpha\}) \\ &\quad + s(\inf\{t>0 : N(x-Tx,t) \geq \alpha\}) + \theta.\end{aligned}$$

Since $x, y \in F^z_{\alpha,\epsilon}(A,B)$, it follows that

$$\inf\{t>0 : N(x-Tx,t) \geq \alpha\} \leq \inf\{t>0 : N(A-B,t) \geq \alpha\} + \epsilon,$$

and

$$\inf\{t>0 : N(y-Ty,t) \geq \alpha\} \leq \inf\{t>0 : N(A-B,t) \geq \alpha\} + \epsilon.$$

Hence

$$\begin{gathered}\inf\{t>0 : N(x-y,t) \geq \alpha\} \leq \\ 2s(\inf\{t>0 : N(x-y,t) \geq \alpha\}) + 2s(\inf\{t>0 : N(A-B,t) \geq \alpha\}) \\ +2s\epsilon + \theta.\end{gathered}$$

Thus

$$\begin{aligned}&(1-2s)(\inf\{t>0 : N(x-y,t) \geq \alpha\}) \\ &\qquad \leq 2s(\inf\{t>0 : N(A-B,t) \geq \alpha\}) + 2s\epsilon + \theta.\end{aligned}$$

Then

$$\begin{aligned}&\inf\{t>0 : N(x-y,t) \geq \alpha\} \\ &\qquad \leq (2s(\inf\{t>0 : N(A-B,t) \geq \alpha\} + \epsilon) + \theta)/(1-2s).\end{aligned}$$

So for every $\theta > 0$ there exists

$$\phi(\theta) = (2s(\inf\{t>0 : N(A-B,t) \geq \alpha\} + \epsilon) + \theta)/(1-2s) > 0,$$

such that

$$\inf\{t>0 : N(x-y,t) \geq \alpha\} - \inf\{t>0 : N(Tx-Ty,t) \geq \alpha\} \leq \theta,$$

which implies that

$$\inf\{t>0 : N(x-y,t) \geq \alpha\} \leq \phi(\theta).$$

Now by Lemma 3.5, we obtain that

$$\delta\left(F^{z}_{\alpha,\epsilon}(A,B)\right) \leq \phi(2\epsilon).$$

Then

$$\delta\left(F^{z}_{\alpha,\epsilon}(A,B)\right) \leq \left(2\epsilon(1+s) + 2s\inf\{t>0 : N(A-B,t) \geq \alpha\}\right)/(1-2s).$$

□

Theorem 5.4. *Let (X,N) be a fuzzy normed space, $\alpha \in (0,1)$ and $(\emptyset \neq) A, B \subseteq X$. Moreover, let $T : A \cup B \to A \cup B$ be an F-Zamfirescu contraction. Then*

$$\begin{aligned}\delta&\left(F^{z}_{\alpha,\epsilon}(A,B)\right)\\ &\leq \left(2\epsilon(1+\eta) + 2\eta\left(\inf\{t>0 : N(A-B,t) \geq \alpha\} + (1/2)\right)\right)/(1-\eta),\end{aligned}$$

for all $\epsilon > 0$, where $\eta = \max\{s_1, s_2/(1-s_2), s_3/(1-s_3)\}$ and s_1, s_2, s_3 as in Definition 4.11.

Proof. Let $x, y \in A \cup B$ and $\alpha \in (0,1)$. Suppose that (F_{Z1}) holds, $C_0 = \{t>0 : N(x-y,t) \geq \alpha\}$ and $C_1 = \{t>0 : N(Tx-Ty,t) \geq \alpha\}$. We have

