Sahand Communications in Mathematical Analysis (SCMA) Vol. 17 No. 1 (2020), 1-22
http://scma.maragheh.ac.ir
DOI: 10.22130/scma.2018.79219.369

Caristi Type Cyclic Contraction and Coupled Fixed Point Results in Bipolar Metric Spaces

Gagula Naveen Venkata Kishore[1,*], Bagathi Srinuvasa Rao[2], Stojan Radenović[3] and Huaping Huang[4]

Abstract. In this paper, we establish the existence of common coupled fixed point results for new Caristi type contraction of three covariant mappings in Bipolar metric spaces. Some interesting consequences of our results are achieved. Moreover, we give an illustration which presents the applicability of the achieved results.

1. Introduction

This work is motivated by the recent work on extension of Banach contraction principle on Bipolar metric spaces, which has been done by Mutlu and Gürdal [2]. Also, they investigated some fixed point and coupled fixed point results on this spaces (see [1, 2]). Subsequently, many authors established coupled fixed point theorems in different spaces (see [3, 8, 10, 11, 14, 15]). Caristi's fixed point theorem ([7]) is a well-known extension of Banach contraction principle ([13]). The proof of Caristi's result has been generalized and extended in many directions (see [9, 12]). It is also a modification of the Ekeland's ε-variational principle (see [6]). Later, many authors referred it to Caristi-Ekeland fixed point results and extended different types of distance spaces ([4, 5]).

The aim of this paper is to initiate the study of a common coupled fixed point results for three mappings under various new Caristi type contractive conditions in bipolar metric spaces. We have also illustrated the validity of the hypotheses of our results.

2010 *Mathematics Subject Classification.* 34B24, 34B27.
Key words and phrases. Bipolar metric space, Compatible mappings, Coupled fixed point, Common fixed point.
Received: 11 January 2018, Accepted: 27 April 2019.
* Corresponding author.

Definition 1.1 ([2])**.** Let A *and* B be two nonempty sets. Suppose that $d : A \times B \to [0, \infty)$ is a mapping satisfying the following properties:

(B_0) If $d(a, b) = 0$ then $a = b$ for all $(a, b) \in A \times B$,
(B_1) If $a = b$ then $d(a, b) = 0$, for all $(a, b) \in A \times B$,
(B_3) If $d(a, b) = d(b, a)$, for all $a, b \in A \cap B$,
(B_4) If $d(a_1, b_2) \leq d(a_1, b_1) + d(a_2, b_1) + d(a_2, b_2)$ for all $a_1, a_2 \in A$, $b_1, b_2 \in B$.

Then the mapping d is called a bipolar-metric on the pair (A, B) and the triple (A, B, d) is called a bipolar-metric space.

Definition 1.2 ([2])**.** Assume (A_1, B_1) and (A_2, B_2) as two pairs of sets.

The function $F : A_1 \cup B_1 \to A_2 \cup B_2$ is said to be a covariant map, if $F(A_1) \subseteq A_2$ and $F(B_1) \subseteq B_2$ and we denote this as

$$F : (A_1, B_1) \rightrightarrows (A_2, B_2).$$

The mapping $F : A_1 \cup B_1 \to A_2 \cup B_2$ is said to be a contravariant map, if $F(A_1) \subseteq B_2$ and $F(B_1) \subseteq A_2$ and we denote this as

$$F : (A_1, B_1) \leftrightarrows (A_2, B_2).$$

In particular, if d_1 and d_2 are bipolar metrics on (A_1, B_1) and (A_2, B_2) respectively, then sometimes we use the notations
$F : (A_1, B_1, d_1) \rightrightarrows (A_2, B_2, d_2)$ and $F : (A_1, B_1, d_1) \leftrightarrows (A_2, B_2, d_2)$.

Definition 1.3 ([2])**.** Let (A, B, d) be a bipolar metric space. A point $v \in A \cup B$ is said to be a left point if $v \in A$, a right point if $v \in B$ and a central point if both $v \in A \cap B$.

Similarly, a sequence $\{a_n\}$ in A $(resp. \{b_n\} in B)$ is called a left $(resp. right)$ sequence.

In a bipolar metric space, sequence is the simple term for a left or right sequence.

A sequence $\{v_n\}$ is convergent to a point v if and only if $\{v_n\}$ is a left sequence, v is a right point and $\lim_{n\to\infty} d(v_n, v) = 0$; or $\{v_n\}$ is a right sequence, v is a left point and $\lim_{n\to\infty} d(v, v_n) = 0$.

A bisequence $(\{a_n\}, \{b_n\})$ on (A, B, d) is a sequence in $A \times B$. If the sequences $\{a_n\}$ and $\{b_n\}$ are convergent, then the bisequence $(\{a_n\}, \{b_n\})$ is said to be convergent. $(\{a_n\}, \{b_n\})$ is a Cauchy sequences if $\lim_{n,m\to\infty} d(a_n, b_m) = 0$.

A bipolar metric space is called complete, if every Cauchy bisequence is convergent, hence biconvergent.

Definition 1.4 ([1])**.** Let (A, B, d) be a bipolar metric space, $F : \left(A^2, B^2\right) \rightrightarrows (A, B)$ be a covariant mapping. If $F(a, b) = a$ and $F(b, a) = b$ for $(a, b) \in A^2 \cup B^2$ then (a, b) is called a coupled fixed point of F.

2. Main Results

In this section, we give some common coupled fixed point theorems for three covariant mappings satisfying various new caristi type contractive conditions in bipolar metric spaces.

Definition 2.1. Let (A,B,d) be a bipolar metric space, $F:\left(A^2,B^2\right)\rightrightarrows(A,B)$ and $f:(A,B)\rightrightarrows(A,B)$ be two covariant mappings. An element (a,b) is said to be a coupled coincident point of F and f if $F\left(a,b\right)=fa$ and $F\left(b,a\right)=fb$.

Definition 2.2. Let (A,B,d) be a bipolar metric space, $F:\left(A^2,B^2\right)\rightrightarrows(A,B)$ and $f:(A,B)\rightrightarrows(A,B)$ be two covariant mappings. An element (a,b) is said to be a common coupled fixed point of F and f. If $F\left(a,b\right)=fa=a$ and $F\left(b,a\right)=fb=b$.

Definition 2.3. Let (A,B,d) be a bipolar metric space, $F:\left(A^2,B^2\right)\rightrightarrows(A,B)$ and $f:(A,B)\rightrightarrows(A,B)$ be two covariant mappings. Then F and f are called ω-compatible if $f\left(F\left(a,b\right)\right)=F\left(fa,fb\right)$ and $f\left(F\left(b,a\right)\right)=F\left(fb,fa\right)$ whenever $F\left(a,b\right)=fa$ and $F\left(b,a\right)=fb$.

Theorem 2.4. *Let (A,B,d) be a bipolar metric space. Suppose that $F:\left(A^2,B^2\right)\rightrightarrows(A,B)$ and $f,g:(A,B)\rightrightarrows(A,B)$ be covariant mappings satisfying*

$$\begin{aligned} d\left(F(a,b),F(p,q)\right) \leq\ & \psi\left(\alpha(fa)\right)\alpha(fa)-\alpha(F(a,b)) \\ & +\psi\left(\beta(fb)\right)\beta(fb)-\beta(F(b,a)) \\ & +\psi\left(\gamma(gp)\right)\gamma(gp)-\gamma(F(p,q)) \\ & +\psi\left(\varphi(gq)\right)\varphi(gq)-\varphi(F(q,p)). \end{aligned} \tag{2.1}$$

for all $a,b\in A$ and $p,q\in B$, where $\alpha,\beta,\gamma,\varphi:A\cup B\to[0,\infty)$ are lower semi-continuous functions and $\psi:(-\infty,\infty)\to(0,1)$ is a continuous function.

a) *$F\left(A^2\cup B^2\right)\subseteq g(A\cup B)$ and $F\left(A^2\cup B^2\right)\subseteq f(A\cup B)$.*
b) *Either (F,f) or (F,g) are ω-compatible.*
c) *Either $f(A\cup B)$ or $g(A\cup B)$ is complete.*

Then the mappings $F:A^2\cup B^2\to A\cup B$ and $f,g:A\cup B\to A\cup B$ have a unique common coupled fixed point of the form (u,u).

Proof. Let $a_0,b_0\in A$ and $p_0,q_0\in B$. From (a), we construct the bisequences $(\{a_{2n}\},\{p_{2n}\})$, $(\{b_{2n}\},\{q_{2n}\})$, $(\{\omega_{2n}\},\{\chi_{2n}\})$ and $(\{\xi_{2n}\},\{\kappa_{2n}\})$ in (A,B) as

$$\begin{aligned} F\left(a_{2n},b_{2n}\right)&=fa_{2n+1}=\omega_{2n},\\ F\left(b_{2n},a_{2n}\right)&=fb_{2n+1}=\xi_{2n},\\ F\left(a_{2n+1},b_{2n+1}\right)&=ga_{2n+2}=\omega_{2n+1}, \end{aligned}$$

$$
\begin{aligned}
&F\left(b_{2n+1}, a_{2n+1}\right) = g b_{2n+2} = \xi_{2n+1},\\
&F\left(p_{2n}, q_{2n}\right) = f p_{2n+1} = \chi_{2n},\\
&F\left(q_{2n}, p_{2n}\right) = f q_{2n+1} = \kappa_{2n},\\
&F\left(p_{2n+1}, q_{2n+1}\right) = g p_{2n+2} = \chi_{2n+1},\\
&F\left(q_{2n+1}, p_{2n+1}\right) = g q_{2n+2} = \kappa_{2n+1},
\end{aligned}
$$

for $n = 0, 1, 2, \ldots$.
Now from (2.1), we have

(2.2)
$$
\begin{aligned}
d\left(\omega_{2n}, \chi_{2n+1}\right) &= d\left(F\left(a_{2n}, b_{2n}\right), F\left(p_{2n+1}, q_{2n+1}\right)\right)\\
&\leq \psi\left(\alpha(f a_{2n})\right)\alpha(f a_{2n}) - \alpha(F(a_{2n}, b_{2n}))\\
&\quad + \psi\left(\beta(f b_{2n})\right)\beta(f b_{2n}) - \beta(F(b_{2n}, a_{2n}))\\
&\quad + \psi\left(\gamma(g p_{2n+1})\right)\gamma(g p_{2n+1}) - \gamma(F(p_{2n+1}, q_{2n+1}))\\
&\quad + \psi\left(\varphi(g q_{2n+1})\right)\varphi(g q_{2n+1}) - \varphi(F(q_{2n+1}, F(p_{2n+1}))\\
&< \alpha(\omega_{2n-1}) - \alpha(\omega_{2n}) + \beta(\xi_{2n-1}) - \beta(\xi_{2n})\\
&\quad + \gamma(\chi_{2n}) - \gamma(\chi_{2n+1}) + \varphi(\kappa_{2n}) - \varphi(\kappa_{2n+1}),
\end{aligned}
$$

