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Continuous k-Fusion Frames in Hilbert Spaces

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Abstract. The study of the ck-fusions frames shows that the emphasis on the measure spaces introduces a new idea, although some similar properties with the discrete case are raised. Moreover, due to the nature of measure spaces, we have to use new techniques for new results. Especially, the topic of the dual of frames which is important for frame applications, have been specified completely for the continuous frames. After improving and extending the concept of fusion frames and continuous frames, in this paper we introduce continuous k-fusion frames in Hilbert spaces. Similarly to the c-fusion frames, dual of continuous k-fusion frames may not be defined, we however define the Q-dual of continuous k-fusion frames. Also some new results and the perturbation of continuous k-fusion frames will be presented.

1. Introduction

After presentation of frames by Duffin and Schaeffer in [10], they have been generalized and expanded as g-frames, c-frames, fusion frames, k-frames etc (see [11, 14, 16, 22]). Frames of subspaces or fusion frames which are the main topic of this paper, were discussed by Casazza and Kutyniok in [5]. They could define frames for closed subspaces of given Hilbert spaces with the help of the orthogonal projections. Then, fusion frames were presented in the continuous case which are called c-fusion frames, by Faroughi and Ahmadi in [13] and [11]. In 2012, Găvruta presented frames for operators (or k-frames) in [16] while studying about the atomic systems with respect to a bounded operator k which had been introduced by Fechtinger and Werther in [15] and showed that the atomic systems for k are equivalent with the k-frames. Recently, Arabyani and

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Arefijamaal have presented k-fusion frames and their duals in [1] and we generate them in the continuous case.

Recently, the researchers are interested in combining different types of frames with each other, which lead to c-fusion frames, g-c-frames, c-g-frames, ck-frames, k-g-fusion frames and etc. For this reason and the importance of continuous frames in harmonic analysis, the purpose of this paper is to introduce and review some of the continuous k-fusion frames which are the result of a combination of c-frames and k-fusion frames. In this Section, we review some definitions and theorems in frames and operator theory. In Section 2, we introduce continuous k-fusion frames and the frame operator. In Section 3, we define Q-dual for continuous k-fusion frame and express some results about them. Finally, in Section 4, perturbation of these frames will be studied.

Throughout this paper, H, K, H_1 and H_2 are Hilbert spaces, $\mathbb{J}$ is a subset of $\mathbb{Z}$, μ is a positive measure and (X,μ) is a measure space, π_V is the orthogonal projection from H onto a closed subspace $V \subset H$ and $\mathcal{B}(H,K)$ is the set of all bounded and linear operators from H to K. If $H = K$ then $\mathcal{B}(H,H)$ will be denoted by $\mathcal{B}(H)$.

Definition 1.1 (frame)**.** Let $\{f_j\}_{j\in\mathbb{J}}$ be a sequence of members of H. We say that $\{f_j\}_{j\in\mathbb{J}}$ is a frame for H if there exist $0 < A \leq B < \infty$ such that for each $f \in H$

$$A\left\|f\right\|^2 \leq \sum_{j\in\mathbb{J}} \left|\langle f, f_j\rangle\right|^2 \leq B\left\|f\right\|^2.$$

Definition 1.2 (k-frame)**.** Let $\{f_j\}_{j\in\mathbb{J}}$ be a sequence of members of H and $k \in \mathcal{B}(H)$. We say that $\{f_j\}_{j\in\mathbb{J}}$ is a k-frame for H if there exist $0 < A \leq B < \infty$ such that for every $f \in H$

$$A\left\|k^* f\right\|^2 \leq \sum_{j\in\mathbb{J}} \left|\langle f, f_j\rangle\right|^2 \leq B\left\|f\right\|^2.$$

If $k = I_H$, then $\{f_j\}_{j\in\mathbb{J}}$ is a frame for H.

Definition 1.3 (c-frame)**.** Let $F : X \to H$ be a mapping such that the mapping $x \to \langle h, F(x)\rangle$ of X to $\mathbb{C}$ is measurable (i.e. weakly measurable) for each $h \in H$. F is called a c-frame for H if there exist $0 < A \leq B < \infty$ such that for each $h \in H$

$$A\left\|h\right\|^2 \leq \int_X \left|\langle h, F(x)\rangle\right|^2 \, d\mu \leq B\left\|h\right\|^2.$$

Definition 1.4 (fusion frame)**.** Let $W := \{W_j\}_{j\in\mathbb{J}}$ be a family of closed subspaces of H and $v := \{v_j\}_{j\in\mathbb{J}}$ be a family of weights (i.e. $v_j > 0$ for any $j \in \mathbb{J}$). We say that W is a fusion frame with respect to v for H if there exist $0 < A \leq B < \infty$ such that for all $h \in H$

$$A \|h\|^2 \leq \sum_{j\in\mathbb{J}} v_j^2 \left\|\pi_{W_j}(h)\right\|^2 \leq B \|h\|^2 .$$