$$\inf C_1 \leq s_1 \wedge [\inf C_0] \leq s_1.$$

Assume that (F_{Z2}) holds, we have

$$\begin{aligned}\inf\{t>0 : N(Tx-Ty,t) \geq \alpha\} &\leq s_2\left[\inf\{t>0 : N(x-Tx,t) \geq \alpha\}\right]\\ &\quad + s_2\left[\inf\{t>0 : N(y-Ty,t) \geq \alpha\}\right]\\ &\leq s_2\left[\inf\{t>0 : N(x-Tx,t) \geq \alpha\}\right]\\ &\quad + s_2\left[\inf\{t>0 : N(y-x,t) \geq \alpha\}\right]\\ &\quad + s_2\left[\inf\{t>0 : N(x-Tx,t) \geq \alpha\}\right]\\ &\quad + s_2\left[\inf\{t>0 : N(Tx-Ty,t) \geq \alpha\}\right]\\ &= 2s_2\left(\inf\{t>0 : N(x-Tx,t) \geq \alpha\}\right)\\ &\quad + s_2\left(\inf\{t>0 : N(x-y,t) \geq \alpha\}\right)\\ &\quad + s_2\left(\inf\{t>0 : N(Tx-Ty,t) \geq \alpha\}\right).\end{aligned}$$

Thus

$$\begin{aligned}\inf C_1 &\leq \left(2s_2/(1-s_2)\right)\left(\inf\{t>0 : N(x-Tx,t) \geq \alpha\}\right)\\ &\quad + \left(s_2/(1-s_2)\right)(\inf C_0).\end{aligned}$$

Suppose that (F_{Z3}) holds, we have

$$\begin{aligned}\inf\{t>0 : N(Tx-Ty,t) \geq \alpha\} &\leq s_3\left[\inf\{t>0 : N(x-Ty,t) \geq \alpha\}\right]\\ &\quad + s_3\left[\inf\{t>0 : N(y-Tx,t) \geq \alpha\}\right]\\ &\leq s_3\left[\inf\{t>0 : N(x-y,t) \geq \alpha\}\right]\end{aligned}$$

$$
\begin{aligned}
&+ s_3 \left[\inf\{t>0 : N(y-Ty,t)\ge\alpha\}\right]\\
&+ s_3 \left[\inf\{t>0 : N(y-Ty,t)\ge\alpha\}\right]\\
&+ s_3 \left[\inf\{t>0 : N(Ty-Tx,t)\ge\alpha\}\right]\\
=\ & s_3 \left(\inf\{t>0 : N(Tx-Ty,t)\ge\alpha\}\right)\\
&+ 2s_3 \left(\inf\{t>0 : N(y-Ty,t)\ge\alpha\}\right)\\
&+ s_3 \left(\inf\{t>0 : N(x-y,t)\ge\alpha\}\right).
\end{aligned}
$$

Thus

$$
\begin{aligned}
\inf C_1 \le\ & (2s_3/(1-s_3))\left(\inf\{t>0 : N(y-Ty,t)\ge\alpha\}\right)\\
&+ (s_3/(1-s_3))(\inf C_0).
\end{aligned}
$$

These imply that

$$\inf C_1 \le 2\eta\left(\inf\{t>0 : N(x-Tx,t)\ge\alpha\}\right) + \eta(\inf C_0) + \eta,$$

where $\eta = \max\{s_1, s_2/(1-s_2), s_3/(1-s_3)\}$.

Now let $\theta>0$ and $x,y\in F^z_{\alpha,\epsilon}(A,B)$. Assume that

$$\inf\{t>0 : N(x-y,t)\ge\alpha\} - \inf\{t>0 : N(Tx-Ty,t)\ge\alpha\} \le \theta.$$

Then

$$
\begin{aligned}
\inf\{t>0 : N(x-y,t)\ge\alpha\} &\le \inf\{t>0 : N(Tx-Ty,t)\ge\alpha\} + \theta\\
&\le 2\eta\left(\inf\{t>0 : N(x-Tx,t)\ge\alpha\}\right)\\
&\quad + \eta\left(\inf\{t>0 : N(x-y,t)\ge\alpha\}\right) + \eta + \theta.
\end{aligned}
$$