and

(2.3)
$$
\begin{aligned}
d\left(\xi_{2n}, \kappa_{2n+1}\right) &= d\left(F\left(b_{2n}, a_{2n}\right), F\left(q_{2n+1}, p_{2n+1}\right)\right)\\
&\leq \psi\left(\alpha(f b_{2n})\right)\alpha(f b_{2n}) - \alpha(F(b_{2n}, a_{2n}))\\
&\quad + \psi\left(\beta(f a_{2n})\right)\beta(f a_{2n}) - \beta(F(a_{2n}, b_{2n}))\\
&\quad + \psi\left(\gamma(g q_{2n+1})\right)\gamma(g q_{2n+1}) - \gamma(F(q_{2n+1}, p_{2n+1}))\\
&\quad + \psi\left(\varphi(g p_{2n+1})\right)\varphi(g p_{2n+1}) - \varphi(F(p_{2n+1}, F(q_{2n+1}))\\
&< \alpha(\xi_{2n-1}) - \alpha(\xi_{2n}) + \beta(\omega_{2n-1}) - \beta(\omega_{2n})\\
&\quad + \gamma(\kappa_{2n}) - \gamma(\kappa_{2n+1}) + \varphi(\chi_{2n}) - \varphi(\chi_{2n+1}).
\end{aligned}
$$

Combining (2.2) and (2.3), we get

(2.4)
$$
\begin{aligned}
d\left(\omega_{2n}, \chi_{2n+1}\right) + d\left(\xi_{2n}, \kappa_{2n+1}\right) &< \alpha(\omega_{2n-1}) - \alpha(\omega_{2n}) + \beta(\omega_{2n-1})\\
&\quad - \beta(\omega_{2n}) + \gamma(\chi_{2n}) - \gamma(\chi_{2n+1})\\
&\quad + \varphi(\chi_{2n}) - \varphi(\chi_{2n+1}) + \alpha(\xi_{2n-1})\\
&\quad - \alpha(\xi_{2n}) + \beta(\xi_{2n-1}) - \beta(\xi_{2n}) + \gamma(\kappa_{2n})\\
&\quad - \gamma(\kappa_{2n+1}) + \varphi(\kappa_{2n}) - \varphi(\kappa_{2n+1}),
\end{aligned}
$$

and

(2.5)
$$
\begin{aligned}
&(\alpha(\omega_{2n}) + \beta(\omega_{2n})) + (\gamma(\chi_{2n+1}) + \varphi(\chi_{2n+1})) + (\alpha(\xi_{2n}) + \beta(\xi_{2n}))\\
&\quad + (\gamma(\kappa_{2n+1}) + \varphi(\kappa_{2n+1}))
\end{aligned}
$$

$$
\begin{aligned}
&\leq \left(\psi\left(\alpha(\omega_{2n-1})\right)\alpha(\omega_{2n-1})+\psi\left(\beta(\omega_{2n-1})\right)\beta(\omega_{2n-1})\right)\\
&\quad+\left(\psi\left(\gamma(\chi_{2n})\right)\gamma(\chi_{2n})+\psi\left(\varphi(\chi_{2n})\right)\varphi(\chi_{2n}\right)\\
&\quad+\left(\psi\left(\alpha(\xi_{2n-1})\right)\alpha(\xi_{2n-1})+\psi\left(\beta(\xi_{2n-1})\right)\beta(\xi_{2n-1})\right)\\
&\quad+\left(\psi\left(\gamma(\kappa_{2n})\right)\gamma(\kappa_{2n})+\psi\left(\varphi(\kappa_{2n})\right)\varphi(\kappa_{2n})\right)\\
&< \left(\alpha(\omega_{2n-1})+\beta(\omega_{2n-1})\right)+\left(\gamma(\chi_{2n})+\varphi(\chi_{2n})\right)\\
&\quad+\left(\alpha(\xi_{2n-1})+\beta(\xi_{2n-1})\right)+\left(\gamma(\kappa_{2n})+\varphi(\kappa_{2n})\right).
\end{aligned}
$$

On the other hand

$$
\begin{aligned}
d\left(\omega_{2n+1},\chi_{2n}\right) &= d\left(F\left(a_{2n+1},b_{2n+1}\right),F\left(p_{2n},q_{2n}\right)\right)\\
&\leq \psi\left(\alpha(fa_{2n+1})\right)\alpha(fa_{2n+1})-\alpha(F(a_{2n+1},b_{2n+1}))\\
&\quad+\psi\left(\beta(fb_{2n+1})\right)\beta(fb_{2n+1})-\beta(F(b_{2n+1},a_{2n+1}))\\
&\quad+\psi\left(\gamma(gp_{2n})\right)\gamma(gp_{2n})-\gamma(F(p_{2n},q_{2n}))\\
&\quad+\psi\left(\varphi(gq_{2n})\right)\varphi(gq_{2n})-\varphi(F(q_{2n},F(p_{2n}))\\
&< \alpha(\omega_{2n})-\alpha(\omega_{2n+1})+\beta(\xi_{2n})-\beta(\xi_{2n+1})\\
&\quad+\gamma(\chi_{2n-1})-\gamma(\chi_{2n})+\varphi(\kappa_{2n-1})-\varphi(\kappa_{2n}),
\end{aligned}
\tag{2.6}
$$

and

$$
\begin{aligned}
d\left(\xi_{2n+1},\kappa_{2n}\right) &= d\left(F\left(b_{2n+1},a_{2n+1}\right),F\left(q_{2n},p_{2n}\right)\right)\\
&\leq \psi\left(\alpha(fb_{2n+1})\right)\alpha(fb_{2n+1})-\alpha(F(b_{2n+1},a_{2n+1}))\\
&\quad+\psi\left(\beta(fa_{2n+1})\right)\beta(fa_{2n+1})-\beta(F(a_{2n+1},b_{2n+1}))\\
&\quad+\psi\left(\gamma(gq_{2n})\right)\gamma(gq_{2n})-\gamma(F(q_{2n},p_{2n}))\\
&\quad+\psi\left(\varphi(gp_{2n})\right)\varphi(gp_{2n})-\varphi(F(p_{2n},F(q_{2n}))\\
&< \alpha(\xi_{2n})-\alpha(\xi_{2n+1})+\beta(\omega_{2n})-\beta(\omega_{2n+1})\\
&\quad+\gamma(\kappa_{2n-1})-\gamma(\kappa_{2n})+\varphi(\chi_{2n-1})-\varphi(\chi_{2n}).
\end{aligned}
\tag{2.7}
$$

Combining (2.6) and (2.7), we get

$$
\begin{aligned}
d\left(\omega_{2n+1},\chi_{2n}\right)+d\left(\xi_{2n+1},\kappa_{2n}\right) &< \alpha(\omega_{2n})-\alpha(\omega_{2n+1})+\beta(\omega_{2n})\\
&\quad-\beta(\omega_{2n+1})+\gamma(\chi_{2n-1})-\gamma(\chi_{2n})\\
&\quad+\varphi(\chi_{2n-1})-\varphi(\chi_{2n})+\alpha(\xi_{2n})\\
&\quad-\alpha(\xi_{2n+1})+\beta(\xi_{2n})-\beta(\xi_{2n+1})\\
&\quad+\gamma(\kappa_{2n-1})-\gamma(\kappa_{2n})+\varphi(\kappa_{2n-1})-\varphi(\kappa_{2n}),
\end{aligned}
\tag{2.8}
$$

and

$$
\begin{aligned}
&\left(\alpha(\omega_{2n+1})+\beta(\omega_{2n+1})\right)+\left(\gamma(\chi_{2n})+\varphi(\chi_{2n})\right)+\left(\alpha(\xi_{2n+1})+\beta(\xi_{2n+1})\right)\\
&\qquad+\left(\gamma(\kappa_{2n})+\varphi(\kappa_{2n})\right)
\end{aligned}
\tag{2.9}
$$

$$
\begin{aligned}
&\leq \left(\psi\left(\alpha(\omega_{2n})\right)\alpha(\omega_{2n}) + \psi\left(\beta(\omega_{2n})\right)\beta(\omega_{2n})\right)\\
&\quad + \left(\psi\left(\gamma(\chi_{2n-1})\right)\gamma(\chi_{2n-1}) + \psi\left(\varphi(\chi_{2n-1})\right)\varphi(\chi_{2n-1})\right)\\
&\quad + \left(\psi\left(\alpha(\xi_{2n})\right)\alpha(\xi_{2n}) + \psi\left(\beta(\xi_{2n})\right)\beta(\xi_{2n})\right)\\
&\quad + \left(\psi\left(\gamma(\kappa_{2n-1})\right)\gamma(\kappa_{2n-1}) + \psi\left(\varphi(\kappa_{2n-1})\right)\varphi(\kappa_{2n-1})\right)\\
&< (\alpha(\omega_{2n}) + \beta(\omega_{2n})) + (\gamma(\chi_{2n-1}) + \varphi(\chi_{2n-1}))\\
&\quad + (\alpha(\xi_{2n}) + \beta(\xi_{2n})) + (\gamma(\kappa_{2n-1}) + \varphi(\kappa_{2n-1})).
\end{aligned}
$$