For fusion frames, the analysis and the synthesis operators are defined by:

$$T_{W,v} : \left(\sum_{j\in\mathbb{J}} \oplus W_j\right)_{\ell_2} \to H, \qquad T_{W,v}(f) = \sum_{j\in\mathbb{J}} v_j f_j,$$

and

$$T^*_{W,v} : H \to \left(\sum_{j\in\mathbb{J}} \oplus W_j\right)_{\ell_2}, \qquad T^*_{W,v}(\{f_j\}_{j\in\mathbb{J}}) = \{v_j \pi_{W_j} f_j\}_{j\in\mathbb{J}}.$$

Hence, the fusion frame operator is given by, for all $f \in H$,

$$\begin{aligned} S_{W,v}(f) &= T_{W,v} T^*_{W,v}(f) \\ &= \sum_{j\in\mathbb{J}} v_j^2 \pi_{W_j}(f). \end{aligned}$$

Now, we present some theorems in operator theory which will be needed in the next sections.

Lemma 1.5 ([9])**.** *Let $L_1 \in \mathcal{B}(H_1, H)$ and $L_2 \in \mathcal{B}(H_2, H)$. Then the following assertions are equivalent:*

(I) $\mathcal{R}(L_1) \subseteq \mathcal{R}(L_2)$;
(II) $L_1 L_1^* \leq \lambda L_2 L_2^*$ *for some* $\lambda > 0$;
(III) *there exists* $u \in \mathcal{B}(H_1, H_2)$ *such that* $L_1 = L_2 u$.

Moreover, if the above conditions are valid then there exists a unique operator u such that

(a) $\|u\|^2 = \inf\{\alpha > 0 \mid L_1 L_1^* \leq \alpha L_2 L_2^*\}$;
(b) $\ker L_1 = \ker u$;
(c) $\mathcal{R}(u) \subseteq \overline{\mathcal{R}(L_2^*)}$.

For the proof of the following lemma, we refer to [5].

Lemma 1.6. *Let $V \subseteq H$ be a closed subspace and u be a linear bounded operator on H. Then*

$$\pi_V u^* = \pi_V u^* \pi_{\overline{uV}}.$$

If an operator u has closed range, then there exists a right-inverse operator $u^\dagger$ (pseudo-inverse of u) in the following sense (see [8]):

Lemma 1.7. *Let $u \in \mathcal{B}(K,H)$ be a bounded operator with closed range $\mathcal{R}_u$. Then there exists a bounded operator $u^\dagger \in \mathcal{B}(H,K)$ for which*

$$uu^\dagger x = x, \quad x \in \mathcal{R}_u.$$

Lemma 1.8. *Let $u \in \mathcal{B}(K,H)$. Then the following assertions holds:*

(I) *$\mathcal{R}_u$ is closed in H if and only if $\mathcal{R}_{u^*}$ is closed in K;*
(II) *$(u^*)^\dagger = (u^\dagger)^*$;*
(III) *The orthogonal projection of H onto $\mathcal{R}_u$ is given by $uu^\dagger$;*
(IV) *The orthogonal projection of K onto $\mathcal{R}_{u^\dagger}$ is given by $u^\dagger u$;*
(V) *On $\mathcal{R}(u)$, the operator $u^\dagger$ is given explicitly by $u^\dagger = u^*(uu^*)^{-1}$.*

2. Continuous k-Fusion Frames

Throughout this paper, $\mathbb{H}$ is the collection of all closed subspaces of H and $k \in \mathcal{B}(H)$. Suppose that $F : X \to \mathbb{H}$ and we denote by $\mathscr{L}^2(X,F)$ the class of all weakly measurable mappings $f : X \to H$ (i.e. for all $h \in H$, the mapping $x \to \langle f(x), h\rangle$ is measurable) such that for any $x \in X$, $f(x) \in F(x)$ and

$$\int_X \|f(x)\|^2 \, d\mu < \infty.$$

It can be a Hilbert space with the inner product defined by

$$\langle f, g\rangle = \int_X \langle f(x), g(x)\rangle \, d\mu, \quad f, g \in \mathscr{L}^2(X,F).$$

Definition 2.1. Assume that $F : X \to \mathbb{H}$ such that for each $h \in H$, the mapping $x \to \pi_{F(x)}(h)$ is measurable (i.e. is weakly measurable) and $v : X \to \mathbb{R}^+$ be a measurable function. Then F is called a continuous k-fusion frame (or ck-fusion frame) with respect to v for H if there exist $0 < A \le B < \infty$ such that for each $h \in H$

$$A \|k^* h\|^2 \le \int_X v^2(x) \left\|\pi_{F(x)}(h)\right\|^2 \, d\mu \le B \|h\|^2 . \tag{2.1}$$