Since $x,y\in F^z_{\alpha,\epsilon}(A,B)$, it follows that

$$\inf\{t>0 : N(x-Tx,t)\ge\alpha\} \le \inf\{t>0 : N(A-B,t)\ge\alpha\} + \epsilon,$$

and

$$\inf\{t>0 : N(y-Ty,t)\ge\alpha\} \le \inf\{t>0 : N(A-B,t)\ge\alpha\} + \epsilon.$$

Hence

$$
\begin{aligned}
(1-\eta)\inf C_0 &\le 2\eta\left(\inf\{t>0 : N(x-Tx,t)\ge\alpha\}\right) + \eta + \theta\\
&\le 2\eta\left(\inf\{t>0 : N(A-B,t)\ge\alpha\}\right) + 2\eta\epsilon + \eta + \theta.
\end{aligned}
$$

Therefore

$$\inf C_0 \le \left(2\eta\left(\inf\{t>0 : N(A-B,t)\ge\alpha\} + \epsilon + (1/2)\right) + \theta\right)/(1-\eta).$$

So for every $\theta>0$ there exists

$$\phi(\theta) = \left(2\eta\left(\inf\{t>0 : N(A-B,t)\ge\alpha\} + \epsilon + (1/2)\right) + \theta\right)/(1-\eta) > 0,$$

such that

$$\inf\{t>0 : N(x-y,t)\ge\alpha\} - \inf\{t>0 : N(Tx-Ty,t)\ge\alpha\} \le \theta,$$

which implies that

$$\inf\{t>0:N(x-y,t)\geq\alpha\}\leq\phi(\theta).$$

Now by Lemma 3.5, we obtain that

$$\delta\left(F^{z}_{\alpha,\epsilon}(A,B)\right)\leq\phi(2\epsilon).$$

Thus

$$\begin{aligned}&\delta\left(F^{z}_{\alpha,\epsilon}(A,B)\right)\\&\quad\leq(2\epsilon(1+\eta)+2\eta(\inf\{t>0:N(A-B,t)\geq\alpha\}+(1/2)))/(1-\eta).\end{aligned}$$

□

Theorem 5.5. *Let (X,N) be a fuzzy normed space, $\alpha\in(0,1)$ and $(\emptyset\neq)A,B\subseteq X$. Moreover, let $T:A\cup B\to A\cup B$ be an F-weak contraction. Then*

$$\delta\left(F^{z}_{\alpha,\epsilon}(A,B)\right)\leq s_1+s_2+2\epsilon,\ \textit{for all } \epsilon>0.$$

Proof. Let $\epsilon>0$, $\alpha\in(0,1)$ and $x,y\in F^{z}_{\alpha,\epsilon}(A,B)$.

Assume that $\theta>0$ and

$$\inf\{t>0:N(x-y,t)\geq\alpha\}-\inf\{t>0:N(Tx-Ty,t)\geq\alpha\}\leq\theta.$$

Since $T:A\cup B\to A\cup B$ is an F-weak contraction,

$$\begin{aligned}\inf\{t>0:N(x-y,t)\geq\alpha\}&\leq\inf\{t>0:N(Tx-Ty,t)\geq\alpha\}+\theta\\&\leq s_1\wedge(\inf\{t>0:N(x-y,t)\geq\alpha\})\\&\quad+s_2\wedge(\inf\{t>0:N(y-Tx,t)\geq\alpha\})\\&\quad+\theta\\&\leq s_1+s_2+\theta.\end{aligned}$$