Moreover,

$$
\begin{aligned}
d\left(\omega_{2n},\chi_{2n}\right) &= d\left(F\left(a_{2n},b_{2n}\right),F\left(p_{2n},q_{2n}\right)\right)\\
&\leq \psi\left(\alpha(fa_{2n})\right)\alpha(fa_{2n}) - \alpha(F(a_{2n},b_{2n}))\\
&\quad + \psi\left(\beta(fb_{2n})\right)\beta(fb_{2n}) - \beta(F(b_{2n},a_{2n}))\\
&\quad + \psi\left(\gamma(gp_{2n})\right)\gamma(gp_{2n}) - \gamma(F(p_{2n},q_{2n}))\\
&\quad + \psi\left(\varphi(gq_{2n})\right)\varphi(gq_{2n}) - \varphi(F(q_{2n},F(p_{2n}))\\
&\leq \psi\left(\alpha(\omega_{2n-1})\right)\alpha(\omega_{2n-1}) - \alpha(\omega_{2n})\\
&\quad + \psi\left(\beta(\xi_{2n-1})\right)\beta(\xi_{2n-1}) - \beta(\xi_{2n})\\
&\quad + \psi\left(\gamma(\chi_{2n-1})\right)\gamma(\chi_{2n-1}) - \gamma(\chi_{2n})\\
&\quad + \psi\left(\varphi(\kappa_{2n-1})\right)\varphi(\kappa_{2n-1}) - \varphi(\kappa_{2n})\\
&< \alpha(\omega_{2n-1}) - \alpha(\omega_{2n}) + \beta(\xi_{2n-1}) - \beta(\xi_{2n})\\
&\quad + \gamma(\chi_{2n-1}) - \gamma(\chi_{2n}) + \varphi(\kappa_{2n-1}) - \varphi(\kappa_{2n}),
\end{aligned} \tag{2.10}
$$

and

$$
\begin{aligned}
d\left(\xi_{2n},\kappa_{2n}\right) &= d\left(F\left(b_{2n},a_{2n}\right),F\left(q_{2n},p_{2n}\right)\right)\\
&\leq \psi\left(\alpha(fb_{2n})\right)\alpha(fb_{2n}) - \alpha(F(b_{2n},a_{2n}))\\
&\quad + \psi\left(\beta(fa_{2n})\right)\beta(fa_{2n}) - \beta(F(a_{2n},b_{2n}))\\
&\quad + \psi\left(\gamma(gq_{2n})\right)\gamma(gq_{2n}) - \gamma(F(q_{2n},p_{2n}))\\
&\quad + \psi\left(\varphi(gp_{2n})\right)\varphi(gp_{2n}) - \varphi(F(p_{2n},F(q_{2n}))\\
&< \alpha(\xi_{2n-1}) - \alpha(\xi_{2n}) + \beta(\omega_{2n-1}) - \beta(\omega_{2n})\\
&\quad + \gamma(\kappa_{2n-1}) - \gamma(\kappa_{2n}) + \varphi(\chi_{2n-1}) - \varphi(\chi_{2n}).
\end{aligned} \tag{2.11}
$$

Combining (2.10) and (2.11), we get

$$
\begin{aligned}
d\left(\omega_{2n},\chi_{2n}\right) + d\left(\xi_{2n},\kappa_{2n}\right) &< \alpha(\omega_{2n-1}) - \alpha(\omega_{2n}) + \beta(\omega_{2n-1}) - \beta(\omega_{2n})\\
&\quad + \gamma(\chi_{2n-1}) - \gamma(\chi_{2n}) + \varphi(\chi_{2n-1}) - \varphi(\chi_{2n})\\
&\quad + \alpha(\xi_{2n-1}) - \alpha(\xi_{2n}) + \beta(\xi_{2n-1}) - \beta(\xi_{2n})\\
&\quad + \gamma(\kappa_{2n-1}) - \gamma(\kappa_{2n}) + \varphi(\kappa_{2n-1}) - \varphi(\kappa_{2n}),
\end{aligned} \tag{2.12}
$$

and

$$\begin{aligned}
(2.13)\qquad & (\alpha(\omega_{2n}) + \beta(\omega_{2n})) + (\gamma(\chi_{2n}) + \varphi(\chi_{2n})) + (\alpha(\xi_{2n}) + \beta(\xi_{2n})) \\
&\quad + (\gamma(\kappa_{2n}) + \varphi(\kappa_{2n})) \\
&\leq (\psi(\alpha(\omega_{2n-1}))\,\alpha(\omega_{2n-1}) + \psi(\beta(\omega_{2n-1}))\,\beta(\omega_{2n-1})) \\
&\quad + (\psi(\gamma(\chi_{2n-1}))\,\gamma(\chi_{2n-1}) + \psi(\varphi(\chi_{2n-1}))\,\varphi(\chi_{2n-1})) \\
&\quad + (\psi(\alpha(\xi_{2n-1}))\,\alpha(\xi_{2n-1}) + \psi(\beta(\xi_{2n-1}))\,\beta(\xi_{2n-1})) \\
&\quad + (\psi(\gamma(\kappa_{2n-1}))\,\gamma(\kappa_{2n-1}) + \psi(\varphi(\kappa_{2n-1}))\,\varphi(\kappa_{2n-1})) \\
&< (\alpha(\omega_{2n-1}) + \beta(\omega_{2n-1})) + (\gamma(\chi_{2n-1}) + \varphi(\chi_{2n-1})) \\
&\quad + (\alpha(\xi_{2n-1}) + \beta(\xi_{2n-1})) + (\gamma(\kappa_{2n-1}) + \varphi(\kappa_{2n-1})).
\end{aligned}$$

From (2.5), (2.9) and (2.13), it follows that the bisequences

$$\begin{aligned}
&(\{\alpha(\omega_{2n})\}, \{\beta(\omega_{2n})\}), \qquad (\{\alpha(\xi_{2n})\}, \{\beta(\xi_{2n})\}), \\
&(\{\gamma(\chi_{2n})\}, \{\varphi(\chi_{2n})\}), \qquad (\{\gamma(\kappa_{2n})\}, \{\varphi(\kappa_{2n})\}),
\end{aligned}$$

and

$$\begin{aligned}
&(\{\alpha(\omega_{2n+1})\}, \{\beta(\omega_{2n+1})\}), \qquad (\{\alpha(\xi_{2n+1})\}, \{\beta(\xi_{2n+1})\}), \\
&(\{\gamma(\chi_{2n+1})\}, \{\varphi(\chi_{2n+1})\}), \qquad (\{\gamma(\kappa_{2n+1})\}, \{\varphi(\kappa_{2n+1})\}),
\end{aligned}$$

are non-increasing bisequences of non-negative real numbers. So they must converges to $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7, \lambda_8 \geq 0$, respectively.

Suppose $\lambda_1 > 0$ or $\lambda_2 > 0$ or $\lambda_3 > 0$ or $\lambda_4 > 0$ or $\lambda_5 > 0$ or $\lambda_6 > 0$ or $\lambda_7 > 0$ or $\lambda_8 > 0$. Letting $n \to \infty$ in equations (2.4), (2.8) and (2.12), we get a contradiction. Therefore,

$$\begin{aligned}
&\lim_{n\to\infty} d(\alpha(\omega_{2n}), \beta(\omega_{2n})) = \lim_{n\to\infty} d(\gamma(\chi_{2n}), \varphi(\chi_{2n})) = 0, \\
&\lim_{n\to\infty} d(\alpha(\xi_{2n}), \beta(\xi_{2n})) = \lim_{n\to\infty} d(\gamma(\kappa_{2n}), \varphi(\kappa_{2n})) = 0, \\
&\lim_{n\to\infty} d(\alpha(\omega_{2n+1}), \beta(\omega_{2n+1})) = \lim_{n\to\infty} d(\gamma(\chi_{2n+1}), \varphi(\chi_{2n+1})) = 0, \\
(2.14)\qquad &\lim_{n\to\infty} d(\alpha(\xi_{2n+1}), \beta(\xi_{2n+1})) = \lim_{n\to\infty} d(\gamma(\kappa_{2n+1}), \varphi(\kappa_{2n+1})) = 0.
\end{aligned}$$

Now, from (2.4), we have

$$\begin{aligned}
&\sum_{2n=1}^{2m} (d(\omega_{2n}, \chi_{2n+1}) + d(\xi_{2n}, \kappa_{2n+1})) \\
&\qquad = (d(\omega_1, \chi_2) + d(\xi_1, \kappa_2)) + (d(\omega_2, \chi_3) + d(\xi_2, \kappa_3)) + \cdots \\
&\qquad\quad + (d(\omega_{2m}, \chi_{2m+1}) + d(\xi_{2m}, \kappa_{2m+1})) \\
&\qquad < \alpha(\omega_0) - \alpha(\omega_1) + \beta(\omega_0) - \beta(\omega_1) + \gamma(\chi_1) - \gamma(\chi_2) + \varphi(\chi_1) \\
&\qquad\quad - \varphi(\chi_2) + \alpha(\xi_0) - \alpha(\xi_1) + \beta(\xi_0) - \beta(\xi_1) + \gamma(\kappa_1) - \gamma(\kappa_2) \\
&\qquad\quad + \varphi(\kappa_1) - \varphi(\kappa_2) + \alpha(\omega_1) - \alpha(\omega_2) + \beta(\omega_1) - \beta(\omega_2) + \gamma(\chi_2) \\
&\qquad\quad - \gamma(\chi_3) + \varphi(\chi_2) - \varphi(\chi_3) + \alpha(\xi_1) - \alpha(\xi_2) + \beta(\xi_1) - \beta(\xi_2)
\end{aligned}$$

$$
\begin{aligned}
&+\gamma(\kappa_2)-\gamma(\kappa_3)+\varphi(\kappa_2)-\varphi(\kappa_3)\\
&\vdots\\
&+\alpha(\omega_{n-1})-\alpha(\omega_n)+\beta(\omega_{n-1})-\beta(\omega_n)+\gamma(\chi_n)-\gamma(\chi_{n+1})\\
&+\varphi(\chi_n)-\varphi(\chi_{n+1})+\alpha(\xi_{n-1})-\alpha(\xi_n)+\beta(\xi_{n-1})-\beta(\xi_n)\\
&+\gamma(\kappa_n)-\gamma(\kappa_{n+1})+\varphi(\kappa_n)-\varphi(\kappa_{n+1})\\
<&\ (\alpha(\omega_0)+\beta(\omega_0))+(\gamma(\chi_1)+\varphi(\chi_1))\\
&+(\alpha(\chi_0)+\beta(\chi_0))+(\gamma(\kappa_1)+\varphi(\kappa_1)).
\end{aligned}
$$

This shows that

$$\sum_{2n=1}^{2m}\left(d\left(\omega_{2n},\chi_{2n+1}\right)+d\left(\xi_{2n},\kappa_{2n+1}\right)\right),$$

is a biconvergent series.