Since each ck-fusion frame is a c-fusion Bessel, so the synthesis, analysis, and ck-fusion frame operators are defined. Indeed, the synthesis operator is defined weakly as follows (for more details see [13]):

$$T_F : \mathscr{L}^2(X,F) \to H,$$
$$T_F(f) = \int_X v f \, d\mu,$$

where for each $h \in H$

$$\langle T_F(f), h\rangle = \int_X v(x) \langle f(x), h\rangle \, d\mu.$$

The analysis operator is given by

$$T_F^*: H \longrightarrow \mathscr{L}^2(X, F),$$
$$T_F^*(h) = v\pi_F(h).$$

Finally, the ck- fusion frame operator $S_F := T_F T_F^*$ is defined by

$$S_F : H \longrightarrow H,$$
$$S_F(h) = \int_X v^2 \pi_F(h)\, d\mu.$$

Hence for each $h_1, h_2 \in H$

$$\langle S_F(h_1), h_2\rangle = \int_X v^2(x) \left\langle \pi_{F(x)} h_1, h_2 \right\rangle \, d\mu.$$

Notice that for each $h \in H$

$$\begin{aligned}\langle S_F(h), h\rangle &= \|T_F^*(h)\|^2 \\ &= \int_X v^2(x) \left\|\pi_{F(x)}(h)\right\|^2 \, d\mu \\ &\geq A \|k^* h\|^2 \\ &= \langle Akk^* h, h\rangle .\end{aligned}$$

Therefore,

$$Akk^* \leq S_F \leq BI.$$

By Lemma 1.5, we can conclude that $\mathcal{R}(k) \subseteq \mathcal{R}(T_F)$.

The ck-fusion frame operator (like the k-fusion frame operator) S_F is not invertible. In the following remark a condition for invertibility is presented:

Proposition 2.2. *If $k \in \mathcal{B}(H)$ has closed range, then the operator S_F is an invertible operator on a subspace of $\mathcal{R}(k) \subset H$.*

Proof. For any $h \in \mathcal{R}(k)$, we have

$$\begin{aligned}\|h\|^2 &= \left\|(k^\dagger|_{\mathcal{R}(k)})^* k^* h\right\|^2 \\ &\leq \left\|k^\dagger\right\|^2 . \|k^* h\|^2 .\end{aligned}$$

Now, we obtain

$$A \left\|k^\dagger\right\|^{-2} \|h\|^2 \leq \langle S_F(h), h\rangle \leq B \|h\|^2 ,$$

which implies that $S_F : \mathcal{R}(k) \to S_F(\mathcal{R}(k))$ is a homeomorphism, furthermore, for any $h \in S_F(\mathcal{R}(k))$, we have

$$B^{-1} \|h\|^2 \leq \left\langle (S_F|_{\mathcal{R}(k)})^{-1}(h), h \right\rangle \leq A^{-1} \left\|k^\dagger\right\|^2 \|h\|^2 . \tag{2.2}$$

□

Remark 2.3. We saw that $S_F \in \mathcal{B}(H)$ and S_F is positive and self-adjoint. Since $\mathcal{B}(H)$ is a C^*-algebra, then

$$(S_F^{-1})^* = (S_F^*)^{-1} = S_F^{-1},$$

whenever $k \in \mathcal{B}(H)$ has closed range. Thus, S_F^{-1} is self-adjoint and positive too. Now, for any $h \in S_F(\mathcal{R}(k))$, we can write

$$\begin{aligned}\langle kh, h\rangle &= \left\langle kh, S_F S_F^{-1}(h)\right\rangle \\ &= \left\langle S_F(kh), S_F^{-1}(h)\right\rangle \\ &= \int_X v^2(x)\left\langle \pi_{F(x)}(kh), S_F^{-1}(h)\right\rangle\, d\mu \\ &= \int_X v^2(x)\left\langle S_F^{-1}\left(\pi_{F(x)}(kh)\right), h\right\rangle\, d\mu.\end{aligned}$$

Proposition 2.4. *Suppose that μ is a σ-finite measure and F is a ck-fusion frame for a dense subset V of H with bounds A and B, respectively. Then F is a ck-fusion frame for H with same bounds.*

Proof. Suppose that F is not a c-fusion Bessel mapping for H. Then, there exists $h \in H$ such that

$$\int_X v^2(x)\left\|\pi_{F(x)}(h)\right\|^2\, d\mu > B\left\|h\right\|^2.$$

Assume that $\{X_n\}_{n=1}^{\infty}$ be a sequence of measurable and mutually disjoint subsets of X such that $X = \cup_{n=1}^{\infty} X_n$ and $\mu(X_n) < \infty$ for each $n \in \mathbb{N}$. Let

$$\Delta_m = \{x \in X : m \le \|f(x)\| < m+1,\ \forall f \in \mathscr{L}^2(X,F)\}, \quad m \ge 0.$$

It is easy to check that $\Delta_m \subseteq X$ is a measurable set for each $m \ge 0$ and $X = \cup_{m=0}^{\infty} \cup_{n=1}^{\infty} (X_n \cap \Delta_m)$. Therefore,

$$\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\int_{X_n\cap\Delta_m} v^2(x)\left\|\pi_{F(x)}(h)\right\|^2\, d\mu > B\left\|h\right\|^2.$$

Thus, there exist finite subspaces $\mathbb{I}$, $\mathbb{J}$ such that

$$\sum_{m\in\mathbb{I}}\sum_{n\in\mathbb{J}}\int_{X_n\cap\Delta_m} v^2(x)\left\|\pi_{F(x)}(h)\right\|^2\, d\mu > B\left\|h\right\|^2. \tag{2.3}$$