So for every $\theta>0$ there exists $\phi(\theta)=s_1+s_2+\theta>0$ such that

$$\inf\{t>0:N(x-y,t)\geq\alpha\}-\inf\{t>0:N(Tx-Ty,t)\geq\alpha\}\leq\theta,$$

which implies that

$$\inf\{t>0:N(x-y,t)\geq\alpha\}\leq\phi(\theta).$$

Now by Lemma 3.5, we obtain that

$$\delta\left(F^{z}_{\alpha,\epsilon}(A,B)\right)\leq\phi(2\epsilon).$$

Hence

$$\delta\left(F^{z}_{\alpha,\epsilon}(A,B)\right)\leq s_1+s_2+2\epsilon.$$

□

References

1. T. Bag and S.K. Samanta, *Finite dimensional fuzzy normed linear spaces*, J. Fuzzy Math., 11 (2003), pp. 687-705.
2. T. Bag and S.K. Samanta, *Fuzzy bounded linear operators*, Fuzzy Sets and Systems, 151 (2005), pp. 513-547.
3. T. Bag and S.K. Samanta, *Some fixed point theorems in fuzzy normed linear spaces*, Inf. Sci., 177 (2007), pp. 3271-3289.
4. M. Berinde, *Approximate fixed point theorems*, Mathematica, LI (1) (2006).
5. V. Berinde, *Approximating fixed points of weak contractions using the Picard iteration*, Nonlinear Anal. Forum, 9 (2004).
6. V. Berinde, *On the approximation of fixed points of weak contractive mappings*, Math., 19 (2003), pp. 7-22.
7. S.K. Chatterjea, *Fixed-point theorems*, C.R. Acad. Bulgare Sci. 25 (1972), pp. 727-730.
8. S.C. Cheng and J.N. Mordeson, *Fuzzy linear operator and fuzzy normed linear spaces*, Bull. Calcutta Math. Soc., 86 (1994), pp. 429-436.
9. A. Chitra and P.Y. Mordeson, *Fuzzy linear operators and fuzzy normed linear spaces*, Bull. Calcutta Math. Soc., 74 (1969), pp. 660-665.
10. Lj. B. Ciric, *A generalization of Banachs contraction principle*, Proc. Am. Math. Soc., 45 (1974), pp. 267-273.
11. C. Felbin, *Finite dimensional fuzzy normeded linear spaces*, Fuzzy Sets and Systems, 48 (1992), pp. 239-248.
12. O. Kaleva and S. Seikkala, *On fuzzy metric spaces*, Fuzzy Sets and Systems, 12 (1984), pp. 215-229.
13. R. Kannan, *Some results on fixed points*, Bull. Calcutta Math. Soc., 10 (1968), pp. 71-76.
14. A.K. Katsaras, *Fuzzy topological vector spaces II*, Fuzzy Sets and Systems, 12 (1984), pp. 143-154.
15. I. Kramosil and J. Michalek, *Fuzzy metric and statistical metric spaces*, Kybernetica, 11 (1975), pp. 326-334.
16. S.A.M. Mohsenalhosseini, *Approximate fixed point theorems in fuzzy norm spaces for an operator*, Adv. Fuzzy Syst., 2013, Article ID 613604, 8 pages.
17. S.A.M. Mohsenalhosseini, *Approximate best proximity pairs in metric space for Contraction Maps,* Advances in Fixed Point Theory, 4 (2014), pp. 310-324.
18. S.A.M. Mohsenalhosseini, H. Mazaheri, and M.A. Dehghan, *Approximate best proximity pairs in metric space*, Abstract and Applied Analysis, (2011), Article ID 596971, 9 pages.

19. S.A.M. Mohsenalhosseini, H. Mazaheri, and M.A. Dehghan, *Approximate fixed point in fuzzy normed spaces for nonlinear maps*, Iranin Journal of Fuzzy Systems, 10 (2013), pp. 135-142.

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[1] Faculty of Mathematics, Yazd University, Yazd, Iran and Department of Mathematics, Vali-e-Asr University of Rafsanjan, Rafsanjan, Iran.

E-mail address: `mohsenhosseini@yazd.ac.ir`

[2] Department of Mathematics, Vali-e-Asr University of Rafsanjan, Rafsanjan, Iran.

E-mail address: `saheli@vru.ac.ir`