Similarly, we can prove

$$\sum_{2n=1}^{2m}\left(d\left(\omega_{2n+1},\chi_{2n}\right)+d\left(\xi_{2n+1},\kappa_{2n}\right)\right),$$

and

$$\sum_{2n=1}^{2m}\left(d\left(\omega_{2n},\chi_{2n}\right)+d\left(\xi_{2n},\kappa_{2n}\right)\right)$$

are biconvergent series and hence convergent.

Using the property (B_4), we get

(2.15)

$$
\begin{aligned}
d\left(\omega_{2n},\chi_{2m}\right)&\leq d\left(\omega_{2n},\chi_{2n+1}\right)+d\left(\omega_{2n+1},\chi_{2n+1}\right)+\cdots+d\left(\omega_{2m-1},\chi_{2m}\right)\\
d\left(\xi_{2n},\kappa_{2m}\right)&\leq d\left(\xi_{2n},\kappa_{2n+1}\right)+d\left(\xi_{2n+1},\kappa_{2n+1}\right)+\cdots+d\left(\xi_{2m-1},\kappa_{2m}\right),
\end{aligned}
$$

and

(2.16)

$$
\begin{aligned}
d\left(\omega_{2m},\chi_{2n}\right)&\leq d\left(\omega_{2m},\chi_{2m-1}\right)+d\left(\omega_{2m-1},\chi_{2m-1}\right)+\cdots+d\left(\omega_{2n+1},\chi_{2n}\right)\\
d\left(\xi_{2m},\kappa_{2n}\right)&\leq d\left(\xi_{2m},\kappa_{2m-1}\right)+d\left(\xi_{2m-1},\kappa_{2m-1}\right)+\cdots+d\left(\xi_{2n+1},\kappa_{2n}\right),
\end{aligned}
$$

for each $n,m\in\mathcal{N}$ with $n<m$. Then from (2.4), (2.8), (2.12),(2.15) and (2.16), we have

$$
\begin{aligned}
d\left(\omega_{2n},\chi_{2m}\right)+d\left(\xi_{2n},\kappa_{2m}\right)\leq&\ \left(d\left(\omega_{2n},\chi_{2n+1}\right)+d\left(\xi_{2n},\kappa_{2n+1}\right)\right)\\
&+\left(d\left(\omega_{2n+1},\chi_{2n+1}\right)+d\left(\xi_{2n+1},\kappa_{2n+1}\right)\right)\\
&+\cdots+\left(d\left(\omega_{2m-1},\chi_{2m-1}\right)+d\left(\xi_{2m-1},\kappa_{2m-1}\right)\right)\\
&+\left(d\left(\omega_{2m-1},\chi_{2m}\right)+d\left(\xi_{2m-1},\kappa_{2m}\right)\right)\\
<&\ \alpha(\omega_{2n-1})-\alpha(\omega_{2n})+\beta(\omega_{2n-1})-\beta(\omega_{2n})
\end{aligned}
$$

$$
\begin{aligned}
&+\gamma(\chi_{2n})-\gamma(\chi_{2n+1})+\varphi(\chi_{2n})-\varphi(\chi_{2n+1})\\
&+\alpha(\xi_{2n-1})-\alpha(\xi_{2n})+\beta(\xi_{2n-1})-\beta(\xi_{2n})\\
&+\gamma(\kappa_{2n})-\gamma(\kappa_{2n+1})+\varphi(\kappa_{2n})-\varphi(\kappa_{2n+1})\\
&+\alpha(\omega_{2n})-\alpha(\omega_{2n+1})+\beta(\omega_{2n})-\beta(\omega_{2n+1})\\
&+\gamma(\chi_{2n})-\gamma(\chi_{2n+1})+\varphi(\chi_{2n})-\varphi(\chi_{2n+1})\\
&+\alpha(\xi_{2n})-\alpha(\xi_{2n+1})+\beta(\xi_{2n})-\beta(\xi_{2n+1})\\
&+\gamma(\kappa_{2n})-\gamma(\kappa_{2n+1})+\varphi(\kappa_{2n})-\varphi(\kappa_{2n+1})\\
&+\cdots+\alpha(\omega_{2m-2})-\alpha(\omega_{2m-1})+\beta(\omega_{2m-2})\\
&-\beta(\omega_{2m-1})+\gamma(\chi_{2m-2})-\gamma(\chi_{2m-1})\\
&+\varphi(\chi_{2m-2})-\varphi(\chi_{2m-1})+\alpha(\xi_{2m-2})\\
&-\alpha(\xi_{2m-1})+\beta(\xi_{2m-2})-\beta(\xi_{2m-1})\\
&+\gamma(\kappa_{2m-2})-\gamma(\kappa_{2m-1})+\varphi(\kappa_{2m-2})\\
&-\varphi(\kappa_{2m-1})+\alpha(\omega_{2m-2})-\alpha(\omega_{2m-1})\\
&+\beta(\omega_{2m-2})-\beta(\omega_{2m-1})+\gamma(\chi_{2m-1})\\
&-\gamma(\chi_{2m})+\varphi(\chi_{2m-1})-\varphi(\chi_{2m})\\
&+\alpha(\xi_{2m-2})-\alpha(\xi_{2m-1})+\beta(\xi_{2m-2})\\
&-\beta(\xi_{2m-1})+\gamma(\kappa_{2m-1})-\gamma(\kappa_{2m})\\
&+\varphi(\kappa_{2m-1})-\varphi(\kappa_{2m})\to 0 \text{ as } n,m\to\infty.
\end{aligned}
$$

Similarly, we can prove $d(\omega_{2m},\chi_{2n})+d(\xi_{2m},\kappa_{2n})\to 0$ as $n,m\to\infty$. This shows that (ω_{2n},χ_{2m}) and (ξ_{2n},κ_{2m}) are Cauchy bisequences in (A,B).
Therefore,

$$\lim_{n\to\infty} d(\omega_{2n},\chi_{2m})=\lim_{n\to\infty} d(\xi_{2n},\kappa_{2m})=0.$$

Since $f(A\cup B)$ is a complete subspace of (A,B,d), so $\{\omega_{2n+1}\}$, $\{\chi_{2m+1}\}$, $\{\xi_{2n+1}\}$, $\{\kappa_{2m+1}\}\subseteq f(A\cup B)$ are converge in the complete bipolar metric space $(f(A),f(B),d)$. Therefore, there exist $u,v\in f(A)$ and $w,z\in f(B)$ with

(2.17)
$$\lim_{n\to\infty}\omega_{2n+1}=w,\quad \lim_{n\to\infty}\xi_{2n+1}=z,\quad \lim_{n\to\infty}\chi_{2n+1}=u,\quad \lim_{n\to\infty}\kappa_{2n+1}=v.$$

Since $f: A\cup B\to A\cup B$ and $u,v\in f(A)$, $w,z\in f(B)$, there exist $l,m\in A$, $r,s\in B$ such that $fl=u$, $fm=v$ and $fr=w$, $fs=z$. Since $\alpha,\beta,\gamma,\varphi$ are lower semi-continuous functions, hence

$$
\begin{aligned}
&\lim_{n\to\infty}\alpha(\omega_{2n})=\alpha(w), &&\lim_{n\to\infty}\gamma(\chi_{2n})=\gamma(u),\\
&\lim_{n\to\infty}\beta(\xi_{2n})=\beta(z), &&\lim_{n\to\infty}\varphi(\kappa_{2n})=\varphi(v).
\end{aligned}
$$

From (2.14), we get $\alpha(w) = \beta(z) = \gamma(u) = \varphi(v) = 0$. From (2.1) and (B_4), we have

$$
\begin{aligned}
d\left(F(l,m),w\right) &\leq d\left(F(l,m),\chi_{2n+1}\right) + d\left(\omega_{2n+1},\chi_{2n+1}\right) + d\left(\omega_{2n+1},w\right)\\
&\leq d\left(F(l,m),F(p_{2n+1},q_{2n+1})\right)\\
&\quad + d\left(\omega_{2n+1},\chi_{2n+1}\right) + d\left(\omega_{2n+1},w\right)\\
&\leq \psi(\alpha(fl))\alpha(fl) - \alpha(F(l,m)) + \psi(\beta(fm))\beta(fm)\\
&\quad - \beta(F(m,l)) + \psi(\gamma(gp_{2n+1}))\gamma(gp_{2n+1})\\
&\quad - \gamma(F(p_{2n+1},q_{2n+1})) + \psi(\varphi(gq_{2n+1}))\varphi(gq_{2n+1})\\
&\quad - \varphi(F(q_{2n+1},p_{2n+1})) + d\left(\omega_{2n+1},\chi_{2n+1}\right)\\
&\quad + d\left(\omega_{2n+1},w\right)\\
&< \alpha(w) - \alpha(F(l,m)) + \beta(z) - \beta(F(m,l))\\
&\quad + \gamma(\chi_{2n}) - \gamma(\chi_{2n+1}) + \varphi(\kappa_{2n}) - \varphi(\kappa_{2n+1})\\
&\quad + d\left(\omega_{2n+1},\chi_{2n+1}\right) + d\left(\omega_{2n+1},w\right) \to 0 \text{ as } n \to \infty.
\end{aligned}
$$

Therefore, $d\left(F(l,m),w\right) = 0$ implies $F(l,m) = w = fr$.
Similarly, we can prove $F(m,l) = z = fs$, $F(r,s) = u = fl$ and $F(s,r) = v = fm$. Since (F,f) is ω-compatible mapping, we have $F(u,v) = fu$, $F(v,u) = fv$, $F(w,z) = fw$ and $F(z,w) = fz$. We shall prove $fu = u$, $fv = v$, $fw = w$ and $fz = z$.
Now consider

$$
\begin{aligned}
d\left(fu,\chi_{2n}\right) &= d\left(F(u,v),F(p_{2n},q_{2n})\right)\\
&\leq \psi(\alpha(fu))\alpha(fu) - \alpha(F(u,v)) + \psi(\beta(fv))\beta(fv)\\
&\quad - \beta(F(v,u)) + \psi(\gamma(gp_{2n}))\gamma(gp_{2n}) - \gamma(F(p_{2n},q_{2n}))\\
&\quad + \psi(\varphi(gq_{2n}))\varphi(gq_{2n}) - \varphi(F(q_{2n},p_{2n}))\\
&< \alpha(fu) - \alpha(fu) + \beta(fv) - \beta(fv)\\
&\quad + \gamma(\chi_{2n-1}) - \gamma(\chi_{2n}) + \varphi(\kappa_{2n-1}) - \varphi(\kappa_{2n}),
\end{aligned}
$$

letting $n \to \infty$, we get

$$d\left(fu,u\right) < \gamma(u) - \gamma(u) + \varphi(v) - \varphi(v) = 0,$$

therefore, $fu = u$.