Let $\{h_j\}_{j=1}^{\infty}$ be a sequence in V such that $\lim_{j\to\infty} h_j = h$. So, for any $j \ge 1$, we have

$$\sum_{m\in\mathbb{I}}\sum_{n\in\mathbb{J}}\int_{X_n\cap\Delta_m} v^2(x)\left\|\pi_{F(x)}(h_j)\right\|^2\, d\mu \le B\left\|h_j\right\|^2,$$

and by the Lebesgue's Dominated Convergence Theorem, it is a contradiction with (2.3) and F is a c-fusion Bessel mapping for H. Now, the analysis operator T_F^* is well-defined for H. Let $h \in H$ be arbitrary and $\{h_j\}_{j=1}^{\infty}$ be a sequence in V such that $\lim_{j\to\infty} h_j = h$; then

$$A \|k^* h_j\|^2 \leq \|T_F^*(h_j)\|^2 .$$

Therefore, if $j \to \infty$, we obtain

$$\begin{aligned} A \|k^* h\|^2 &\leq \|T_F^*(h)\|^2 \\ &= \int_X v^2(x) \left\|\pi_{F(x)}(h)\right\|^2 \, d\mu, \end{aligned}$$

and the proof is completed. □

Proposition 2.5. *Let $u \in \mathcal{B}(H_1, H_2)$ be an invertible operator and F be a c-fusion Bessel mapping with respect to v for H_1 with the bound B. Then uF is a c-fusion Bessel mapping with respect to v for H_2.*

Proof. By applying Lemma 1.6 and the fact that u is invertible, for each $h \in H_2$, we have

$$\begin{aligned} \int_X v^2(x) \left\|\pi_{uF(x)}(h)\right\|^2 \, d\mu &= \int_X v^2(x) \left\|\pi_{uF(x)}(u^{-1})^* u^*(h)\right\|^2 \, d\mu \\ &= \int_X v^2(x) \left\|\pi_{uF(x)}(u^{-1})^* \pi_{F(x)} u^*(h)\right\|^2 \, d\mu \\ &\leq \left\|u^{-1}\right\|^2 \int_X v^2(x) \left\|\pi_{F(x)} u^*(h)\right\|^2 \, d\mu \\ &\leq B \left\|u^{-1}\right\|^2 \|u\|^2 \|h\|^2 . \end{aligned}$$

□

Theorem 2.6. *Let $k \in \mathcal{B}(H)$ has closed range and F be a ck-fusion frame for H with respect to v with bounds A and B, respectively. Then*

(I) *If $\pi_{\mathcal{R}(k)}F$ is a c-fusion Bessel mapping, then $k^\dagger F$ is a c-fusion frame for $\mathcal{R}(k^*)$;*
(II) *If $u \in \mathcal{B}(H)$ is an invertible operator, then uF is a c(uk)-fusion frame for H;*
(III) *If $u \in \mathcal{B}(H)$ is an invertible operator and $ku = uk$, then uF is a ck-fusion frame for H;*
(IV) *If $u \in \mathcal{B}(H)$ and $\mathcal{R}(u) \subseteq \mathcal{R}(k)$, then F is also a cu-fusion frame.*

Proof. (I). Let $h \in \mathcal{R}(k^*)$ and $x \in X$, then by Lemma 1.6, we have

$$\int_X v^2(x) \left\|\pi_{k^\dagger F(x)}(h)\right\|^2 \, d\mu = \int_X v^2(x) \left\|\pi_{k^\dagger F(x)} k^* (k^\dagger)^*(h)\right\|^2 \, d\mu$$

$$
\begin{aligned}
&= \int_X v^2(x) \left\| \pi_{k^\dagger F(x)} k^* \pi_{kk^\dagger F(x)} (k^\dagger)^*(h) \right\|^2 d\mu \\
&\le \|k\|^2 \int_X v^2(x) \left\| \pi_{\pi_{\mathcal{R}(k)} F(x)} (k^\dagger)^*(h) \right\|^2 d\mu.
\end{aligned}
$$

Hence, by the assumption, $k^\dagger F$ is a c-fusion Bessel mapping. On the other hand, there exists $A > 0$ such that

$$
\begin{aligned}
A\|h\|^2 &= A \left\| k^*(k^*)^\dagger h \right\|^2 \\
&\le \int_X v^2(x) \left\| \pi_{F(x)} (k^*)^\dagger (h) \right\|^2 d\mu \\
&= \int_X v^2(x) \left\| \pi_{F(x)} (k^*)^\dagger \pi_{k^\dagger F(x)} (h) \right\|^2 d\mu \\
&\le \left\| k^\dagger \right\|^2 \int_X v^2(x) \left\| \pi_{k^\dagger F(x)} (h) \right\|^2 d\mu,
\end{aligned}
$$

and (I) holds.
(II). Let $u \in \mathcal{B}(H)$ be an invertible operator. By Proposition 2.5, uF is a c-fusion Bessel mapping for H. So, by applying Lemma 1.6, for each $h \in H$, we obtain