Similarly, we can show $fw = w$, $fv = v$ and $fz = z$. Therefore,

$$
\begin{aligned}
F(w,z) &= fw = w = fr = F(l,m),\\
F(u,v) &= fu = u = fl = F(r,s),\\
F(z,w) &= fz = z = fs = F(m,l),\\
F(v,u) &= fv = v = fm = F(s,r).
\end{aligned}
$$

On the other hand, from (2.17), we get

$$\begin{aligned} d\,(fl, fr) &= d\,(u, w) \\ &= d\left(\lim_{n\to\infty} \chi_{2n}, \lim_{n\to\infty} \omega_{2n}\right) \\ &= \lim_{n\to\infty} d(\omega_{2n}, \chi_{2n}) \\ &= 0, \end{aligned}$$

and

$$\begin{aligned} d\,(fm, fs) &= d\,(v, z) \\ &= d\left(\lim_{n\to\infty} \kappa_{2n}, \lim_{n\to\infty} \xi_{2n}\right) \\ &= \lim_{n\to\infty} d(\xi_{2n}, \kappa_{2n}) \\ &= 0. \end{aligned}$$

Since $F(A^2 \cup B^2) \subseteq g(A \cup B)$, so there exist $a, b \in A$ and $x, y \in B$ such that $ga = u$, $gb = v$, $gx = w$and $gy = z$.

Therefore, $F(u, v) = gx = w$, $F(v, u) = gy = z$, $F(w, z) = ga = u$ and $F(z, w) = gb = v$.

Now from (B_4) and (2.1), we have

$$\begin{aligned} d\,(u, F(x, y)) &\leq d\,(u, \chi_{2n}) + d\,(\omega_{2n}, \chi_{2n}) + d\,(\omega_{2n}, F(x, y)) \\ &\leq d\,(u, \chi_{2n}) + d\,(\omega_{2n}, \chi_{2n}) + d\,(F(a_{2n}, b_{2n}), F(x, y)) \\ &< d\,(u, \chi_{2n}) + d\,(\omega_{2n}, \chi_{2n}) + \alpha(\omega_{2n-1}) - \alpha(\omega_{2n}) \\ &\quad + \beta(\xi_{2n-1}) - \beta(\xi_{2n}) + \gamma(u) - \gamma(F(x, y) \\ &\quad + \varphi(v) - \varphi(F(y, x). \end{aligned}$$

Letting $n \to \infty$, we get

$$\begin{aligned} d\,(u, F(x, y)) &< \alpha(w) - \alpha(w) + \beta(z) - \beta(z) + \gamma(u) \\ &\qquad - \gamma(F(x, y) + \varphi(v) - \varphi(F(y, x) \\ &= 0, \end{aligned}$$

that is $d\,(u, F(x, y)) = 0$ implies $F(x, y) = u$,
therefore, $u = F(x, y) = ga$. Similarly, we prove $v = F(y, x) = gb$, $w = F(a, b) = gx$ and $z = F(b, a) = gy$. Since (F, g) is ω-compatible, so we have $F(u, v) = gu$, $F(v, u) = gv$, $F(w, z) = gw$ and $F(z, w) = gz$. Now we prove $gu = u$, $gv = v$ and $gw = w$, $gz = z$.

$$\begin{aligned} d\,(gu, \chi_{2n+1}) &= d\,(F(u, v), F(p_{2n+1}, q_{2n+1})) \\ &\leq \psi(\alpha(fu))\alpha(fu) - \alpha(F(u, v)) \\ &\quad + \psi(\beta(fv))\beta(fv) - \beta(F(v, u)) \\ &\quad + \psi(\gamma(gp_{2n+1}))\gamma(gp_{2n+1}) - \gamma(F(p_{2n+1}, q_{2n+1})) \end{aligned}$$

$$
\begin{aligned}
&+\psi(\varphi(gq_{2n+1}))\varphi(gq_{2n+1})-\varphi(F(q_{2n+1},p_{2n+1}))\\
&<\alpha(fu)-\alpha(fu)+\beta(fv)-\beta(fv)\\
&+\gamma(\chi_{2n})-\gamma(\chi_{2n+1})+\varphi(\kappa_{2n})-\varphi(\kappa_{2n+1}),
\end{aligned}
$$

letting $n \to \infty$, we get

$$
\begin{aligned}
d\,(gu,u) &< \gamma(u)-\gamma(u)+\varphi(v)-\varphi(v)\\
&=0.
\end{aligned}
$$

Therefore, $gu = u$. Similarly, we can show $gw = w$, $gv = v$ and $gz = z$.

$$
\begin{aligned}
F(w,z) &= gw = w = gx = F(a,b), \qquad & F(z,w) &= gz = z = gy = F(b,a),\\
F(u,v) &= gu = u = ga = F(x,y), \qquad & F(v,u) &= gv = v = gb = F(y,x).
\end{aligned}
$$

On the other hand, from (2.17), we get

$$
\begin{aligned}
d\,(ga,gx) &= d\,(u,w)\\
&= d\left(\lim_{n\to\infty}\chi_{2n},\lim_{n\to\infty}\omega_{2n}\right)\\
&= \lim_{n\to\infty} d(\omega_{2n},\chi_{2n})\\
&=0,
\end{aligned}
$$

and

$$
\begin{aligned}
d\,(gb,gy) &= d\,(v,z)\\
&= d\left(\lim_{n\to\infty}\kappa_{2n},\lim_{n\to\infty}\xi_{2n}\right)\\
&= \lim_{n\to\infty} d(\xi_{2n},\kappa_{2n})\\
&=0.
\end{aligned}
$$

So $u = w$ and $v = z$. Therefore, $(u,v) \in A^2 \cap B^2$ is a coupled fixed point of covariant mappings F, f and g. Now we prove the uniqueness. We begin by taking $(u^*,v^*) \in A^2 \cup B^2$ as another fixed point of F, f and g. If $(u^*,v^*) \in A^2$, then we have

$$
\begin{aligned}
d\,(u,u^*) &= d(F(u,v),F(u^*,v^*))\\
&\leq \psi(\alpha(fu))\alpha(fu)-\alpha(F(u,v))+\psi(\beta(fv))\beta(fv)\\
&\quad -\beta(F(v,u))+\psi(\gamma(gu^*))\gamma(gu^*)-\gamma(F(u^*,v^*))\\
&\quad +\psi(\varphi(gv^*))\varphi(gv^*)-\varphi(F(v^*,u^*))\\
&< \alpha(u)-\alpha(u)+\beta(v)-\beta(v)+\gamma(u^*)-\gamma(u^*)+\varphi(v^*)-\varphi(v^*)\\
&=0.
\end{aligned}
$$

Therefore, $d\,(u,u^*) = 0$ implies $u = u^*$. Similarly, we can prove that $v = v^*$. Similarly, if $(u^*,v^*) \in B^2$, then we have $u = u^*$ and $v = v^*$.

Then $(u,v) \in A^2 \cap B^2$ is unique coupled fixed point of covariant mappings F, f and g. Finally we will show $u = v$.

$$\begin{aligned}
d(u,v) &= d(F(u,v), F(v,u)) \\
&\le \psi(\alpha(fu))\alpha(fu) - \alpha(F(u,v)) + \psi(\beta(fv))\beta(fv) - \beta(F(v,u)) \\
&\quad + \psi(\gamma(gv))\gamma(gv) - \gamma(F(v,u)) + \psi(\varphi(gu))\varphi(gu) - \varphi(F(u,v)) \\
&< \alpha(u) - \alpha(u) + \beta(v) - \beta(v) + \gamma(v) - \gamma(v) + \varphi(u) - \varphi(u) \\
&= 0.
\end{aligned}$$

Therefore, $d(u,v) = 0 \Rightarrow u = v$. Hence u is the common fixed point of F, f, g. □

Corollary 2.5. *Let (A,B,d) be a bipolar metric space. Suppose that $F : (A^2, B^2) \rightrightarrows (A,B)$ and $f : (A,B) \rightrightarrows (A,B)$ are covariant mappings satisfying*

$$\begin{aligned}
d(F(a,b), F(p,q)) \le\ & \psi(\alpha(fa))\,\alpha(fa) - \alpha(F(a,b)) \\
&+ \psi(\beta(fb))\,\beta(fb) - \beta(F(b,a)) \\
&+ \psi(\gamma(fp))\,\gamma(fp) - \gamma(F(p,q)) \\
&+ \psi(\varphi(fq))\,\varphi(fq) - \varphi(F(q,p)),
\end{aligned} \tag{2.18}$$

for all $a,b \in A$ and $p,q \in B$, where $\alpha,\beta,\gamma,\varphi : A \cup B \to [0,\infty)$ are lower semi-continuous functions and $\psi : (-\infty,\infty) \to (0,1)$ is a continuous function.

a_1) *$F(A^2 \cup B^2) \subseteq f(A \cup B)$.*
b_1) *(F,f) is ω-compatible.*
c_1) *$f(A \cup B)$ is complete.*

Then the mappings $F : A^2 \cup B^2 \to A \cup B$ and $f : A \cup B \to A \cup B$ have a unique common coupled fixed point of the form (u,u).

Example 2.6. Let $A = (1,\infty)$ and $B = [-1,1]$. Define $d : A \times B \to [0,+\infty)$ as $d(a,b) = |a^2 - b^2|$, for all $(a,b) \in (A,B)$. Then the triple (A,B,d) is a Bipolar-metric space. Suppose $F : A^2 \cup B^2 \to A \cup B$ is a covariant mapping defined as

$$F(a,b) = \begin{cases} \frac{1}{3}(a-b), & a \ge b, \\ 0, & a < b, \end{cases}$$

and define two covariant mappings $f, g : A \cup B \to A \cup B$ as $ga = \frac{1}{3}a$ and $fa = \frac{1}{2}a$. Let $\alpha,\beta,\gamma,\varphi : A \cup B \to [0,+\infty)$ be lower semi-continuous mappings defined as

$$\alpha(w) = \begin{cases} 1, & \text{if } w > 0 \\ 0, & \text{otherwise} \end{cases} \qquad \beta(x) = \begin{cases} 2, & \text{if } x > 0 \\ 0, & \text{otherwise} \end{cases}$$

$$\gamma(y) = \begin{cases} 4, & \text{if } y > 0 \\ 0, & \text{otherwise} \end{cases} \qquad \varphi(z) = \begin{cases} 6, & \text{if } z > 0 \\ 0, & \text{otherwise} \end{cases}$$

and define $\psi : (-\infty, +\infty) \to (0, 1)$ as

$$\psi(t) = \begin{cases} \frac{2}{3}, & \text{if } t > 0, \\ 0, & \text{if } t < 0, \end{cases}$$

then clearly, F, f and g satisfied all the conditions of Theorem 2.4. Then F, f and g have unique common coupled fixed point.