$$
\begin{aligned}
A\|k^* u^* h\|^2 &\le \int_X v^2(x) \left\| \pi_{F(x)} u^*(h) \right\|^2 d\mu \\
&= \int_X v^2(x) \left\| \pi_{F(x)} u^* \pi_{uF(x)} (h) \right\|^2 d\mu \\
&\le \|u\|^2 \int_X v^2(x) \left\| \pi_{uF(x)} (h) \right\|^2 d\mu,
\end{aligned}
$$

and (II) is completed.
(III). By (II) , it is clear.
(IV). Via Lemma 1.5, there exists $\lambda > 0$ such that $uu^* \le \lambda^2 kk^*$. Thus, for any $h \in H$

$$
\begin{aligned}
\|u^* h\|^2 &= \langle uu^* h, h \rangle \\
&\le \lambda^2 \langle kk^* h, h \rangle \\
&= \lambda^2 \|k^* h\|^2 .
\end{aligned}
$$

It follows that

$$
\frac{A}{\lambda^2} \|u^* h\|^2 \le \int_X v^2(x) \left\| \pi_{F(x)} (h) \right\|^2 d\mu,
$$

and this shows that F is a cu-fusion frame. □

Theorem 2.7. *Let k be a closed range operator and F be a c-fusion frame with respect to v for $\mathcal{R}(k^*)$. Then kF is a ck-fusion frame for H.*

Proof. Assume that B is an upper bound for F and $h \in H$. We have $h = f + g$ which $f \in \mathcal{R}(k)$ and $g \in \mathcal{R}(k)^{\perp}$. Thus

$$\begin{aligned}
\int_X v^2(x) \left\|\pi_{kF(x)}(h)\right\|^2 d\mu &= \int_X v^2(x) \left\|\pi_{kF(x)}(k^{\dagger})^* k^* f\right\|^2 d\mu \\
&= \int_X v^2(x) \left\|\pi_{kF(x)}(k^{\dagger})^* \pi_{k^{\dagger}kF(x)} k^* f\right\|^2 d\mu \\
&\leq \left\|k^{\dagger}\right\|^2 \int_X v^2(x) \left\|\pi_{\pi_{\mathcal{R}(k^{\dagger})}F(x)} k^* f\right\|^2 d\mu \\
&\leq \left\|k^{\dagger}\right\|^2 \int_X v^2(x) \left\|\pi_{F(x)} k^* f\right\|^2 d\mu \\
&\leq B \left\|k^{\dagger}\right\|^2 \|k\|^2 \|h\|^2 .
\end{aligned}$$

Hence, kF is a c-fusion Bessel fusion mapping. On the other hand, there exists $A > 0$ such that for any $h \in H$

$$\begin{aligned}
A \|k^* h\|^2 &\leq \int_X v^2(x) \left\|\pi_{F(x)} k^*(h)\right\|^2 d\mu \\
&= \int_X v^2(x) \left\|\pi_{F(x)} k^* \pi_{kF(x)}(h)\right\|^2 d\mu \\
&\leq \|k\|^2 \int_X v^2(x) \left\|\pi_{kF(x)}(h)\right\|^2 d\mu,
\end{aligned}$$

and the proof is completed. □

The following result follows immediately from Theorem (2.7) and the fact that $\mathcal{R}(k^*) \subseteq H$.

Corollary 2.8. *Let k be a closed range operator and F be a c-fusion frame with respect to v for H. Then $kF|_{\mathcal{R}(k^*)}$ is a ck-fusion frame for H.*

3. Q-Duality of Continuous k-fusion Frames

In this section, we shall define the duality of ck-fusion frames and present some properties of them.

Definition 3.1. Suppose that F is a ck-fusion frame with respect to v for H. A c-fusion Bessel mapping G with respect to w is called the Q-dual ck-fusion frame of F (or cQk-dual for F) if there exsits a bounded linear operator $Q : \mathscr{L}^2(X, F) \to \mathscr{L}^2(X, G)$ such that

$$T_F Q^* T_G^* = k. \tag{3.1}$$

The following theorem presents some conditions which are equivalent to the ones given in (3.1).

Theorem 3.2. *Let G be a cQk-dual for F. The following conditions are equivalent:*

(I) $T_F Q^* T_G^* = k$;
(II) $T_G Q T_F^* = k^*$;
(III) *for each $h, h' \in H$, we have*

$$\begin{aligned} \langle kh, h' \rangle &= \langle T_G^*(h), QT_F^*(h') \rangle \\ &= \langle Q^* T_G^*(h), T_F^*(h') \rangle . \end{aligned}$$