Theorem 2.7. *Let (A, B, d) be a bipolar metric space. Suppose that $F : (A^2, B^2) \rightrightarrows (A, B)$ and $f, g : (A, B) \rightrightarrows (A, B)$ are covariant mappings satisfying*

(2.19)
$$\begin{aligned} d\left(F(a,b), F(p,q)\right) \leq\ & \alpha\left(\psi(fa, gp)\right)\psi(fa, gp) - \psi(F(a,b), F(p,q)) \\ & + \alpha\left(\phi(fb, gq)\right)\phi(fb, gq) - \phi(F(b,a), F(q,p)), \end{aligned}$$

for all $a, b \in A$ and $p, q \in B$, where $\psi, \phi : (A \times B) \cup (B \times A) \to [0, \infty)$ are lower semi-continuous functions and $\alpha : (-\infty, \infty) \to (0, 1)$ is a continuous function.

- *a_2) $F\left(A^2 \cup B^2\right) \subseteq g(A \cup B)$ and $F\left(A^2 \cup B^2\right) \subseteq f(A \cup B)$.*
- *b_2) Either (F, f) or (F, g) is ω-compatible.*
- *c_2) Either $f(A \cup B)$ or $g(A \cup B)$ is complete.*

Then the mappings $F : A^2 \cup B^2 \to A \cup B$ and $f, g : A \cup B \to A \cup B$ have a unique common coupled fixed point of the form (u, u).

Corollary 2.8. *Let (A, B, d) be a bipolar metric space. Suppose that $F : (A^2, B^2) \rightrightarrows (A, B)$ and $f : (A, B) \rightrightarrows (A, B)$ are covariant mappings satisfying*

(2.20)
$$\begin{aligned} d\left(F(a,b), F(p,q)\right) \leq\ & \alpha\left(\psi(fa, fp)\right)\psi(fa, fp) - \psi(F(a,b), F(p,q)) \\ & + \alpha\left(\phi(fb, fq)\right)\phi(fb, fq) - \phi(F(b,a), F(q,p)), \end{aligned}$$

for all $a, b \in A$ and $p, q \in B$, where $\psi, \phi : (A \times B) \cup (B \times A) \to [0, \infty)$ are lower semi-continuous functions and $\alpha : (-\infty, \infty) \to (0, 1)$ is a continuous function. Furthermore, assume

- *a_3) $F\left(A^2 \cup B^2\right) \subseteq f(A \cup B)$;*
- *b_3) (F, f) is ω-compatible;*
- *c_3) $f(A \cup B)$ is complete.*

Then the mappings $F : A^2 \cup B^2 \to A \cup B$ and $f : A \cup B \to A \cup B$ have a unique common coupled fixed point of the form (u, u).

Example 2.9. Let $U_m(R)$ and $L_m(R)$ be the set of all $m\times m$ upper and lower triangular matrices over R. Define $d : U_m(R) \times L_m(R) \to [0,\infty)$ as

$$d(P,Q) = \sum_{i,j=1}^{m} |p_{ij} - q_{ij}|,$$

for all $P = (p_{ij})_{m\times m} \in U_m(R)$ and $Q = (q_{ij})_{m\times m} \in L_m(R)$. Then obviously $(U_m(R), L_m(R), d)$ is a Bipolar-metric space.
Define $F : A^2 \cup B^2 \to A \cup B$ as $F(P,Q) = (\frac{p_{ij}-q_{ij}}{5})_{m\times m}$ where $(P = (p_{ij})_{m\times m}, Q = (q_{ij})_{m\times m}) \in U_m(R)^2 \cup L_m(R)^2$ and define $f,g : A\cup B \to A\cup B$ as $f(P) = \frac{1}{2}(p_{ij})_{m\times m}$ and $g(P) = (p_{ij})_{m\times m}$. Let $\psi, \phi : U_m(R)^2 \cup L_m(R)^2 \to [0,\infty)$ be lower semi-continuous mappings defined by

$$\psi(P,Q) = \sum_{i,j=1}^{m} |p_{ij} + q_{ij}|, \qquad \phi(C,D) = \frac{1}{2}\sum_{i,j=1}^{m} |c_{ij} + d_{ij}|.$$

Define $\alpha : (-\infty,+\infty) \to (0,1)$ as

$$\alpha(t) = \begin{cases} \frac{2}{3}, & \text{if } t > 0,\\ 0, & \text{if } t < 0,\end{cases}$$

then clearly, F, f, g satisfy all the conditions of Theorem 2.7 and $(O_{m\times m}, O_{m\times m})$ is the unique coupled fixed point.

Definition 2.10. Let (A,B,d) be a bipolar metric space and $F : (A\times B, B\times A) \rightrightarrows (A,B)$ be a covariant mapping. If $F(a,p) = a$ and $F(p,a) = p$ for $a\in A, p\in B$ then (a,p) is called a coupled fixed point of F.

Theorem 2.11. *Let (A,B,d) be a bipolar metric space and suppose that $F : (A\times B, B\times A) \rightrightarrows (A,B)$ and $f,g : (A,B) \rightrightarrows (A,B)$ are covariant mappings satisfying*

(2.21)
$$\begin{aligned} d\left(F(a,p),F(b,q)\right) \le{} & \psi\left(\alpha(fa)\right)\alpha(fa) - \alpha(F(a,p)) + \psi\left(\beta(fp)\right)\beta(fp)\\ & - \beta(F(p,a)) + \psi\left(\gamma(gb)\right)\gamma(gb) - \gamma(F(b,q))\\ & + \psi\left(\varphi(gq)\right)\varphi(gq) - \varphi(F(q,b)), \end{aligned}$$

for all $a,b\in A$ and $p,q\in B$, where $\alpha,\beta,\gamma,\varphi : A\cup B \to [0,\infty)$ are lower semi-continuous functions and $\psi : (-\infty,\infty) \to (0,1)$ is a continuous function. Furthermore, assume

- a_4) *$F((A\times B)\cup(B\times A)) \subseteq g(A\cup B)$ and $F((A\times B)\cup(B\times A)) \subseteq f(A\cup B)$.*
- b_4) *Either (F,f) or (F,g) is ω-compatible.*
- c_4) *Either $f(A\cup B)$ or $g(A\cup B)$ is complete.*

Then the mappings $F : (A \times B) \cup (B \times A) \to A \cup B$ *and* $f, g : A \cup B \to A \cup B$ *have a unique common fixed point of the form* (u, u).

Theorem 2.12. *Let* (A, B, d) *be a bipolar metric space and suppose that* $F : (A \times B, B \times A) \rightrightarrows (A, B)$ *and* $f, g : (A, B) \rightrightarrows (A, B)$ *are covariant mappings satisfying*

(2.22)
$$\begin{aligned} d\left(F(a,p), F(q,b)\right) \leq \alpha\left(\psi(fa, gp)\right)\psi(fa, gp) - \psi(F(a,p), F(q,b)) \\ + \alpha\left(\phi(fq, gb)\right)\phi(fq, gb) - \phi(F(p,a), F(q,b)), \end{aligned}$$

for all $a, b \in A$ *and* $p, q \in B$, *where* $\psi, \phi : (A \times B) \cup (B \times A) \to [0, \infty)$ *are lower semi-continuous functions and* $\alpha : (-\infty, \infty) \to (0, 1)$ *is a continuous function. Furthermore, assume*

- a_5) $F\left((A \times B) \cup (B \times A)\right) \subseteq g(A \cup B)$ *and* $F\left((A \times B) \cup (B \times A)\right) \subseteq f(A \cup B)$.
- b_5) *Either* (F, f) *or* (F, g) *is* ω*-compatible.*
- c_5) *Either* $f(A \cup B)$ *or* $g(A \cup B)$ *is complete.*

Then the mappings $F : (A \times B) \cup (B \times A) \to A \cup B$ *and* $f, g : A \cup B \to A \cup B$ *have a unique common fixed point of the form* (u, u).

2.1. **Application to Homotopy.**

Theorem 2.13. *Let* (A, B, d) *be a complete bipolar metric space,* (U, V) *be an open subset of* (A, B) *and* $(\overline{U}, \overline{V})$ *be a closed subset of* (A, B) *such that* $(U, V) \subseteq (\overline{U}, \overline{V})$. *Suppose* $H : (\overline{U}^2 \cup \overline{V}^2) \times [0, 1] \to A \cup B$ *is an operator with satisfying the following conditions:*

(2.23)
$$\begin{aligned} d\left(H(u,v,\kappa), H(x,y,\kappa)\right) \leq \alpha(\psi(u,x))\psi(u,x) - \psi(H(u,v,\kappa), H(x,y,\kappa)) \\ + \alpha(\phi(v,y))\psi(v,y) - \psi(H(v,u,\kappa), H(y,x,\kappa)), \end{aligned}$$

for all $u, v \in \overline{U}$, $x, y \in \overline{V}$ *and* $\kappa \in [0, 1]$, *where* $\psi, \phi : (A \times B) \cup (B \times A) \to [0, \infty)$ *are lower semi-continuous functions and* $\alpha : (-\infty, +\infty) \to (0, 1)$ *is a continuous. Furthermore, assume*

- A) $u \neq H(u, v, \kappa)$ *and* $v \neq H(v, u, \kappa)$ *for each* $u, v \in \partial U \cup \partial V$ *and* $\kappa \in [0, 1]$ *(Here* $\partial U \cup \partial V$ *is boundary of* $U \cup V$ *in* $A \cup B$*)*
- B) $\exists\ M \geq 0 \ni$, $d(H(u, v, \kappa), H(x, y, \zeta)) \leq M|\kappa - \zeta|$,

for every $u, v \in \overline{U}$, $x, y \in \overline{V}$ *and* $\kappa, \zeta \in [0, 1]$.

Then $H(., 0)$ *has a fixed point* $\Leftrightarrow$ $H(., 1)$ *has a fixed point.*

Proof. Set

$$X = \left\{\kappa \in [0,1] : u = H(u,v,\kappa), v = H(v,u,\kappa) \text{ for some } (u,v) \in U^2 \cup V^2\right\}.$$

$Y = \left\{\zeta \in [0,1] : x = H(x,y,\zeta), y = H(y,x,\zeta) \text{ for some } (x,y) \in U^2 \cup V^2\right\}.$
Since $H(.,0)$ has a fixed point in $U^2 \cup V^2$, so $(0,0) \in X^2 \cap Y^2$.

Now, we show that $X^2 \cap Y^2$ is both closed and open in $[0,1]$ and hence by the connectedness $X = Y = [0,1]$. Let $(\{\kappa_n\}_{n=1}^{\infty}, \{\zeta_n\}_{n=1}^{\infty}) \subseteq (X,Y)$ with $(\kappa_n, \zeta_n) \to (\kappa, \zeta) \in [0,1]$ as $n \to \infty$.

We must show that $(\kappa, \zeta) \in X^2 \cap Y^2$. Since $(\kappa_n, \zeta_n) \in (X,Y)$ for $n = 0,1,2,3,\ldots$, there exist bisequences $(u_n, x_n), (v_n, y_n)$ with

$$u_{n+1} = H(u_n, v_n, \kappa_n), \qquad v_{n+1} = H(v_n, u_n, \kappa_n),$$

and

$$x_{n+1} = H(x_n, y_n, \zeta_n), \qquad y_{n+1} = H(y_n, x_n, \zeta_n).$$

Consider

(2.24)
$$\begin{aligned} d(u_n, x_{n+1}) &= d(H(u_{n-1}, v_{n-1}, \kappa_{n-1}), H(x_n, y_n, \zeta_n)) \\ &\leq \alpha(\psi(u_{n-1}, x_n))\psi(u_{n-1}, x_n) \\ &\quad - \psi(H(u_{n-1}, v_{n-1}, \kappa_{n-1}), H(x_n, y_n, \zeta_n)) \\ &\quad + \alpha(\phi(v_{n-1}, y_n))\phi(v_{n-1}, y_n) \\ &\quad - \phi(H(v_{n-1}, u_{n-1}, \kappa_{n-1}), H(y_n, x_n, \zeta_n)) \\ &< \psi(u_{n-1}, x_n) - \psi(H(u_{n-1}, v_{n-1}, \kappa_{n-1}), H(x_n, y_n, \zeta_n)) \\ &\quad + \phi(v_{n-1}, y_n) - \phi(H(v_{n-1}, u_{n-1}, \kappa_{n-1}), H(y_n, x_n, \zeta_n)) \\ &= \psi(u_{n-1}, x_n) - \psi(u_n, x_{n+1}) + \phi(v_{n-1}, y_n) - \phi(v_n, y_{n+1}), \end{aligned}$$

and

(2.25)
$$\begin{aligned} d(v_n, y_{n+1}) &= d(H(v_{n-1}, u_{n-1}, \kappa_{n-1}), H(y_n, x_n, \zeta_n)) \\ &\leq \alpha(\psi(v_{n-1}, y_n))\psi(v_{n-1}, y_n) \\ &\quad - \psi(H(v_{n-1}, u_{n-1}, \kappa_{n-1}), H(y_n, x_n, \zeta_n)) \\ &\quad + \alpha(\phi(u_{n-1}, x_n))\phi(u_{n-1}, x_n) \\ &\quad - \phi(H(u_{n-1}, v_{n-1}, \kappa_{n-1}), H(x_n, y_n, \zeta_n)) \\ &< \psi(v_{n-1}, y_n) - \psi(H(v_{n-1}, u_{n-1}, \kappa_{n-1}), H(y_n, x_n, \zeta_n)) \\ &\quad + \phi(u_{n-1}, x_n) - \phi(H(u_{n-1}, v_{n-1}, \kappa_{n-1}), H(x_n, y_n, \zeta_n)) \\ &= \psi(v_{n-1}, y_n) - \psi(v_n, y_{n+1}) + \phi(u_{n-1}, x_n) - \phi(u_n, x_{n+1}). \end{aligned}$$

Combining (2.24) and (2.25), we have

$$\begin{aligned} d(u_n, x_{n+1}) + d(v_n, y_{n+1}) &< \psi(u_{n-1}, x_n) - \psi(u_n, x_{n+1}) + \phi(v_{n-1}, y_n) \\ &\quad - \phi(v_n, y_{n+1}) + \psi(v_{n-1}, y_n) - \psi(v_n, y_{n+1}) \\ &\quad + \phi(u_{n-1}, x_n) - \phi(u_n, x_{n+1}), \end{aligned}$$

letting $n \to \infty$, we get

$$\begin{aligned}\lim_{n\to\infty}(d(u_n, x_{n+1}) + d(v_n, y_{n+1})) \leq &\lim_{n\to\infty}(\psi(u_{n-1}, x_n) - \psi(u_n, x_{n+1})) \\ &+ \lim_{n\to\infty}(\phi(v_{n-1}, y_n) - \phi(v_n, y_{n+1})) \\ &+ \lim_{n\to\infty}(\psi(v_{n-1}, y_n) - \psi(v_n, y_{n+1})) \\ &+ \lim_{n\to\infty}(\phi(u_{n-1}, x_n) - \phi(u_n, x_{n+1})).\end{aligned}$$

It follows $\lim_{n\to\infty}(d(u_n, x_{n+1}) + d(v_n, y_{n+1})) = 0$.

Similarly, we can show $\lim_{n\to\infty}(d(u_{n+1}, x_n) + d(v_{n+1}, y_n)) = 0$ and $\lim_{n\to\infty}(d(u_n, x_n) + d(v_n, y_n)) = 0$ for each $n, m \in \mathcal{N}, n < m$. Using the property (B_4), we have

$$\begin{aligned}d(u_n, x_m) + d(v_n, y_m) \leq &(d(u_n, x_{n+1}) + d(v_n, y_{n+1})) + (d(u_{n+1}, x_{n+1}) \\ &+ d(v_{n+1}, y_{n+1})) + \cdots + (d(u_{m-1}, x_{m-1}) \\ &+ d(v_{m-1}, y_{m-1})) + (d(u_{m-1}, x_m) + d(v_{m-1}, y_m)) \\ \leq &\psi(u_{n-1}, x_n) - \psi(u_n, x_{n+1}) + \phi(v_{n-1}, y_n) \\ &- \phi(v_n, y_{n+1}) + M|\kappa_{n+1} - \zeta_{n+1}| + \cdots \\ &+ M|\kappa_{m-1} - \zeta_{m-1}| + \psi(u_{m-2}, x_{m-1}) \\ &- \psi(u_{m-1}, x_m) + \phi(v_{m-2}, y_{m-1}) \\ &- \phi(v_{m-1}, y_m) \to 0 \text{ as } n, m \to \infty.\end{aligned}$$

It follows

$$\lim_{n\to\infty}(d(u_n, x_m) + d(v_n, y_m)) = 0.$$

Similarly, we can show

$$\lim_{n\to\infty}(d(u_m, x_n) + d(v_m, y_n)) = 0.$$

Therefore, (u_n, x_n) and (v_n, y_n) are Cauchy bisequences in (U, V). By completeness, there exist $\xi, \nu \in U$ and $\delta, \eta \in V$ with

$$\lim_{n\to\infty} u_n = \delta, \quad \lim_{n\to\infty} v_n = \eta, \quad \lim_{n\to\infty} x_n = \xi, \quad \lim_{n\to\infty} y_n = \nu. \tag{2.26}$$

Now consider

$$\begin{aligned}d(H(\xi, \nu, \kappa), \delta) &\leq d(H(\xi, \nu, \kappa), x_{n+1}) + d(u_{n+1}, x_{n+1}) + d(u_{n+1}, \delta) \\ &\leq d(H(\xi, \nu, \kappa), H(x_n, y_n, \zeta_n)) \\ &\quad + d(H(u_n, v_n, \kappa), H(x_n, y_n, \zeta)) + d(u_{n+1}, \delta) \\ &< \psi(\xi, x_n) - \psi(\xi, x_n) + \phi(\nu, y_n) - \phi(\nu, y_n) \\ &\quad + M|\kappa_n - \zeta_n| + d(u_{n+1}, \delta) \to 0 \text{ as } n \to \infty.\end{aligned}$$

It follows $d(H(\xi, \nu, \kappa), \delta) = 0$, which implies $H(\xi, \nu, \kappa) = \delta$. Similarly, we get $H(\nu, \xi, \kappa) = \eta$, $H(\delta, \eta, \zeta) = \xi$ and $H(\eta, \delta, \zeta) = \nu$.

On the other hand, from (2.26), we get

$$d(\xi,\delta) = d(\lim_{n\to\infty} x_n, \lim_{n\to\infty} u_n) = \lim_{n\to\infty} d(u_n,x_n) = 0,$$

$$d(\nu,\eta) = d(\lim_{n\to\infty} y_n, \lim_{n\to\infty} v_n) = \lim_{n\to\infty} d(v_n,y_n) = 0.$$

Therefore, $\xi = \delta$ and $\nu = \eta$ and hence $\kappa = \zeta$. Thus $(\kappa,\zeta) \in X^2 \cap Y^2$. Clearly $X^2 \cap Y^2$ is closed in [0, 1]. Let $(\kappa_0,\zeta_0) \in (X,Y)$, then there exist bisequences $(u_0,x_0),(v_0,y_0)$ with $u_0 = H(u_0,v_0,\kappa_0)$, $v_0 = H(v_0,u_0,\kappa_0)$, $x_0 = H(x_0,y_0,\zeta_0)$ and $y_0 = H(y_0,x_0,\zeta_0)$. Since $U^2 \cup V^2$ is open, then there exists $r > 0$ such that $X_d(u_0,r) \subseteq U^2 \cup V^2$, $X_d(v_0,r) \subseteq U^2 \cup V^2$, $X_d(x_0,r) \subseteq U^2 \cup V^2$ and $X_d(y_0,r) \subseteq U^2 \cup V^2$. Choose $\kappa \in (\zeta_0 - \epsilon, \zeta_0 + \epsilon)$ and $\zeta \in (\kappa_0 - \epsilon, \kappa_0 + \epsilon)$ such that $|\kappa - \zeta_0| \leq \frac{1}{M^n} < \frac{\epsilon}{2}$, $|\zeta - \kappa_0| \leq \frac{1}{M^n} < \frac{\epsilon}{2}$ and $|\kappa_0 - \zeta_0| \leq \frac{1}{M^n} < \frac{\epsilon}{2}$.