Proof. The proof is straightforward. □

Example 3.3. Suppose that $H = \mathbb{R}^2$ with the standard base $\{e_1, e_2\}$ and

$$\mathscr{B} := \{x \in \mathbb{R}^2 : \|x\| \leq 1\},$$

equipped with the Lebesgue measure λ form a measure space. Suppose $\{B_1, B_2\}$ is a partition of $\mathscr{B}$ where $\lambda(B_1) \geq \lambda(B_2) > 1$. Let $\mathbb{H} = \{W_1, W_2\}$ which $W_1 = \text{span}\{e_1\}$ and $W_2 = \text{span}\{e_2\}$. Define

$$F : \mathscr{B} \longrightarrow \mathbb{H},$$

$$F(x) = \begin{cases} W_1, & x \in B_1, \\ W_2, & x \in B_2, \end{cases}$$

and

$$v : \mathscr{B} \longrightarrow [0, \infty),$$

$$v(x) = \begin{cases} \frac{1}{\sqrt{2\lambda B_1}}, & x \in B_1, \\ \frac{1}{\sqrt{\lambda B_2}}, & x \in B_2. \end{cases}$$

Put

$$ke_1 = e_2, \qquad ke_2 = e_1,$$

therefore, F is weakly measurable and is a ck-fusion frame with respect to v with frame bounds $\frac{1}{2}$ and 1. If we define

$$G : \mathscr{B} \longrightarrow \mathbb{H},$$

$$G(x) = \begin{cases} W_2, & x \in B_1, \\ W_1, & x \in B_2, \end{cases}$$

then G is also a ck-fusion frame with the same bounds of F. Let

$$Q : \mathscr{L}^2(\mathscr{B}, F) \longrightarrow \mathscr{L}^2(\mathscr{B}, G),$$

$$Q(x) = \begin{cases} 2e_2, & x \in B_1, \\ e_1, & x \in B_2. \end{cases}$$

So, Q is a linear bounded operator. Now, we have

$$T_G Q T_F^* e_1 = e_2 = k^* e_1,$$

and

$$T_G Q T_F^* e_2 = e_1 = k^* e_2.$$

Thus, G is a Q-dual of F by Theorem 3.2.

Theorem 3.4. *If G is a cQk-dual for F, then G is a ck^*-fusion frame with respect to w for H.*

Proof. Let $h \in H$ and B be an upper bound of F. Therefore

$$\begin{aligned}
\|kh\|^4 &= |\langle kh, kh\rangle|^2 \\
&= |\langle T_F Q^* T_G^*(h), kh\rangle|^2 \\
&= |\langle T_G^*(h), Q T_F^*(kh)\rangle|^2 \\
&\leq \|T_G^*(h)\|^2 \|Q\|^2 B \|kh\|^2 \\
&= \|Q\|^2 B \|kh\|^2 \int_X w^2(x) \left\|\pi_{G(x)}(h)\right\|^2 \, d\mu.
\end{aligned}$$

Now, by definition, the proof is completed. □

Corollary 3.5. *If C_{op} and D_{op} are the optimal bounds of G, respectively, then*

$$C_{op} \geq B_{op}^{-1} \|Q\|^{-2}, \qquad D_{op} \geq A_{op}^{-1} \|Q\|^{-2},$$

in which A_{op} and B_{op} are the optimal bounds of F, respectively.

We want to show a necessary condition on Q which satisfies the above definition. For that, we need the following lemma in [20].

Lemma 3.6. *Let F be a c-fusion Bessel frame with respect to v for H with bounds of B and $\dim H < \infty$. Then*

$$A \leq \int_X v^2(x) \, d\mu \leq B \dim H.$$

Theorem 3.7. *Let G be a cQk-dual for F with upper bounds of B_G and B_F, respectively. If $n := \dim H < \infty$, then*

$$\|Q\| \geq \frac{\|kh\|^2}{n \|k\| \|h\|^2} (B_F B_G)^{-\frac{1}{2}}.$$

Proof. Let $h \in H$. By (III) in Theorem 3.2, we can write

$$\begin{aligned}
\|kh\|^2 &\leq \|T_G^* h\| \|Q\| \|T_F^* kh\| \\
&= \|Q\| \left(\int_X w^2(x) \left\|\pi_{G(x)} h\right\|^2 \, d\mu\right)^{\frac{1}{2}} \left(\int_X v^2(x) \left\|\pi_{F(x)} kh\right\|^2 \, d\mu\right)^{\frac{1}{2}} \\
&\leq \|Q\| \|k\| \|h\|^2 \left(\int_X w^2(x) \, d\mu\right)^{\frac{1}{2}} \left(\int_X v^2(x) \, d\mu\right)^{\frac{1}{2}}.
\end{aligned}$$

Now, by Lemma 3.6 the proof is completed. □

Suppose that F is a ck-fusion frame with respect to v for H. Since $S_F \geq Akk^*$, then by Lemma 1.5, there exists an operator $u \in \mathcal{B}\left(H, \mathscr{L}^2(X,F)\right)$ such that

$$T_F u = k. \tag{3.2}$$

By this operator, we may construct some cQk-duals for H.