Then we have

$$\begin{aligned} x &\in \overline{B_{X\cup Y}(u_0,r)} = \{x, x_0 \in V / d(u_0,x) \leq r + d(u_0,x_0)\}, \\ y &\in \overline{B_{X\cup Y}(v_0,r)} = \{y, y_0 \in V / d(v_0,y) \leq r + d(v_0,y_0)\}, \end{aligned}$$

and

$$\begin{aligned} u &\in \overline{B_{X\cup Y}(r,x_0)} = \{u, u_0 \in U / d(u,x_0) \leq r + d(u_0,x_0)\}, \\ v &\in \overline{B_{X\cup Y}(r,y_0)} = \{v, v_0 \in U / d(v,y_0) \leq r + d(v_0,y_0)\}. \end{aligned}$$

Also

$$\begin{aligned} d(H(u,v,\kappa),x_0) &= d(H(u,v,\kappa),H(x_0,y_0,\zeta_0)) \\ &\leq d(H(u,v,\kappa),H(x,y,\zeta_0)) + d(H(u_0,v_0,\kappa),H(x,y,\zeta_0)) \\ &\quad + d(H(u_0,v_0,\kappa),H(x_0,y_0,\zeta_0)) \\ &< \frac{2}{M^{n-1}} + \psi(u_0,x) - \psi(H(u_0,v_0,\kappa),H(x,y,\zeta_0)) \\ &\quad + \phi(v_0,y) - \phi(H(v_0,u_0,\kappa),H(y,x,\zeta_0)). \end{aligned}$$

Letting $n \to \infty$, we get

$$\begin{aligned} d(H(u,v,\kappa),x_0) &< \psi(u_0,x) - \psi(H(u_0,v_0,\kappa),H(x,y,\zeta_0)) \\ &\quad + \phi(v_0,y) - \phi(H(v_0,u_0,\kappa),H(y,x,\zeta_0)) \\ &< \psi(u_0,x) + \phi(v_0,y) \leq d(u_0,x) \leq r + d(u_0,x_0). \end{aligned}$$

Similarly, we can prove $d(H(v,u,\kappa),y_0) \leq d(v_0,y) \leq r + d(v_0,y_0)$, $d(u_0,H(x,y,\zeta),) \leq d(u,x_0) \leq r + d(u_0,x_0)$ and $d(v_0,H(y,x,\zeta),) \leq d(v,y_0) \leq r + d(v_0,y_0)$.

On the other hand

$$d(u_0,x_0) = d(H(u_0,v_0,\kappa_0),H(x_0,y_0,\zeta_0)) \leq M|\kappa_0 - \zeta_0|$$

$$\leq M\frac{1}{M^n} \leq \frac{1}{M^{n-1}} \to 0 \text{ as } n \to \infty,$$

and

$$\begin{aligned} d(v_0, y_0) &= d(H(v_0, u_0, \kappa_0), H(y_0, x_0, \zeta_0)) \\ &\leq M|\kappa_0 - \zeta_0| \\ &\leq M\frac{1}{M^n} \\ &\leq \frac{1}{M^{n-1}} \to 0 \text{ as } n \to \infty. \end{aligned}$$

So $u_0 = x_0$ and $v_0 = y_0$ and hence $\kappa = \zeta$. Thus for each fixed $\kappa \in (\kappa_0 - \epsilon, \kappa_0 + \epsilon)$, $H(., \kappa) : \overline{B_{X\cup Y}(u_0, r)} \to \overline{B_{X\cup Y}(u_0, r)}$ and $H(., \kappa) : \overline{B_{X\cup Y}(v_0, r)} \to \overline{B_{X\cup Y}(v_0, r)}$.

Then all the conditions of Theorem (2.13) are satisfied. Thus we conclude that $H(., \kappa)$ has a coupled fixed point in $\overline{U}^2 \cap \overline{V}^2$. But this must be in $U^2 \cap V^2$. Therefore, $(\kappa, \kappa) \in X^2 \cap Y^2$ for $\kappa \in (\kappa_0 - \epsilon, \kappa_0 + \epsilon)$. Hence $(\kappa_0 - \epsilon, \kappa_0 + \epsilon) \subseteq X^2 \cap Y^2$. Clearly $X^2 \cap Y^2$ is open in $[0, 1]$. To prove the reverse, we can use the similar process. □

Theorem 2.14. *Let (A, B, d) be a complete bipolar metric space, (U, V) be an open subset of (A, B) and $(\overline{U}, \overline{V})$ be a closed subset of (A, B) such that $(U, V) \subseteq (\overline{U}, \overline{V})$. Suppose $H : (\overline{U}^2 \cup \overline{V}^2) \times [0, 1] \to A \cup B$ is an operator satisfying the following conditions:*

$$\begin{aligned} d\left(H(u, v, \kappa), H(x, y, \kappa)\right) \leq\ & \psi\left(\alpha(u)\right) \alpha(u) - \alpha(H(u, v, \kappa)) \\ &+ \psi\left(\beta(v)\right) \beta(v) - \beta(H(v, u, \kappa)) \\ &+ \psi\left(\gamma(x)\right) \gamma(x) - \gamma(H(x, y, \kappa)) \\ &+ \psi\left(\varphi(y)\right) \varphi(y) - \varphi(H(y, x, \kappa)), \end{aligned} \tag{2.27}$$

for all $u, v \in \overline{U}$, $x, y \in \overline{V}$ and $\kappa \in [0, 1]$, where $\alpha, \beta, \gamma, \varphi : A\cup B \to [0, \infty)$ are lower semi-continuous functions and $\psi : (-\infty, +\infty) \to (0, 1)$ is a continuous. Furthermore, assume

C) $u \neq H(u, v, \kappa)$ and $v \neq H(v, u, \kappa)$for each $u, v \in \partial U \cup \partial V$ and $\kappa \in [0, 1]$ (Here $\partial U \cup \partial V$ is boundary of $U \cup V$ in $A \cup B$)

D) $\exists\ M \geq 0 \ni$, $d(H(u, v, \kappa), H(x, y, \zeta)) \leq M|\kappa - \zeta|$ for every $u, v \in \overline{U}$, $x, y \in \overline{V}$ and $\kappa, \zeta \in [0, 1]$.

Then $H(., 0)$ has a fixed point $\Leftrightarrow$ $H(., 1)$ has a fixed point.

3. Conclusions

In this paper, we obtain the existence and uniqueness solution for three covariant mappings in a complete bipolar metric space under new caristi type contraction with an example. Also, we have provided some applications to Homotopy theory by using fixed point theorems in bipolar metric spaces.

4. Declaration

Competing Interesting. The authors declare that they have no competing interest.

Funding. No funding.

Author contributions. All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

Acknowledgment. The authors are very thankful to the reviewers and editors for valuable comments, remarks and suggestions for improving the content of the paper.

References

1. A. Mutlu, K. Özkan and U. Gürdal, *Coupled fixed point theorems on bipolar metric spaces*, Eur. J. Pure Appl. Math., 10 (2017), pp. 655-667.
2. A. Mutlu and U. Gürdal, *Bipolar metric spaces and some fixed point theorems*, J. Nonlinear Sci. Appl., 9 (2016), pp. 5362-5373.
3. B. Samet and C. Vetro, *Coupled fixed point theorem for multivalued nonlinear contraction mappings in partially ordered metric spaces*, Nonlinear Anal., 74 (2011), pp. 4260-4268.
4. I. Ekeland, *Nonconvex minimization problems* , Bull. Am. Math. Soc., 1 (1979), pp.443-474.
5. I. Ekeland, *On the variational principle* , J. Math. Anal. Appl., 47 (1974), pp. 324-353.
6. I. Ekeland, *Sur les problems variationnels*, C.R.Acad.Sci.Paris, 275 (1972), pp. 1057-1059.
7. J. Caristi, *Fixed point Theorems for mappings satisfying inwardness condition*, Trans. Amer. Math. Soc., 215 (1976), pp. 241-251.
8. M. Abbas, M. Alikhan and S. Radenović, *Common coupled fixed point theorems in cone metric spaces for ω-compatible mappings*, Appl. Math. Comput., 217 (2010), pp. 195-202.
9. M.A. Khamsi, *Remarks on Caristi's fixed point theorem*, Nonlinear Anal.TMA., 71 (2009), pp. 227-231.
10. N.V. Can, V. Berinde, N.V. Luong and N.X. Thuan, *A Coupled coincidence point theorem in partially ordered metric spaces*, Kragujevac J. Math., 37 (2013), pp. 103-119.

11. N.V. Luong and N.X. Thuan, *Coupled fixed point in partially ordered metric spaces and application*, Nonlinear Anal., 74 (2011), pp. 983-992.
12. R.P. Agarwal and M. A. Khamsi, *Extension of Caristi's fixed point theorem to vector valued metric space*, Nonlinear Anal., 74 (2011), pp. 141-145.
13. S. Banach, *Sur les operations dans les ensembles abstraits etleur applications aux equations integrales*, Fund. Math., 3 (1922), pp. 133-181.
14. V. Lakshmikantham and Lj. Ćirić, *Coupled fixed point theorems for non linear contractions in partially ordered metric spaces*, Nonlinear Anal., 70 (2009), pp. 4341-4349.
15. W. Sintunavarat, Y.J. Ch and P. Kumam, *Coupled fixed point theorems for Nonlinear contractions without mixed monotone property*, Fixed Point Theory Appl., (2012), 2012:170.

[1] Department of Mathematics, Sagi Rama Krishnam Raju Engineering College, Bhimavaram, West Godavari - 534 204, Andhra Pradesh, India.
E-mail address: kishore.apr2@gmail.com

[2]Department of Mathematics, Dr.B.R.Ambedkar University, Srikakulam, Etcherla - 532410, Andhra Pradesh, India.
E-mail address: srinivasabagathi@gmail.com

[3]Department Faculty of Mechanical Engineering, University of Belgrade, Belgrade.
E-mail address: stojan.radenovic@tdt.edu.vn

[4] School of Mathematical Sciences, Beijing Normal University, Laboratory of Mathematics and Complex Systems, Ministry of Education, Beijing, 100875, China.
E-mail address: mathhhp@163.com