Theorem 3.8. *Let F be a ck-fusion frame with respect to v for H. If u is an operator as in (3.2) and $\widehat{F}(x) := u^*uF(x)$, for all $x \in X$, is a c-fusion Bessel mapping with respect to v, then $\widehat{F}$ is a cQk-dual for F.*

Proof. We define the mapping

$$\begin{aligned}\Phi_0 : \mathcal{R}(T^*_{\widehat{F}}) \to \mathscr{L}^2(X,F),\\ \Phi_0(T^*_{\widehat{F}}h) = uh.\end{aligned}$$

Then Φ_0 is well-defined. Indeed, if $h_1, h_2 \in H$ and $T^*_{\widehat{F}}h_1 = T^*_{\widehat{F}}h_2$, then $\pi_{\widehat{F}}(h_1 - h_2) = 0$. Thus

$$h_1 - h_2 \in (\widehat{F})^{\perp} = \mathcal{R}(u^*)^{\perp} = \ker u,$$

then $uh_1 = uh_2$. It is clear that Φ_0 is bounded and linear. Therefore, it has a unique linear extension (also denoted by Φ_0) to $\overline{\mathcal{R}(T^*_{\widehat{F}})}$. Define Φ on $\mathscr{L}^2(X,\widehat{F})$ by setting

$$\Phi = \begin{cases}\Phi_0, & \text{on } \overline{\mathcal{R}(T^*_{\widehat{F}})},\\ 0, & \text{on } \overline{\mathcal{R}(T^*_{\widehat{F}})}^{\perp},\end{cases}$$

and let $Q := \Phi^*$. This implies that $Q^* \in \mathcal{B}\left(\mathscr{L}^2(X,\widehat{F}), \mathscr{L}^2(X,F)\right)$ and

$$\begin{aligned}T_F Q^* T^*_{\widehat{F}} &= T_F \Phi T^*_{\widehat{F}}\\ &= T_F u\\ &= k.\end{aligned}$$

□

Proposition 3.9. *Let F be a ck-fusion frame with optimal bounds of A_{op} and B_{op}, respectively and k has closed range. Then*

$$B_{op} = \|S_F\| = \|T_F\|^2, \qquad A_{op} = \|u_0\|^{-2},$$

in which u_0 is a unique solution of equation (3.2).

Proof. By using Lemma 1.5, the equation (3.2) has a unique solution as u_0 such that

$$\begin{aligned}\|u_0\|^2 &= \inf\{\alpha > 0 \ : kk^* \leq \alpha T_F T^*_F\}\\ &= \inf\{\alpha > 0 \ : \|k^*h\|^2 \leq \alpha \|T^*_F h\|^2, h \in H\}.\end{aligned}$$

Now, we obtain

$$\begin{aligned} A_{op} &= \sup\{A > 0 : A\,\|k^*h\|^2 \leq \|T_F^*h\|^2\,, h \in H\} \\ &= \left(\inf\{\alpha > 0 : \|k^*h\|^2 \leq \alpha\,\|T_F^*h\|^2\,, h \in H\}\right)^{-1} \\ &= \|u_0\|^{-2}. \end{aligned}$$

□

4. Perturbation of ck-fusion Frames

The perturbation of frames have been discussed by Cazassa and Christensen in [4]. For k-fusion frames, it is studied by Arabyani and Arefijamaal in [1]. We aim to present it for ck-fusion frames.

Theorem 4.1. *Let F be a ck-fusion frame with respect to $v \in \mathscr{L}^2(\mu)$ for H with bounds A and B, respectively, $G : X \to \mathbb{H}$ be weakly measurable and $w : X \to \mathbb{R}^+$ be a measurable function. If for some $0 < \lambda_1, \lambda_2 < 1$ and $\varepsilon > 0$*

$$\begin{aligned} \left\|\left(v(x)\pi_{F(x)} - w(x)\pi_{G(x)}\right)(h)\right\| &\leq \lambda_1 \left\|v(x)\pi_{F(x)}(h)\right\| + \lambda_2 \left\|w(x)\pi_{G(x)}(h)\right\| \\ &\quad + \varepsilon v(x)\,\|k^*h\|, \end{aligned}$$

for any $h \in H$ and $x \in X$ such that

$$\varepsilon < \frac{(1-\lambda_1)\sqrt{A}}{\|k\|}\left(\int_X v^2(x)\,d\mu\right)^{-\frac{1}{2}}, \tag{4.1}$$

then G is a ck-fusion frame with respect to w with bounds

$$\left(\frac{\sqrt{A}(1-\lambda_1)(\|k\|-1)}{\|k\|\,(1+\lambda_2)}\right)^2, \qquad \left(\frac{(1+\lambda_1)\sqrt{B} + (1-\lambda_1)\sqrt{A}}{(1-\lambda_2)}\right)^2.$$

Proof. Suppose that $h \in H$ and $x \in X$. We have

$$\begin{aligned} &\left(\int_X w^2(x)\left\|\pi_{G(x)}(h)\right\|^2 d\mu\right)^{\frac{1}{2}} \\ &= \left(\int_X \left\|w(x)\pi_{G(x)}(h) - v(x)\pi_{F(x)}(h) + v(x)\pi_{F(x)}(h)\right\|^2 d\mu\right)^{\frac{1}{2}} \\ &\leq \left(\int_X \left\{(1+\lambda_1)\left\|v(x)\pi_{F(x)}(h)\right\| + \lambda_2\left\|w(x)\pi_{G(x)}(h)\right\| + \varepsilon v(x)\,\|k^*h\|\right\}^2 d\mu\right)^{\frac{1}{2}} \\ &\leq (1+\lambda_1)\left(\int_X \left\|v(x)\pi_{F(x)}(h)\right\|^2 d\mu\right)^{\frac{1}{2}} + \lambda_2\left(\int_X \left\|w(x)\pi_{G(x)}(h)\right\|^2 d\mu\right)^{\frac{1}{2}} \\ &\quad + \varepsilon\,\|k^*h\|\left(\int_X v^2(x)\,d\mu\right)^{\frac{1}{2}}. \end{aligned}$$

Using (4.1) we have

$$\int_X w^2(x)\left\|\pi_{G(x)}(h)\right\|^2 d\mu \leq \left(\frac{(1+\lambda_1)\sqrt{B}+(1-\lambda_1)\sqrt{A}}{(1-\lambda_2)}\right)^2 \|h\|^2 .$$

This means that G is a c-fusion Bessel mapping with respect w. Now, for the lower bound, we can write

$$\begin{aligned}
&\left(\int_X w^2(x)\left\|\pi_{G(x)}(h)\right\|^2 d\mu\right)^{\frac{1}{2}}\\
&\quad= \left(\int_X \left\|w(x)\pi_{G(x)}(h)-v(x)\pi_{F(x)}(h)+v(x)\pi_{F(x)}(h)\right\|^2 d\mu\right)^{\frac{1}{2}}\\
&\quad\geq \left(\int_X \left\{(1-\lambda_1)\left\|v(x)\pi_{F(x)}(h)\right\|-\lambda_2\left\|w(x)\pi_{G(x)}(h)\right\|-\varepsilon v(x)\|k^*h\|\right\}^2 d\mu\right)^{\frac{1}{2}}\\
&\quad\geq (1-\lambda_1)\left(\int_X \left\|v(x)\pi_{F(x)}(h)\right\|^2 d\mu\right)^{\frac{1}{2}}-\lambda_2\left(\int_X \left\|w(x)\pi_{G(x)}(h)\right\|^2 d\mu\right)^{\frac{1}{2}}\\
&\qquad-\varepsilon\|k^*h\|\left(\int_X v^2(x)\,d\mu\right)^{\frac{1}{2}}.
\end{aligned}$$

Thus

$$\int_X w^2(x)\left\|\pi_{G(x)}(h)\right\|^2 d\mu \geq \left(\frac{\sqrt{A}(1-\lambda_1)(\|k\|-1)}{\|k\|(1+\lambda_2)}\right)^2 \|k^*h\|^2 ,$$

and the proof is completed. □

In case of $\mu(X)<\infty$, we can show a simple case of perturbation in the following proposition.

Proposition 4.2. *Let F be a ck-fusion frame with respect to v for H with bounds A and B, respectively and $G: X\to\mathbb{H}$ be weakly measurable. If for each $h\in H$ and $x\in X$*

$$\left\|v(x)\left(\pi_{F(x)}-\pi_{G(x)}\right)(h)\right\| \leq \varepsilon\|k^*h\| ,$$

for some $0<\varepsilon<\mu^{-1}\sqrt{A}$ and $\mu:=\mu(X)<\infty$, then G is a ck-fusion frame with respect to v for H with bounds

$$(A-\mu^2\varepsilon^2), \qquad (B+\mu^2\varepsilon^2\|k\|^2).$$

Proof. Let $h\in H$ and $x\in X$, then

$$\begin{aligned}
\int_X v^2(x)\left\|\pi_{G(x)}(h)\right\|^2 d\mu &\leq \int_X v^2(x)\left\|\pi_{F(x)}(h)\right\|^2 d\mu\\
&\quad+\int_X v^2(x)\left\|(\pi_{F(x)}-\pi_{G(x)})(h)\right\|^2 d\mu
\end{aligned}$$

$$\leq \left(B + \mu^2 \varepsilon^2 \|k\|^2\right) \|h\|^2 .$$

Therefore, G is a c-fusion Bessel mapping. On the other hand

$$\begin{aligned}\int_X v^2(x) \left\|\pi_{G(x)}(h)\right\|^2 d\mu &\geq \int_X v^2(x) \left\|\pi_{F(x)}(h)\right\|^2 d\mu \\ &\quad - \int_X v^2(x) \left\|(\pi_{F(x)} - \pi_{G(x)})(h)\right\|^2 d\mu \\ &\geq \left(A - \mu^2 \varepsilon^2\right) \|k^* h\|^2 .\end{aligned}$$

This completes the proof. □

5. Conclusions

The study of the ck-fusions shows that the emphasis on the measure spaces introduces a new idea, although some similar properties with the discrete case are raised. Moreover, due to the nature of measure spaces, we have to use new techniques for new results. Especially, the topic of the dual of frames which is important for frame applications, has been specified completely for the ck-fusions. Observance of perturbation for ck-fusions are raised and the required results have been obtained. For an open problem in the continuation of this research work, the problem of the optimal deletion of the measure spaces and reproduction of the ck-fusions and the design of Kadison-Singer problem can be mentioned (see [7, 18]).

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